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Block-Error Performance of Root-LDPC Codes

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***Abstract*—This paper[1] investigates the error rate of root-LDPC (RLDPC) codes. These codes were introduced in [1], as a class of codes achieving full diversity D over a nonergodic block-fading transmission channel, and hence with an error probability decreasing as SNR^{-D} at high signal-to-noise ratios. As for their structure, root-LDPC codes can be viewed as a special case of multiedge-type LDPC codes [2]. However, RLDPC code optimization for nonergodic channels does not follow the same criteria as those applied for standard ergodic erasure or Gaussian channels. While previous analyses of RLDPC codes were based on their asymptotic *bit* threshold for information variables under iterative decoding, in this work we investigate asymptotic *block* threshold. A stability condition is first derived for a given fading channel realization. Then, in a similar way as for unstructured LDPC codes [3], with the help of Bhattacharyya parameter, we state a sufficient condition for a vanishing block-error probability with the number of decoding iterations.**

I. Introduction and motivation of our work

When a block of encoded data is sent, after being split into F subblocks, through F independent slow-fading channels, the appropriate channel model is nonergodic. This model may correspond to a parallel (MIMO systems) or to a sequential (HARQ protocols) data-transmission scheme.

It turns out that special design criteria are needed for codes to be used with such a model — in particular, full transmit diversity is sought, which guarantees that, at large signal-to-noise ratios (SNR), the error probability of the transmission scheme scales as $1/\mathrm{SNR}^D$, with D the maximum diversity order achievable. It has been shown in [1] that standard sparse-graph code ensembles allow one to obtain error probabilities decreasing only as $1/\mathrm{SNR}$, and hence they are not full-diversity ensembles. Even infinite-length random code ensembles cannot achieve full diversity, as shown via a diversity population evolution technique in [4].

The key idea for codes achieving full diversity is to ensure that each information node is receiving multiple messages affected by independent fading coefficients. This idea has been implemented in RLDPC codes [1] designed for block-fading channels with $F = 2$ by introducing the concept of *root checknodes*. A root checknode protects a message received from the second subchannel when the variable node is received from the first subchannel. RLDPC codes are full-diversity codes (thus, they are also Maximum Distance Separable in the Singleton-bound sense) and can be devised for any diversity order.

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In this paper we focus on rate-$1/2$, diversity-2 RLDPC codes, and study their stability under iterative decoding. We also derive a sufficient condition for vanishing block-error probability. As expected, since root checknodes occupy a single edge in each information variable, stability and block-error performance of RLDPC codes depend on the fraction of variables with degrees 2 and 3.

II. Transmission model

Under our assumptions, a block of encoded data (a codeword) is divided into two equal subblocks, each one being transmitted over an independent Rayleigh fading channel with SNR$=\gamma$ and fading coefficients α_1 and α_2. Therefore, the observation y corresponding to the binary transmitted symbol $x = \pm 1$ received from the i-th channel is $y = \alpha_i x + z$, where $\alpha_i \in [0, +\infty)$, and $z \sim \mathcal{N}(0, \sigma^2)$ with $\sigma^2 = 1/\gamma$.

III. RLDPC Codes: Definition and Density Evolution

A. Definition

Given an initial (λ, ρ) LDPC ensemble, one defines a (λ, ρ) RLDPC ensemble with diversity 2 through the multinomials $\lambda_{\mathrm{root}}(\underline{\mu}, \underline{x})$ and $\rho_{\mathrm{root}}(\underline{\mu}, \underline{x})$, with $\underline{\mu} \triangleq (\mu_1, \mu_2)$ and $\underline{x} \triangleq (x_1, x_2, x_3, x_4, x_5, x_6)$:

$$\lambda_{\mathrm{root}}(\underline{\mu}, \underline{x}) \triangleq \frac{1}{2}\sum_i \left(\frac{\lambda_i}{i}\mu_1 x_1^i + \frac{(i-1)\lambda_i}{i}\mu_1 x_2^i + \lambda_i \mu_1 x_3^i + \lambda_i \mu_2 x_4^i + \frac{(i-1)\lambda_i}{i}\mu_2 x_5^i + \frac{\lambda_i}{i}\mu_2 x_6^i \right), \quad (1)$$

$$\rho_{\mathrm{root}}(\underline{\mu}, \underline{x}) \triangleq \frac{1}{2}\sum_i \rho_i \left(x_1 \sum_j \binom{i}{j} f_e^j x_4^j g_e^{i-j} x_5^{i-j} + x_6 \sum_k \binom{i}{k} f_e^k x_3^k g_e^{i-k} x_2^{i-k} \right), \quad (2)$$

where the fractions f_e and g_e will be defined in next subsection. In words, the structure of the RLDPC ensemble consists of four types of variable nodes ($1i$, $1p$, $2i$, $2p$), two sets of check nodes ($1c$, $2c$), and 6 different edge classes (see Fig.1a). Permutations of edges within edge classes are chosen uniformly at random. Variable nodes $1i$ and $1p$ correspond to information and redundancy bits, respectively, in a codeword sent through the first fading subchannel. Similarly, variable nodes $2i$ and $2p$ correspond to bits sent through the second subchannel. Note that the information variable nodes ki ($k = 1, 2$) are connected to check nodes of the same type, kc, through exactly one edge; all other edges are connected to check nodes of the other type. Redundancy variable nodes are always connected to check nodes of different type. In (1)-(2), μ_1 and μ_2 correspond to two fading subchannels, and the variables $x_1, x_2, \ldots, x_6$ to the following edge classes: $1i \to 1c$, $1i \to 2c$, $1p \to 2c$, $2p \to 1c$, $2i \to 1c$, and $2i \to 2c$.

We have thus obtained a code ensemble of rate $1/2$. As shown in [1], such a construction guarantees transmit diversity 2, which is the maximum we can obtain with two independent transmission subchannels.

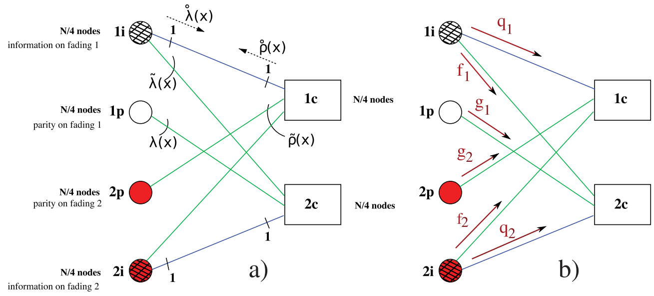

Fig. 1. Structure of a (λ, ρ) RLDPC code ensemble of diversity 2.

B. Density Evolution

RLDPC codes are decoded, as standard LDPC codes, using an iterative algorithm. An asymptotic analysis of iterative decoding is provided in [1], [4] and we shall summarize it here, after giving some notation. We denote the probability density functions (pdfs) of channel LLR outputs from the two transmission subchannels by $\mu_1(x)$ and $\mu_2(x)$, respectively. These are normal pdfs with means $2\alpha_1^2/\gamma$ and $2\alpha_2^2/\gamma$ and variances $4\alpha_1^2/\gamma$ and $4\alpha_2^2/\gamma$, respectively. Further, we denote by $\otimes$ the operation of convolution of two pdfs. We also define the following operation:

Definition 1: The R-convolution of two pdfs $\alpha(x)$ and $\beta(x)$ is

$$\alpha \odot \beta(x) = f(\hat{\alpha}(x) \otimes \hat{\beta}(x)),$$

where

$$\hat{\alpha}(x) \triangleq \frac{2\alpha(2\text{th}^{-1}(x))}{1-x^2}, \quad \hat{\beta}(x) = \frac{2\beta(2\text{th}^{-1}(x))}{1-x^2}$$

and

$$f(x) = \cosh^2\left(\frac{\hat{\alpha} \otimes \hat{\beta}(x)}{2}\right) \text{th}^{-1}(\hat{\alpha} \otimes \hat{\beta}(x)).$$

Note that the R-convolution of pdfs corresponds to the following operation over the corresponding random variables A and B:

$$2\text{th}^{-1}\big(\text{th}(A/2) + \text{th}(B/2)\big),$$

which is exactly the operation performed at the check nodes.

Let us denote the average pdfs for 6 edge sets by $q_1(x)$, $f_1(x)$, $g_1(x)$, $g_2(x)$, $f_2(x)$, and $q_2(x)$ as shown in Fig.1b. Then the evolution of the pdfs at the iteration $m+1$ can be described by the following recursions:

$$\begin{aligned}
q_1^{m+1}(x) &= \mu_1(x) \otimes \mathring{\lambda}(\tilde{\rho}(q_2^m(x), f_e f_1^m(x) + g_e g_1^m(x))) \\
f_1^{m+1}(x) &= \mu_1(x) \otimes \tilde{\lambda}(\tilde{\rho}(q_2^m(x), f_e f_1^m(x) + g_e g_1^m(x))) \\
&\quad \otimes \mathring{\rho}(f_e f_2^m(x) + g_e g_2^m(x)) \\
g_1^{m+1}(x) &= \mu_1(x) \otimes \lambda(\tilde{\rho}(q_2^m(x), f_e f_1^m(x) + g_e g_1^m(x))) \\
g_2^{m+1}(x) &= \mu_2(x) \otimes \lambda(\tilde{\rho}(q_1^m(x), f_e f_2^m(x) + g_e g_2^m(x))) \\
f_2^{m+1}(x) &= \mu_2(x) \otimes \tilde{\lambda}(\tilde{\rho}(q_1^m(x), f_e f_2^m(x) + g_e g_2^m(x))) \\
&\quad \otimes \mathring{\rho}(f_e f_1^m(x) + g_e g_1^m(x)) \\
q_2^{m+1}(x) &= \mu_2(x) \otimes \mathring{\lambda}(\tilde{\rho}(q_1^m(x), f_e f_2^m(x) + g_e g_2^m(x)))
\end{aligned}$$

where we have borrowed from [4] the following notation:

$$\tilde{\lambda}(x) \triangleq \frac{\bar{d}_b}{\bar{d}_b - 1} \sum_i \frac{\lambda_i(i-1)}{i} x^{\otimes(i-2)}; \qquad \bar{d}_b \triangleq 1/\sum_i \lambda_i/i;$$

$$\tilde{\rho}(x) \triangleq \frac{\bar{d}_c}{\bar{d}_c - 1} \sum_i \frac{\rho_i(i-1)}{i} x^{\odot(i-2)}; \qquad \bar{d}_c \triangleq 1/\sum_i \rho_i/i;$$

$$f_e \triangleq \frac{\sum_i (i-1)\frac{\lambda_i}{i}}{\sum_i (i-1)\frac{\lambda_i}{i} + 1} = \frac{\bar{d}_b - 1}{2\bar{d}_b - 1}; \qquad g_e \triangleq 1 - f_e;$$

$$\mathring{\lambda}(x) \triangleq \bar{d}_b \sum_i \frac{\lambda_i}{i} x^{\otimes(i-1)}; \qquad \mathring{\rho}(x) \triangleq \bar{d}_c \sum_i \frac{\rho_i}{i} x^{\odot(i-1)}.$$

Also, we define

$$\tilde{\rho}(q, x) \triangleq \frac{\bar{d}_c}{\bar{d}_c - 1} \sum_i \frac{\rho_i(i-1)}{i} q \odot x^{\odot(i-3)}.$$

IV. Stability Conditions

We are interested in defining stability conditions for RLDPC codes. The main difficulty here lies in the fact that not all messages need be recovered exactly (or, in LDPC jargon, not all pdfs converge to δ_∞). It is not hard to prove that only the pdfs responsible for the convergence of *information* messages, i.e., f_1 and f_2, need to converge for exact recovery of the information bits (this condition is also sufficient). The main concept of the proof is that f_1 and f_2 are strictly "better" than q_1 and q_2.

In this section we derive the stability condition for RLDPC codes based on the recovery of information bits only. Before starting our derivation, let us first apply the traditional stability condition [2] to RLDPC codes, assuming that *all* the code bits should be recovered. In such case the RLDPC codes are simply viewed as a multi-edge code ensemble,

A. RLDPCs as Multi-Edge Codes

The stability condition for multi-edge codes consists in ensuring that the spectral radius of a matrix M is < 1, where $M \triangleq B(\underline{\mu})\Lambda P$, with $B(\underline{\mu})$ the vector of Bhattacharyya parameters for all transmission channels, the Λ matrix corresponding to the variable node side of the graph, and P corresponding to the check node side. Applying the expressions derived in [2], we find that

$$B(\underline{\mu}) = \begin{pmatrix} B(\mu_1) & B(\mu_1) & B(\mu_1) & B(\mu_2) & B(\mu_2) & B(\mu_2) \end{pmatrix}^T$$
$$\Lambda = \begin{pmatrix} \frac{\bar{d}_b\lambda_2}{2} & \frac{\bar{d}_b\lambda_2}{2\bar{d}_b-2} & \lambda_2 & \lambda_2 & \frac{\bar{d}_b\lambda_2}{2\bar{d}_b-2} & \frac{\bar{d}_b\lambda_2}{2} \end{pmatrix} \cdot I$$
$$P = \begin{pmatrix} P_2 & P_1 & P_2 & P_3 & P_4 & P_3 \end{pmatrix}^T$$

with

$$P_1 \triangleq \begin{pmatrix} 0 & 0 & 0 & (\bar{d}_c-1)g_e & (\bar{d}_c-1)f_e & 0 \end{pmatrix}$$
$$P_2 \triangleq \begin{pmatrix} 0 & \tilde{\rho}'(1)f_e & \tilde{\rho}'(1)g_e & 0 & 0 & \tilde{\rho}(1) \end{pmatrix}$$
$$P_3 \triangleq \begin{pmatrix} \tilde{\rho}(1) & 0 & 0 & \tilde{\rho}'(1)f_e & \tilde{\rho}'(1)g_e & 0 \end{pmatrix}$$
$$P_4 \triangleq \begin{pmatrix} 0 & (\bar{d}_c-1)f_e & (\bar{d}_c-1)g_e & 0 & 0 & 0 \end{pmatrix}$$

Note that two eigenvalues of M are already 0.

B. RLDPCs as Full-Diversity Codes

By looking at RLDPC as at full-diversity codes, we only ask for the convergence of f_1 and f_2 to δ_∞. To derive a stability condition for this case, assume that, at iteration $m-1$,

$$f_1^{m-1} = \epsilon_1\delta_0 + (1-\epsilon_1)\delta_\infty, \quad f_2^{m-1} = \epsilon_2\delta_0 + (1-\epsilon_2)\delta_\infty.$$

and find an approximation of messages f_1 and f_2 at the next iteration which is linear in ϵ.

To do this, let us first find a linear approximation of $\rho(f_e f(x) + g_e g(x)$:

$$\rho(f_e f(x)+g_e g(x)) = \rho(f_e\epsilon\delta_0 + f_e(1-\epsilon)\delta_\infty + g_e g(x)) = g(x)^{\odot j-1}$$
$$= \sum_j \rho_j \left(g_e^{j-1} g(x)^{\odot j-1} + (j-1)f_e\epsilon \sum_{k=0}^{j-2} \binom{j-2}{k} f_e^{j-2-k} g_e^k \cdot g(x)^{\odot k} \right)$$
$$+ c\cdot\delta_\infty = \rho(g_e g(x)) + \epsilon f_e F(g_e g(x)) + c\cdot\delta_\infty,$$

where c is a constant, and $\rho(g_e g(x))$ denotes the first term in the sum, while $F(g_e g(x))$ denotes the second one. Over the binary erasure channel, $\rho(g_e g(x))$ and $F(g_e g(x))$ can be computed explicitly, while, in the general case, the two functions should be computed by running the density evolution iterations. Also note that one can bound the pdf of $g(x)$ by the initial pdf corresponding to the channel estimate. If the transmission channel is bad, the bound will be quite tight. Next,

$$\tilde{\rho}(q_2(x), f_e f(x) + g_e g(x)) = \sum_j \rho_j g_e^{j-2} q_2(x) \odot \left(g(x)^{\odot j-2}\right)$$
$$+ \sum_j \rho_j (j-2) f_e \epsilon \sum_{k=0}^{j-3} \binom{j-3}{k} f_e^{j-3-k} g_e^k \cdot q_2(x) \odot \left(g(x)^{\odot k}\right)$$
$$+ c\cdot\delta_\infty = \tilde{\rho}(q_2(x), g_e g(x)) + \epsilon f_e F(q_2(x), g_e g(x)) + c\cdot\delta_\infty.$$

Further calculations yield

$$\tilde{\lambda}(\tilde{\rho}(q_2(x), f_e f(x) + g_e g(x)) =$$
$$= \tilde{\lambda}(\tilde{\rho}(q_2(x), f_e\epsilon\delta_0 + f_e(1-\epsilon)\delta_\infty + g_e g(x))$$
$$= \tilde{\lambda}_1 + \tilde{\lambda}_2 \left(\tilde{\rho}(q_2(x), g_e g(x)) + \epsilon f_e F(q_2(x), g_e g(x))\right) + c\cdot\delta_\infty.$$

Finally, the approximation of f_1 linear in ϵ is obtained as

$$f_1 = \mu_1(x) \otimes \left(\tilde{\lambda}_1 + \tilde{\lambda}_2\tilde{\rho}(q_2(x), g_e g_1(x)) + \tilde{\lambda}_2\epsilon_1 f_e F(q_2(x), g_e g_1(x))\right)$$
$$\otimes (\rho(g_e g_2(x)) + \epsilon_2 f_e F(g_e g_2(x))) + const\cdot\delta_\infty$$
$$= \mu_1(x) \otimes \left([\tilde{\lambda}_1 + \tilde{\lambda}_2\rho(q_2(x), g_e g_1(x))] \otimes \rho(g_e g_2(x))\right.$$
$$+ \epsilon_2 f_e [\tilde{\lambda}_1 + \tilde{\lambda}_2\tilde{\rho}(q_2(x), g_e g_1(x))] \otimes F(g_e g_2(x))$$
$$\left.+ \tilde{\lambda}_2\epsilon_1 f_e F(q_2(x), g_e g_1(x)) \otimes (\rho(g_e g_2(x)))\right) + c\cdot\delta_\infty$$
$$= \mu_1(x) \otimes (C_0(x) + \epsilon_1 f_e C_1(x) + \epsilon_2 f_e C_2(x)) + c\cdot\delta_\infty$$

where

$$C_0(x) \triangleq [\tilde{\lambda}_1 + \tilde{\lambda}_2\tilde{\rho}(q_2(x), g_e g_1(x))] \otimes \rho(g_e g_2(x))$$
$$C_1(x) \triangleq \tilde{\lambda}_2 F(q_2(x), g_e g_1(x)) \otimes \rho(g_e g_2(x))$$
$$C_2(x) \triangleq [\tilde{\lambda}_1 + \tilde{\lambda}_2\tilde{\rho}(q_2(x), g_e g_1(x))] \otimes F(g_e g_2(x))$$

Similarly,

$$f_2 = \mu_2(x) \otimes \left(\tilde{C}_0(x) + \epsilon_1 f_e\tilde{C}_1(x) + \epsilon_2 f_e\tilde{C}_2(x)\right) + c\cdot\delta_\infty$$

with

$$\tilde{C}_0(x) \triangleq [\tilde{\lambda}_1 + \tilde{\lambda}_2\tilde{\rho}(q_1(x), g_e g_2(x))] \otimes \rho(g_e g_1(x))$$
$$\tilde{C}_1(x) \triangleq [\tilde{\lambda}_1 + \tilde{\lambda}_2\tilde{\rho}(q_1(x), g_e g_2(x))] \otimes F(g_e g_1(x))$$
$$\tilde{C}_2(x) \triangleq \tilde{\lambda}_2 F(q_1(x), g_e g_2(x)) \otimes \rho(g_e g_1(x))$$

Therefore, we have the following relation:

$$\begin{pmatrix} f_1^m \\ f_2^m \end{pmatrix} = \underline{a} + f_e A \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix},$$

with

$$\underline{a} \triangleq \begin{pmatrix} \mu_1(x) \otimes C_0(x) \\ \mu_2(x) \otimes \tilde{C}_0(x) \end{pmatrix}$$

and

$$A \triangleq \begin{pmatrix} \mu_1(x) \otimes C_1(x) & \mu_1(x) \otimes C_2(x) \\ \mu_2(x) \otimes \tilde{C}_1(x) & \mu_2(x) \otimes \tilde{C}_2(x) \end{pmatrix}$$

Denote now by $q(f)$ the Bhattacharyya parameter related to the pdf f,

$$B(f) \triangleq \int_R e^{-x/2} f(x)dx.$$

B is closely related to the bit error probability P_b corresponding to $f(x)$, and it has been shown in [5] that $P_b \to 0 \Leftrightarrow B(f) \to 0$. Knowing this, and taking into account the properties of convolution and of R-convolution, we obtain that

$$\begin{pmatrix} B(f_1^m) \\ B(f_2^m) \end{pmatrix} \leq [C + f_e B(A)] \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix},$$

where

$$C = \begin{pmatrix} 0 & B(\mu_2)\frac{\lambda_2 g_e}{2}\rho'(g_e) \\ B(\mu_1)\frac{\lambda_2 g_e}{2}\rho'(g_e) & 0 \end{pmatrix}.$$

Note that we simplified the expressions by bounding $1 - B(g_1) \leq 1 - B(q_1)$ and $1 - B(g_2) \leq 1 - B(q_2)$, and further bounding $C_0(x)$ and $\tilde{C}_0(x)$.

Define next $D \triangleq C + f_e B(A)$. Then the following recurrence relation can be obtained:

$$\begin{pmatrix} B(f_1^m) \\ B(f_2^m) \end{pmatrix} \leq D \cdot \begin{pmatrix} B(f_1^{m-1}) \\ B(f_2^{m-1}) \end{pmatrix},$$

and hence, if we perform m iterations of density evolution, we obtain that

$$\begin{pmatrix} B(f_1^m) \\ B(f_2^m) \end{pmatrix} \leq D^m \cdot \begin{pmatrix} B(f_1^0) \\ B(f_2^0) \end{pmatrix},$$

where we assume that the messages q and g for any iteration are bounded by q^0 and g^0. We are interested in the case of $B(f^\infty)$ decreasing to 0.

Taking all the above into account, we have the following sufficient stability condition for full-diversity codes:

Theorem 1 (Sufficiency part of the stability condition): The bit error probability P_e for a full-diversity RLDPC ensemble converges to 0 if all the absolute values of the eigenvalues of D are < 1.

Notice that the usual stability condition mentioned in Section IV-A depends on λ_2, while the stability condition derived here depends on both λ_2 and λ_3, "hidden" in $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$.

V. Block-Error Rate of RLDPC Codes

The main result of this paper is the study of the block-error probability P_B of RLDPC codes. Using the sufficient part of the stability condition derived above, we can link P_B to the bit-error probability P_b, and show in which cases $P_b \to 0$ implies $P_B \to 0$.

Using a union bound at some iteration m, we obtain

$$\begin{aligned} P_B^m &\leq \frac{n}{4} P_b^l(1i) + \frac{n}{4} P_b^l(2i) \\ &\leq \frac{1}{4} (\max M_l(1i))^{6+\varepsilon} P_b^l(1i) + \frac{1}{4} (\max M_l(2i))^{6+\varepsilon} P_b^l(2i), \end{aligned}$$

where n is the code length, and $\max M_m(1i)$ ($\max M_m(2i)$) is the maximum number of variable nodes in a computation tree of a variable node from the set $1i$ ($2i$) in the bipartite graph, after m iterations. The second inequality follows from the same reasoning used in [3, Section II], to which we refer the reader desiring a detailed proof. Now, to ensure that, as $m \to \infty$, P_B^m decreases to 0 while $P_b^m \to 0$, one has to ensure that P_b^m decreases with m faster than the maximum number of variable nodes in the computation tree.

A. Case of $\lambda_2 = \lambda_3 = 0$

Let us consider the simple case of both λ_2 and λ_3 being 0. (this is similar to the case of standard LDPC codes with $\lambda_2 = 0$). Repeating the calculations of [3, Section VI.A], we obtain

$$\begin{aligned} &P_B(m+k) \leq \\ &\frac{1}{4} (d_v^{max} d_c^{max})^{6(1+\varepsilon)(m+k)} [B(f_1^m)^{(3/2)^k} + B(f_2^m)^{(3/2)^k}], \end{aligned}$$

which decreases to 0 as $k \to \infty$.

B. General case

Given that

$$B(\text{output}) = \Pi_i B(\text{input}_i)$$

for variable nodes and

$$1 - B(\text{output}) \geq \Pi_i (1 - B(\text{input}_i))$$

for check nodes, and since $B(q^m)$ and $B(g^m)$ for any m, q, and g are no greater than the corresponding $B(\mu)$, one can bound

$$\begin{aligned} B(C_1) &\leq \tilde{\lambda}_2 g_e \rho'(g_e) B(\mu_2) \max\{B(\mu_1), B(\mu_2)\} \\ B(C_2) &\leq (\tilde{\lambda}_1 + \lambda_2 \rho(g_e)) g_e B(\mu_2) \\ B(\tilde{C}_1) &\leq (\tilde{\lambda}_1 + \lambda_2 \rho(g_e)) g_e B(\mu_1) \\ B(\tilde{C}_2) &\leq \tilde{\lambda}_2 g_e \rho'(g_e) B(\mu_1) \max\{B(\mu_1), B(\mu_2)\} \end{aligned}$$

and obtain

$$\begin{aligned} B(f_1^m) &\leq B(\mu_2)(w_1 B(f_1^{m-1}) + w_2 B(f_2^{m-1})) \\ B(f_2^m) &\leq B(\mu_1)(w_2 B(f_1^{m-1}) + w_1 B(f_2^{m-1})) \end{aligned}$$

with $w_1 \triangleq f_e \tilde{\lambda}_2 \rho'(g_e)$ and $w_2 \triangleq f_e(\tilde{\lambda}_1 + \tilde{\lambda}_2 \rho(g_e)) + \frac{\lambda_2 g_e}{2} \rho'(g_e)$. Thus, with a linear approximation,

$$\begin{aligned} B(f_1^{m+2k}) &\leq B(\mu_2)^{2k} w_1^{2k} B(f_1^m) \\ &\quad + B(\mu_2)^k B(\mu_2)^k w_1^k w_2^k B(f_2^m) \\ B(f_2^{m+2k}) &\leq B(\mu_1)^{2k} w_1^{2k} B(f_2^m) \\ &\quad + B(\mu_2)^k B(\mu_2)^k w_1^k w_2^k B(f_1^m). \end{aligned}$$

Consequently, the block error probability

$$\begin{aligned} &P_B(m+k) \leq \\ &\frac{1}{4} (d_v^{max} d_c^{max})^{6(1+\varepsilon)(m+k)} [B(\mu_2)^{2k} w_1^{2k} B(f_1^m) + \\ &B(\mu_1)^{2k} w_1^{2k} B(f_2^m) + B(\mu_2)^k B(\mu_2)^k w_1^k w_2^k \{B(f_1^m) + B(f_2^m)\}], \end{aligned}$$

can be seen to decrease to 0, as $k \to \infty$, if the following conditions are satisfied:

$$\begin{aligned} B(\mu_2) w_1 &\leq (d_v^{max} d_c^{max})^{-3}, \\ B(\mu_1) w_2 &\leq (d_v^{max} d_c^{max})^{-3}. \end{aligned}$$

VI. Conclusion

In this paper we have derived the conditions under which the block-error rate of a RLDPC code ensemble decreases to 0 as the bit-error rate does the same. The interest of our findings lies in the fact that results existing in the literature deal with errors related to all the of code bits, while for RLDPC only errors affecting information bits should be considered.

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