DISS. ETH NO. 18290

Ricci Flow Coupled with Harmonic Map Heat Flow

A dissertation submitted to
ETH ZURICH

for the degree of
Doctor of Sciences

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2009

Abstract

In this work, we investigate a new geometric flow which consists of a coupled system of the Ricci flow on a closed manifold M with the harmonic map flow of a map ϕ from M to some closed target manifold N. It is defined via the following system of nonlinear partial differential equations

$$\begin{cases} \frac{\partial}{\partial t} g = -2\mathrm{Rc} + 2\alpha \nabla\phi \otimes \nabla\phi, \\ \frac{\partial}{\partial t}\phi = \tau_g \phi, \end{cases}$$

where α is a (possibly time-dependent) coupling constant which plays an important role in the analysis of the behavior of the flow. This system can be interpreted as the gradient flow of an *energy* functional $\mathcal{F}_\alpha$ which is a modification of Perelman's energy $\mathcal{F}$ for the Ricci flow, including a Dirichlet energy type term for the map ϕ.

Surprisingly, the coupled system may behave less singular than the Ricci flow or the standard harmonic map flow alone. In fact, we can always rule out energy concentration of ϕ a-priori without any assumptions on the curvature of the target manifold N whenever α is chosen large enough. Moreover, we prove that as long as α is bounded away from zero it suffices to bound the curvature of $(M, g(t))$ to also obtain control of ϕ and all its derivatives – a result which is clearly not true for $\alpha = 0$. To obtain this criterion for long-time existence, we need to derive *interior-in-time gradient estimates* for the curvature of M and for the map ϕ along the flow.

Besides these new phenomena, the flow shares many good properties with the Ricci flow. Making use of *DeTurck's trick*, we obtain short-time existence for smooth initial data. Moreover, we derive the monotonicity of an *entropy* functional $\mathcal{W}_\alpha$ – similar to Perelman's Ricci flow entropy $\mathcal{W}$ – which is built from $\mathcal{F}_\alpha$ by explicitly introducing a scaling factor. With the monotonicity of energy and entropy, we are able to rule out nontrivial generalised periodic solutions for the flow, so-called breathers. Finally, we define so-called *reduced volume* functionals and prove their monotonicity along the flow. This result holds for the more general case of a geometric flow $\frac{\partial}{\partial t} g_{ij} = -2S_{ij}$ in the direction of a symmetric two-tensor S_{ij} that satisfies a certain evolution inequality. In view of the monotonicity in the case of our coupled flow, we conclude a *non-collapsing* theorem in the spirit of Perelman's result for the Ricci flow.

Zusammenfassung

Wir untersuchen einen neuen Geometrischen Fluss, bestehend aus einem gekoppelten System aus dem Riccifluss auf einer geschlossenen Mannigfaltigkeit M und dem Harmonischen Wärmefluss für eine Abbildung ϕ von M in eine geschlossene Zielmannigfaltigkeit N. Der Fluss ist durch folgendes System nichtlinearer partieller Differentialgleichungen gegeben

$$\begin{cases} \frac{\partial}{\partial t} g = -2\mathrm{Rc} + 2\alpha \nabla\phi \otimes \nabla\phi, \\ \frac{\partial}{\partial t}\phi = \tau_g \phi, \end{cases}$$

wobei α eine (möglicherweise zeitabhängige) Koppelungs-Konstante ist, welche bei der Analyse des Verhaltens des Flusses eine wichtige Rolle spielt. Das Gleichungssystem kann als Gradientenfluss eines *Energie*-Funktionals $\mathcal{F}_\alpha$ interpretiert werden, einer Modifikation von Perelmans Energie $\mathcal{F}$ für den Riccifluss die einen zusätzlichen Dirichlet-Energie Term für die ϕ enthält.

Überraschenderweise kann sich das gekoppelte System regulärer verhalten als der Riccifluss oder der Harmonische Wärmefluss alleine. Für genügend grosse α können nämlich Energie-Konzentrationen von ϕ a-priori ausgeschlossen werden ohne zusätzliche Bedingungen an die Krümmung von N zu stellen. Eine Krümmungsschranke für $(M, g(t))$ reicht zudem aus, um auch die Abbildung ϕ und ihre Ableitungen zu kontrollieren, falls α von Null weg beschränkt ist. Um dieses Kriterium für Langzeit-Existenz zu beweisen, benötigen wir *innere Gradientenabschätzungen* für die Krümmung von M und die Abbildung ϕ entlang des Flusses.

Neben diesen neuen Phänomenen teilt der Fluss viele gute Eigenschaften mit dem Riccifluss. Mit Hilfe des *DeTurck Tricks* erhalten wir Kurzzeitexistenz für glatte Anfangsbedingungen. Weiter beweisen wir die Monotonie eines *Entropie*-Funktionals $\mathcal{W}_\alpha$ – ähnlich der Riccifluss-Entropie von Perelman. Mit Energie und Entropie ist es möglich, nicht-triviale verallgemeinerte periodische Lösungen, sogenannte *breathers*, auszuschliessen. Schliesslich definieren wir sogenannte *Reduzierte Volumina* und beweisen die Monotonie dieser Funktionale. Dieses Ergebnis gilt im allgemeineren Fall eines Geometrischen Flusses $\frac{\partial}{\partial t} g_{ij} = -2S_{ij}$ in Richtung eines symmetrischen 2-Tensors, der eine bestimmte Evolutions-Ungleichung erfüllt. Mit der Monotonie für den Fall unseres gekoppelten Flusses folgern wir ein *non-collapsing* Theorem im Sinne von Perelmans Resultat für den Riccifluss.

Contents

Introduction

Harmonic map heat flow and Ricci flow

Let (M^m, g) and (N^n, γ) be smooth Riemannian manifolds without boundary. According to Nash's embedding theorem [46] we can assume that N is isometrically embedded into some Euclidean space $(N^n, \gamma) \hookrightarrow \mathbb{R}^d$ for a sufficiently large d. If $e_N : N \to \mathbb{R}^d$ denotes this embedding, we identify maps $\phi : M \to N$ with $e_N \circ \phi : M \to \mathbb{R}^d$, such maps may thus be written as $\phi = (\phi^\lambda)_{1\leq\lambda\leq d}$. Harmonic maps $\phi : M \to N$ are extremal maps (i.e. critical points in the sense of calculus of variations) for the energy functional

$$E(\phi) = \int_M |\nabla\phi|^2 dV. \tag{0.1}$$

Here, $|\nabla\phi|^2 := 2e(\phi) = g^{ij}\nabla_i\phi^\lambda\nabla_j\phi^\lambda$ denotes the local energy density, where we use the convention that repeated Latin indices are summed over from 1 to m and repeated Greek indices are summed over from 1 to d. Harmonic maps generalize the concept of harmonic functions and in particular include closed geodesics (i.e. harmonic maps from S^1 to a target manifold N) and minimal surfaces.

A basic existence problem for harmonic maps is the *homotopy problem*: is there a harmonic map ϕ homotopic to a given map $\phi_0 : M \to N$? Unfortunately, the answer may be negative, as the following counterexamples illustrate. Lemaire [38] proved that if $\phi : B_1(0) \subset \mathbb{R}^2 \to S^2$ is harmonic and $\phi|_{\partial B_1(0)} \equiv const.$, then ϕ has to be constant. Eells and Wood [20] showed that if $\phi : T^2 \to S^2$ is harmonic, then it cannot have degree ± 1. These examples indicate that the direct variational methods may fail, which is why Eells and Sampson [19] proposed a different way to attack the homotopy problem in 1964. They introduced the harmonic map heat flow, i.e. the L^2-gradient flow of the energy functional (0.1) above. Under this flow, a given map $\phi_0 : M \to N$ is deformed according to the evolution equation

$$\tfrac{\partial}{\partial t}\phi = \tau_g\phi, \qquad \phi(0) = \phi_0, \tag{0.2}$$

where $\tau_g\phi$ denotes the intrinsic Laplacian of ϕ, often called the tension field of ϕ. Using their flow approach, they obtained an affirmative answer to the homotopy problem if the target manifold N has non-positive sectional curvature. Indeed, in this situation, there always exists a unique, global, smooth solution of (0.2) which converges smoothly to a harmonic map $\phi_\infty : M \to N$ homotopic to ϕ_0 as $t \to \infty$ suitably. On the other hand, as the counterexamples above indicate, there are initial data for which the flow cannot both exist for all time and converge. In particular, the solution may blow up in finite or infinite time. In dimension $m = 2$ for example, solutions that become singular in finite time were first constructed by Chang, Ding and Ye [8] by assuming enough symmetry to reduce the system to a scalar equation.

Important applications of harmonic maps are in understanding the Weil-Peterson metric on Teichmüller space (see for example Earle-Eells [16], Fisher-Tromba [22] and Tromba [63]) or in proving rigidity results for Kähler manifolds (see Mostow [44] and Siu [57]). Comprehensive surveys about harmonic maps and the harmonic map heat flow are given in Eells-Lemaire [17, 18], Jost [35] and Struwe [59]. The harmonic map flow was the first appearance of a nonlinear heat flow in Riemannian geometry. Today, geometric heat flows have become an intensely studied topic in geometric analysis.

Another fundamental problem in differential geometry is to find canonical metrics on Riemannian manifolds, i.e. metrics which are highly symmetrical, for example metrics with constant curvature in some sense. Using the idea of evolving an object to such an ideal state by a nonlinear heat flow, Richard Hamilton [23] invented the Ricci flow in 1981. Hamilton's idea was to smooth out irregularities of the curvature by evolving a given Riemannian metric on a manifold M with respect to the nonlinear weakly parabolic equation

$$\tfrac{\partial}{\partial t} g = -2\mathrm{Rc}, \qquad g(0) = g_0, \tag{0.3}$$

where g denotes the Riemannian metric and Rc its Ricci curvature. Unlike the harmonic map flow above, the Ricci flow is *not* the gradient flow of any functional $\mathcal{F}(g) = \int_M F(\partial^2 g, \partial g, g)dV$ (see for example Müller [45, Proposition 1.7]), but in 2002, Perelman [48] showed that it is gradient-like nevertheless. Indeed, he presented a new functional which may be regarded as an improved version of the Einstein-Hilbert functional $E(g) = \int_M R\,dV$, namely

$$\mathcal{F}(g, f) := \int_M \Big(R + |\nabla f|^2\Big) e^{-f} dV. \tag{0.4}$$

This new energy functional has the gradient flow system

$$\begin{cases} \tfrac{\partial}{\partial t} g_{ij} = -2(R_{ij} + \nabla_i \nabla_j f), \\ \ \tfrac{\partial}{\partial t} f = -\triangle f - R. \end{cases}$$

After a pull-back with the family of diffeomorphisms generated by ∇f, this gradient flow system changes into the Ricci flow equation and the adjoint heat equation for e^{-f} under the Ricci flow. In this sense, the Ricci flow is a gradient flow modulo pull-back with diffeomorphisms.

After developing a maximum principle for tensors, Hamilton [23, 24] showed that the Ricci flow preserves the positivity of the Ricci tensor in dimension three and of the curvature operator in all dimensions. Furthermore, he showed a pointwise pinching result for the eigenvalues of the curvature where the curvature is getting large. This result allowed him to conclude that evolving metrics of positive Ricci curvature in dimension three or positive curvature operator in dimension four converge (modulo scaling) to metrics of constant positive curvature. In particular, a three-manifold with positive Ricci curvature must be diffeomorphic to the three-sphere or a quotient of it by a finite group of isometries. But again, in the general case, the solution of the Ricci flow (0.3) may behave much more complicatedly and develop singularities in finite time, in particular the curvature may become arbitrarily large in some region while staying bounded in its complement. In such a case, one often considers a blow-up of the solution for times close to the singular time and at a point where the curvature is large. Under the additional assumption that the injectivity radii of the blown up metrics are bounded away from zero (i.e. if the solution is not *geometrically collapsing*), there exists a blow-up limit according to a compactness result by Hamilton [26]. From a curvature pinching theorem for three-manifolds by Ivey [34] and Hamilton [27], such a blow-up limit must have non-negative sectional curvature. Using his Harnack inequalities for the Ricci flow from [25], Hamilton [27] was able to classify all possible blow-up limits for three-dimensional singularities – modulo the control of the injectivity radius.

The most natural example for a finite time singularity of a three-dimensional Ricci flow is obtained if one starts with an almost round cylindrical neck connecting two large pieces of low curvature. Then, one expects the neck to shrink and pinch off. An existence proof and detailed analysis of such neckpinches can be found in the book by Chow and Knopf [9], the first rigorous examples were constructed by Angenent and Knopf in [1]. In order to deal with such neckpinches, Hamilton [29] invented a topological surgery where one cuts the neck open and glues small caps to each of the boundaries in such a way that one can continue running the Ricci flow. Hamilton proposed a surgery procedure for four-manifolds that satisfy certain curvature assumptions and conjectured that a similar surgery would also work for three-manifolds with no a-priori assumptions at all. This led him to a program of attacking the Poincaré conjecture [51] and William Thurston's geometrization conjecture [61], which states that every closed three-manifold can be decomposed along spheres S^2 or tori T^2 into pieces that admit one of eight different geometric structures. In this context, neckpinch surgery corresponds to the topological decomposition along two-spheres into such pieces. A good source for Hamilton's program is his survey [27] from 1995. It is not surprising that obtaining control of the injectivity radius turned out to be one of the most severe problems. Fortunately, Perelman [48] was able to rule out geometric collapsing using the monotonicity of a so-called *reduced volume* quantity, which he defined with the help of a special length functional $\mathcal{L}$ on Ricci flow space-times. In [49], Perelman developed a surgery procedure for the Ricci flow of three-manifolds under which the curvature pinching conditions mentioned above, as well as the monotonicity of his reduced volume were preserved. This led to a completion of Hamilton's program and a complete proof of Thurston's geometrization conjecture and (using a finite extinction result from Perelman [50] or Colding and Minicozzi [13, 14]) of the Poincaré conjecture.

Introductory surveys on the Ricci flow and Perelman's functionals can be found in the books by Chow and Knopf [9], Chow, Lu and Ni [12], Müller [45] and Topping [62]. More advanced explanations of Perelman's proof of the two conjectures are given in Cao and Zhu [5] Chow et al. [10, 11], Kleiner and Lott [36] and Morgan and Tian [42, 43]. A good survey on Perelman's work is given in Tao [60]. From Hamilton's original sources, we particularly recommend [23, 24, 27].

The coupled system $(RH)_\alpha$ and our main results

The goal of this thesis is to study a coupled system of the two flows (0.2) and (0.3). Again, we let (M^m, g) and (N^n, γ) be smooth manifolds without boundary and with $(N^n, \gamma) \hookrightarrow \mathbb{R}^d$. Throughout this thesis, we will assume in addition that M and N are compact, hence closed. However, many of our results hold for more general manifolds.

Let $g(t)$ be a family of Riemannian metrics on M and $\phi(t)$ a family of smooth maps from M to N. We call $(g(t), \phi(t))_{t\in[0,T)}$ a solution to the coupled system of Ricci flow and harmonic map heat flow with coupling constant $\alpha(t)$, the $(RH)_\alpha$ flow for short, if it satisfies

$$\begin{cases} \frac{\partial}{\partial t} g = -2\mathrm{Rc} + 2\alpha \nabla\phi \otimes \nabla\phi, \\ \frac{\partial}{\partial t}\phi = \tau_g \phi. \end{cases} \tag{$(RH)_\alpha$}$$

Here, Rc is the (time-dependent) Ricci curvature of (M, g), $\tau_g\phi$ denotes the tension field of the map ϕ with respect to the evolving metric g, and α denotes a (time-dependent) coupling constant $\alpha(t) \geq 0$. Finally, $\nabla\phi \otimes \nabla\phi$ has the components $(\nabla\phi \otimes \nabla\phi)_{ij} = \nabla_i \phi^\lambda \nabla_j \phi^\lambda$. In particular, $|\nabla\phi|^2$ as defined above is the trace of $\nabla\phi \otimes \nabla\phi$ with respect to g.

In Chapter 1 of this thesis, we prove, using techniques of Perelman [48] for the Ricci flow, that for constant coupling functions $\alpha(t) \equiv \alpha > 0$ the $(RH)_\alpha$ flow can be interpreted as a gradient flow for the energy functional

$$\mathcal{F}_\alpha(g, \phi, f) = \int_M \Big(R + |\nabla f|^2 - \alpha |\nabla\phi|^2\Big) e^{-f} dV$$

modified by a family of diffeomorphisms generated by ∇f. Proposition 1.1 states that if $(g(t), \phi(t))_{t\in[0,T)}$ solves $(RH)_\alpha$ and e^{-f} is a solution to the adjoint heat equation under the flow, then

$$\frac{d}{dt}\mathcal{F}_\alpha = \int_M \Big(2\left|\mathrm{Rc} - \alpha\nabla\phi \otimes \nabla\phi + \mathrm{Hess}(f)\right|^2 + 2\alpha \left|\tau_g\phi - \langle \nabla\phi, \nabla f\rangle\right|^2\Big) e^{-f} dV,$$

in particular $\mathcal{F}_\alpha$ is non-decreasing and constant if and only if $(g(t), \phi(t))$ is a steady gradient soliton (as introduced in Chapter 4). In the more general case where $\alpha(t)$ is a positive function, we can still show that the above functional $\mathcal{F}_\alpha = \mathcal{F}(\alpha, g, \phi, f)$ is monotone non-decreasing along the flow whenever $\alpha(t)$ is non-increasing.

In the second chapter, we prove short-time existence for the flow again using a method from Ricci flow theory known as DeTurck's trick (cf. [15]), i.e. we transform the weakly parabolic system $(RH)_\alpha$ into a strictly parabolic one by pushing it forward with a family of diffeomorphisms. As for the Ricci flow, one needs to break the symmetry of the equation since the invariance of the flow under diffeomorphisms makes it degenerate. Moreover, we also compute the evolution equations for the Ricci and scalar curvature, the gradient of ϕ and combinations thereof. In particular, the evolution equations for the symmetric tensor $S_{ij} := R_{ij} - \alpha\nabla_i\phi\nabla_j\phi$ and its trace $S = R - \alpha|\nabla\phi|^2$ will be very useful. Using the maximum principle, we show that $\min_{x\in M} S(x,t)$ is non-decreasing along the flow. This has some rather surprising consequences for the existence or non-existence of certain types of singularities. In particular, we see in Corollary 2.8 that if $|\nabla\phi|^2(x_k,t_k) \to \infty$ for $t_k \nearrow T$, then $R(x_k,t_k)$ blows up as well, i.e. $g(t_k)$ must become singular as $t_k \nearrow T$. Conversely, if $|\mathrm{Rm}|$ stays bounded along the flow, $|\nabla\phi|^2$ must stay bounded, too. This leads to the conjecture that a uniform Riemann-bound is enough to conclude long-time existence. This conjecture is proved in Theorem 3.12.

In Chapter 3, we study the short-time smoothing effect of the flow. To this end, we first compute estimates for the Riemannian curvature tensor, its derivatives and the higher derivatives of ϕ. Using these estimates, we follow Bando's [3] and Shi's [55] results for the Ricci flow (based on Bernstein's principle for parabolic equations, cf. [4]) to derive interior-in-time gradient estimates for Rm and ϕ along the $(RH)_\alpha$ flow. These then yield the desired long-time existence result for the case where the Riemannian curvature tensor stays bounded, Theorem 3.12 mentioned above.

The fourth chapter is devoted to the study of explicit examples of solutions $(g(t),\phi(t))$ evolving under the $(RH)_\alpha$ flow as well as soliton solutions which are generalised fixed points of the flow modulo diffeomorphisms and scaling. The stationary solutions of $(RH)_\alpha$ satisfy $\mathrm{Rc} = \alpha\nabla\phi\otimes\nabla\phi$, where $\phi : (M,g) \to (N,\gamma)$ is a harmonic map. Then either $\nabla\phi = 0$, i.e. ϕ is constant in space, or else $\alpha(t) \equiv \alpha$ is constant in time. To prevent $(M,g(t))$ from shrinking to a point or blowing up, it is convenient to introduce a volume preserving version of the flow.

In Chapter 5, we introduce an important monotone quantity for $(RH)_\alpha$, the entropy functional

$$\mathcal{W}_\alpha(g,\phi,f,\tau) = \int_M \Big(\tau\big(R + |\nabla f|^2 - \alpha|\nabla\phi|^2\big) + f - m\Big)(4\pi\tau)^{-m/2}e^{-f}dV,$$

which corresponds to Perelman's shrinker entropy for the Ricci flow [48, Section 3]. Here $\tau = T - t$ denotes a backwards time. Similarly to the result in Chapter 1, we prove that for $\alpha(t) \equiv \alpha > 0$ our flow can be interpreted as the gradient flow of $\mathcal{W}_\alpha$. The entropy functional is non-decreasing (Proposition 5.1) and constant exactly on shrinking solitons. Again, the entropy is monotone if we allow non-increasing positive coupling functions $\alpha(t)$ instead of constant ones. Furthermore, we introduce the notion of a breather, that is a generalised periodic solution of $(RH)_\alpha$ modulo diffeomorphisms and scaling. Using $\mathcal{F}_\alpha$ and $\mathcal{W}_\alpha$ we can exclude nontrivial breathers, i.e. we show in Theorem 5.4 that a breather has to be a gradient soliton as defined in Chapter 4. In the case of a steady or expanding breather the result is even stronger, namely we can show that $\phi(t)$ has to be harmonic in these cases for all t.

Finally in the last chapter, we prove the monotonicity of a forwards and a backwards reduced volume quantity for the $(RH)_\alpha$ flow with positive non-increasing $\alpha(t)$. The monotonicity result, Theorem 6.4, is proved in a more general context, for geometric flows of the form $\frac{\partial}{\partial t}g_{ij} = -2S_{ij}$ with a symmetric tensor S_{ij} that satisfies the evolution inequality

$$\tfrac{\partial}{\partial t}S - \triangle S - 2\left|S_{ij}\right|^2 + 4(\nabla_i S_{ij})X_j - 2(\nabla_j S)X_j + 2R_{ij}X_iX_j - 2S_{ij}X_iX_j \geq 0$$

for all vector fields $X \in \Gamma(TM)$ along the flow. Other examples where the monotonicity holds (for example the mean curvature flow on Lorentzian manifolds) are collected in the appendix. For the $(RH)_\alpha$ flow, we apply the monotonicity result to deduce a local non-collapsing result, Theorem 6.13.

The appendix contains a short collection of important facts from Riemannian geometry and analysis. In addition to the standard formulas like Bianchi identities for the curvature tensor of M and commutator identities for tensors in $(T^*M)^{\otimes p} \otimes (TM)^{\otimes q}$, we also compute the commutator identities for sections of bundles like $T^*M \otimes \phi^*TN$. We also state the weak maximum principle in the simplest form that is still general enough for the purpose of this thesis. Moreover, some of the less important proofs are collected in the appendix to make the main body of this work more readable and compact.

Relation to other extended Ricci flow systems

The idea of coupling the Ricci flow with another flow is not new. However, Perelman's work provoked intense new studies of such coupled systems. Here are a few old and new related flow systems.

i) Relation to the Ricci-DeTurck flow if $\alpha \equiv 0$. In the case $\alpha(t) \equiv 0$, the above system reduces to

$$\begin{cases} \frac{\partial}{\partial t}g = -2\mathrm{Rc}, \\ \frac{\partial}{\partial t}\phi = \tau_g\phi, \end{cases} \qquad (RH)_0$$

which – under the additional assumption that $\phi(0)$ is a diffeomorphism from M to N – was studied for example by Hamilton [27] and Simon [56]. In particular, if one looks at the push-forward of the evolving metric with maps ϕ satisfying the second equation, one gets the so-called dual Ricci-Harmonic map flow for which it is easy to show short-time existence since it is strictly parabolic. This result implies short-time existence and uniqueness for the first equation of $(RH)_0$, which is simply the Ricci flow equation. In the special case where $N = M$ and $\phi(0) = \mathrm{id}$, the dual Ricci-Harmonic map flow is known as Ricci-DeTurck flow (see DeTurck [15] and Simon [56]). We will use similar ideas to show short-time existence for arbitrary N and $\alpha(t)$ in the second chapter of this thesis.

ii) Relation to the Ricci flow of warped products if $N = \mathbb{R}$. Let $\tilde{M}$ be the product space $\tilde{M} = S^1 \times M^m$, where (M^m, g_{ij}) is a smooth Riemannian manifold and S^1 denotes a circle. Then the Ricci flow equation $\frac{\partial}{\partial t}\tilde{g} = -2\,\mathrm{Rc}(\tilde{g})$ for the warped product metric $\tilde{g} = \psi^2(dx^0)^2 + g_{ij}\,dx^i \otimes dx^j$ on $S^1 \times M^m$, where $0 < \psi \in C^\infty(M)$, can be written as

$$\begin{cases} \frac{\partial}{\partial t} g = -2\mathrm{Rc} + \frac{2}{\psi}\,\mathrm{Hess}(\psi), \\ \frac{\partial}{\partial t}\psi = \triangle_g \psi, \end{cases} \tag{0.5}$$

see Lemma A.2 in the appendix. Modulo the action of a family of diffeomorphisms, if one sets $\psi = e^{\sqrt{2}\phi}$, this system is equivalent to

$$\begin{cases} \frac{\partial}{\partial t} g = -2\mathrm{Rc} + 4\nabla\phi \otimes \nabla\phi, \\ \frac{\partial}{\partial t}\phi = \triangle_g \phi, \end{cases} \tag{0.6}$$

i.e. $(RH)_\alpha$ with $\alpha \equiv 2$ and $N = \mathbb{R}$ (and thus $\tau_g = \triangle_g$ is the Laplace-Beltrami operator), see appendix, Lemma A.3, for a proof. The system (0.6) was studied in detail by List in his dissertation [41] for complete manifolds M. However, the behavior of our system $(RH)_\alpha$ may look different from that of warped products or List's flow when the assumption $N = \mathbb{R}$ is dropped – even for constant α. On the one hand, the class of singularities caused by a concentration of the energy density $|\nabla\phi|^2$ at a point can be excluded for List's flow but not for $(RH)_\alpha$, on the other hand, the two flows may also behave rather differently in the case where no singularities occur. In fact, if a solution of (0.6) exists for all time, the function $\phi : M \to \mathbb{R}$ must converge to a constant and the expression $\nabla\phi \otimes \nabla\phi$ vanishes asymptotically. Thus, for large times t, the behavior of the first equation is controlled by the behavior of the Ricci flow. In our case where the target manifold N may have nontrivial topology, the map ϕ may converge to a nontrivial harmonic map; in particular the expression $\nabla\phi \otimes \nabla\phi$ does not have to go to zero.

iii) Relation to the Ricci Yang-Mills flow. Let $E \to M$ be a vector bundle with a connection A and curvature 2-form $F = dA + A \wedge A$. The Ricci Yang-Mills flow, introduced by Streets in [58], is the system

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2R_{ij} + g^{kl}F_{ik}F_{jl}, \\ \frac{\partial}{\partial t} A = -d^*F. \end{cases} \tag{0.7}$$

This system has a very similar structure to the system $(RH)_\alpha$ and indeed the flow can also be interpreted as the gradient flow of an energy functional which looks similar to our energy, namely

$$\mathcal{F}(g, A, f) = \int_M \Big(R + |\nabla f|^2 - \tfrac{1}{4}|F|^2\Big)e^{-f}dV.$$

Therefore, it is not surprising that the two flows share many common properties. However, there are also major differences. Indeed, since R and $\frac{1}{4}|F|^2$ scale differently, a direct analogue to the entropy functional $\mathcal{W}_\alpha$ does not exist for the Ricci Yang-Mills flow. Also, the monotonicity of the reduced volume formulas fails if these formulas are defined in an analogous way as for our flow. Moreover, the Ricci Yang-Mills flow on a surface M^2 is a conformal flow (and hence the evolution of the metric reduces to a scalar evolution equation for the conformal factor) while the $(RH)_\alpha$ flow on surfaces is not conformal unless $\phi(t)$ is a conformal map for all times t.

iv) Relation to problems in physics. Systems similar to $(RH)_\alpha$ may arise naturally in physical problems. We illustrate this with two particular examples.

a) List's motivation to study the system (0.6) was the following connection to general relativity. Let (L,h) be a Lorentzian 4-manifold solving the Einstein vacuum equations and let M^3 be a spacelike slice of L. If (L,h) is static, there exists a timelike Killing field K and L satisfies the Frobenius integrability condition. In local coordinates this is equivalent to h taking the simple form

$$h = -|\psi|^2 dt^2 + \sum_{i=1}^{3} g_{ij}\, dx^i \otimes dx^j \tag{0.8}$$

in a neighborhood of any point $p \in M^3$, where g_{ij} is a Riemannian metric on M^3 and $\psi := \sqrt{-|K|^2}$ is the so-called lapse function that measures the speed of the evolution of the slice M^3 in time direction. The Einstein vacuum equations $\mathrm{Rc}(h) = 0$ are transformed to the constrained equations for the induced Riemannian metric g

$$\begin{cases} \mathrm{Rc}(g) = 2\nabla\phi \otimes \nabla\phi, \\ \triangle_g \phi = 0, \end{cases} \tag{0.9}$$

where ϕ denotes the logarithm of the lapse function ψ. Clearly, these equations are solved by stationary solutions of the system (0.6) above. Getting (0.9) from the (non-compact Lorentzian) warped product structure (0.8) is analogous to the derivation of (0.6) for the (compact Riemannian) warped product metric $\tilde{g}$ as in Lemma A.3. See [41] for details.

b) In quantum field theory, one may consider the so-called renormalization group (RG) flow. In the special case of a worldsheet nonlinear sigma model with purely gravitational target space, the 1-loop (order α') approximation of the RG flow is simply the Ricci flow for the target space metric. In the more general case, the RG flow also depends on a 2-form field B. Writing its 1-loop approximation in Hamilton gauge and denoting $H := dB$, the evolution equations read

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -\alpha' R_{ij} + \frac{\alpha'}{4} H_{ik\ell} H_j{}^{k\ell}, \\ \frac{\partial}{\partial t} H = \frac{\alpha'}{2} \triangle_g H, \end{cases} \tag{0.10}$$

where $\triangle_g$ now denotes the Laplace-Beltrami operator acting on the 3-form H, see Oliynyk, Suneeta and Woolgar [47]. Since these equations (and other RG flow type evolution systems) have a structure similar to $(RH)_\alpha$ and similar to the Ricci Yang-Mills flow above, we hope our system can be used as a basic model for the behavior of this type of equations. However, as for the Ricci Yang-Mills flow above, the lack of scaling covariance makes it impossible to construct a $\mathcal{W}_\alpha$-type entropy for (0.10) and it seems necessary to invent new tools to study these flows.

CHAPTER 1

The $(RH)_\alpha$ flow as a gradient flow

In this chapter, we introduce an energy functional $\mathcal{F}_\alpha$ for the $(RH)_\alpha$ flow, which corresponds to Perelman's $\mathcal{F}$-energy for the Ricci flow introduced in [48, Section 1]. For a detailed study of Perelman's functional, we refer to Chow et al. [10, Chapter 5], Müller [45, Chapter 3], or Topping [62, Chapter 6]. The corresponding functional for the flow (0.6), which is a special case of $(RH)_\alpha$, was introduced by List [41]. Since there is almost no difference between the calculations in the case where ϕ is a function and the more general case where $\phi : M \to N$ is a map, we will follow List's work closely in the first part of this chapter.

The energy functional and its first variation

Let $g = g_{ij} \in \Gamma\big(\operatorname{Sym}^2_+(T^*M)\big)$ be a Riemannian metric on a closed manifold M, $f : M \to \mathbb{R}$ a smooth function and $\phi \in C^\infty(M,N) := \{\phi \in C^\infty(M,\mathbb{R}^d) \mid \phi(M) \subseteq N\}$. For a constant $\alpha(t) \equiv \alpha > 0$, we set

$$\mathcal{F}_\alpha(g,\phi,f) := \int_M \Big(R_g + |\nabla f|_g^2 - \alpha|\nabla\phi|_g^2\Big)e^{-f}dV_g. \tag{1.1}$$

Take variations

$$\begin{aligned} g_{ij}^\varepsilon &= g_{ij} + \varepsilon h_{ij}, & h_{ij} &\in \Gamma\big(\operatorname{Sym}^2(T^*M)\big),\\ f^\varepsilon &= f + \varepsilon\ell, & \ell &\in C^\infty(M),\\ \phi^\varepsilon &= \pi_N(\phi + \varepsilon\vartheta), & \vartheta &\in C^\infty(M,\mathbb{R}^d) \text{ with } \vartheta(x) \in T_{\phi(x)}N, \end{aligned}$$

where π_N is the smooth nearest-neighbour projection defined on a tubular neighbourhood of $N \subset \mathbb{R}^d$. Note that we used the identification $T_pN \subset T_p\mathbb{R}^d \cong \mathbb{R}^d$. We denote by δ the derivative $\delta = \frac{d}{d\varepsilon}\big|_{\varepsilon=0}$, i.e. we have $\delta g = h$, $\delta f = \ell$ and $\delta\phi = (d\pi_N \circ \phi)\vartheta = \vartheta$. Our goal is to compute

$$\begin{aligned} \delta\mathcal{F}_{\alpha,g,\phi,f}(h,\vartheta,\ell) &:= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{F}_\alpha(g+\varepsilon h, \pi_N(\phi+\varepsilon\vartheta), f+\varepsilon\ell)\\ &= \underbrace{\delta\int_M \big(R + |\nabla f|^2\big)e^{-f}dV}_{=:\,I} - \alpha\cdot\underbrace{\delta\int_M |\nabla\phi|^2 e^{-f}dV}_{=:\,II}. \end{aligned}$$

For the first integral, we know from Ricci flow theory that

$$I = \int_M \Big(- h^{ij}\big(R_{ij} + \nabla_i\nabla_j f\big) + \big(\tfrac{1}{2}\operatorname{tr}_g h - \ell\big)\big(2\triangle f - |\nabla f|^2 + R\big)\Big) e^{-f} dV,$$

see Perelman [48, Section 1.1], or Müller [45, Lemma 3.3], for a detailed proof. We continue by computing the variation of the second integral

$$\begin{aligned} II &= \delta \int_M g^{ij}\nabla_i\phi^\lambda\nabla_j\phi^\lambda e^{-f} dV \\ &= \underbrace{\int_M 2g^{ij}\nabla_i\phi^\lambda\nabla_j\vartheta^\lambda e^{-f} dV}_{=:\, III} + \int_M \Big(- h^{ij}\nabla_i\phi^\lambda\nabla_j\phi^\lambda + |\nabla\phi|_g^2\big(\tfrac{1}{2}\operatorname{tr}_g h - \ell\big)\Big) e^{-f} dV, \end{aligned}$$

where a partial integration yields

$$III = - \int_M 2\vartheta^\lambda \big(\triangle_g \phi^\lambda - \big\langle \nabla\phi^\lambda, \nabla f\big\rangle_g\big) e^{-f} dV.$$

Hence, by putting everything together, we find

$$\begin{aligned} \delta\mathcal{F}_{\alpha,g,\phi,f}(h,\vartheta,\ell) = &\int_M -h^{ij}\big(R_{ij} + \nabla_i\nabla_j f - \alpha\nabla_i\phi\nabla_j\phi\big)e^{-f} dV \\ &+ \int_M \big(\tfrac{1}{2}\operatorname{tr}_g h - \ell\big)\big(2\triangle f - |\nabla f|^2 + R - \alpha|\nabla\phi|^2\big)e^{-f} dV \qquad (1.2) \\ &+ \int_M 2\alpha\vartheta\big(\tau_g\phi - \langle\nabla\phi, \nabla f\rangle\big)e^{-f} dV, \end{aligned}$$

since $\vartheta\triangle_g\phi = \vartheta\tau_g\phi$, where $\tau_g\phi := \triangle_g\phi - A(\phi)(\nabla\phi,\nabla\phi)_M$ denotes the tension field of ϕ. Here, we used the definition

$$A(\phi)(\nabla\phi,\nabla\phi)_M = \sum_{\beta=n+1}^{d} g^{ij}A^\beta(\phi)(\nabla_i\phi,\nabla_j\phi)(\nu_\beta\circ\phi),$$

where $\nu_{n+1}\dots,\nu_d$ denotes a local orthonormal frame for $(T_pN)^\perp \subseteq \mathbb{R}^d$ and $A^\beta(X,Y) = \langle -(\nabla_X^{\mathbb{R}^d}\nu_\beta)^\top, Y\rangle$ denotes the second fundamental form of N with respect to ν_β. For more background on the harmonic map heat flow, see for example Struwe's survey [59].

Gradient flow for fixed background measure

Now, we fix the measure $d\mu = e^{-f}dV$, i.e. let $f = -\log\big(\frac{d\mu}{dV}\big)$, where $\frac{d\mu}{dV}$ denotes the Radon-Nikodym differential of measures. Then, from $0 = \delta d\mu = \big(\frac{1}{2}\operatorname{tr}_g h - \ell\big)\, d\mu$ we get $\ell = \frac{1}{2}\operatorname{tr}_g h$. Thus, for a fixed measure μ the functional $\mathcal{F}_\alpha$ and its variation $\delta\mathcal{F}_\alpha$ depend only on g and ϕ and their variations $\delta g = h$ and $\delta\phi = \vartheta$. In the following we write

$$\mathcal{F}_\alpha^\mu(g,\phi) := \mathcal{F}_\alpha\big(g,\phi,-\log\big(\tfrac{d\mu}{dV}\big)\big) \qquad (1.3)$$

and

$$\delta\mathcal{F}^{\mu}_{\alpha,g,\phi}(h,\vartheta) := \delta\mathcal{F}_{\alpha,g,\phi,-\log(\frac{d\mu}{dV})}\left(h,\vartheta,\tfrac{1}{2}\operatorname{tr}_g h\right).$$

Equation (1.2) reduces to

$$\begin{aligned}\delta\mathcal{F}^{\mu}_{\alpha,g,\phi}(h,\vartheta) = &\int_M -h^{ij}\big(R_{ij}+\nabla_i\nabla_j f-\alpha\nabla_i\phi\nabla_j\phi\big)d\mu\\ &+\int_M 2\alpha\vartheta\big(\tau_g\phi-\langle\nabla\phi,\nabla f\rangle\big)d\mu.\end{aligned}\tag{1.4}$$

Let $(g,\phi)\in\Gamma(\mathrm{Sym}^2_+(T^*M))\times C^\infty(M,N)$ and define on $H := H_{g,\phi} = \Gamma(\mathrm{Sym}^2(T^*M))\times T_\phi C^\infty(M,N)$ an inner product depending on α and the measure μ by

$$\langle (k_{ij},\psi),(h_{ij},\vartheta)\rangle_{H,\alpha,\mu} := \int_M \big(\tfrac{1}{2}h^{ij}k_{ij}+2\alpha\psi\vartheta\big)d\mu.$$

From $\delta\mathcal{F}^{\mu}_{\alpha,g,\phi}(h,\vartheta) = \langle \operatorname{grad}\mathcal{F}^{\mu}_{\alpha}(g,\phi),(h,\vartheta)\rangle_{H,\alpha,\mu}$ we then deduce

$$\operatorname{grad}\mathcal{F}^{\mu}_{\alpha}(g,\phi) = \big(-2(R_{ij}+\nabla_i\nabla_j f-\alpha\nabla_i\phi\nabla_j\phi),\ \tau_g\phi-\langle\nabla\phi,\nabla f\rangle\big).\tag{1.5}$$

Let π_1, π_2 denote the natural projections of H onto its first and second factor, respectively. Then, the gradient flow of $\mathcal{F}^{\mu}_{\alpha}$ is

$$\begin{cases}\frac{\partial}{\partial t}g_{ij} = \pi_1(\operatorname{grad}\mathcal{F}^{\mu}_{\alpha}(g,\phi)),\\ \frac{\partial}{\partial t}\phi = \pi_2(\operatorname{grad}\mathcal{F}^{\mu}_{\alpha}(g,\phi)).\end{cases}$$

Thus, recalling the equation $\frac{\partial}{\partial t}f = \ell = \frac{1}{2}\operatorname{tr}_g\big(\frac{\partial}{\partial t}g_{ij}\big)$, we obtain the gradient flow system

$$\begin{cases}\frac{\partial}{\partial t}g_{ij} = -2(R_{ij}+\nabla_i\nabla_j f-\alpha\nabla_i\phi\nabla_j\phi),\\ \frac{\partial}{\partial t}\phi = \tau_g\phi-\langle\nabla\phi,\nabla f\rangle,\\ \frac{\partial}{\partial t}f = -R-\triangle f+\alpha|\nabla\phi|^2.\end{cases}\tag{1.6}$$

Pulling back with diffeomorphisms

As one can do for the Ricci flow (see Perelman [48, Section 1], or Müller [45, page 52]), we now pull back a solution (g,ϕ,f) of (1.6) with a family of diffeomorphisms generated by $X=\nabla f$. Indeed, recalling the formulas for the Lie derivatives $(\mathcal{L}_{\nabla f}g)_{ij} = 2\nabla_i\nabla_j f$, $\mathcal{L}_{\nabla f}\phi = \langle\nabla\phi,\nabla f\rangle$ and $\mathcal{L}_{\nabla f}f = |\nabla f|^2$, we can rewrite (1.6) in the form

$$\begin{cases}\frac{\partial}{\partial t}g = -2\mathrm{Rc}+2\alpha\nabla\phi\otimes\nabla\phi-\big(\mathcal{L}_{\nabla f}g\big),\\ \frac{\partial}{\partial t}\phi = \tau_g\phi-\big(\mathcal{L}_{\nabla f}\phi\big),\\ \frac{\partial}{\partial t}f = -\triangle f+|\nabla f|^2-R+\alpha|\nabla\phi|^2-\big(\mathcal{L}_{\nabla f}f\big).\end{cases}$$

Hence, if ψ_t is the one-parameter family of diffeomorphisms induced by the vector field $X(t) = \nabla f(t) \in \Gamma(TM)$, $t \in [0, T)$, i.e. if $\frac{\partial}{\partial t}\psi_t = X(t) \circ \psi_t$, $\psi_0 = \mathrm{id}$, then the pulled-back quantities $\tilde{g} = \psi_t^* g$, $\tilde{\phi} = \psi_t^* \phi$, $\tilde{f} = \psi_t^* f$ satisfy

$$\begin{cases} \frac{\partial}{\partial t}\tilde{g} = -2\tilde{\mathrm{Rc}} + 2\alpha \nabla\tilde{\phi} \otimes \nabla\tilde{\phi}, \\ \frac{\partial}{\partial t}\tilde{\phi} = \tau_{\tilde{g}}\tilde{\phi}, \\ \frac{\partial}{\partial t}\tilde{f} = -\triangle_{\tilde{g}}\tilde{f} + |\nabla\tilde{f}|^2_{\tilde{g}} - \tilde{R} + \alpha|\nabla\tilde{\phi}|^2_{\tilde{g}}. \end{cases}$$

Here, $\tilde{\mathrm{Rc}}$ and $\tilde{R}$ denote the Ricci and scalar curvature of $\tilde{g}$ and $\triangle$, τ and the norms are also computed with respect to $\tilde{g}$. In the following, we will usually consider the pulled-back gradient flow system and therefore drop the tildes for convenience of notation.

Note that the formal adjoint of the heat operator $\Box = \frac{\partial}{\partial t} - \triangle$ under the flow $\frac{\partial}{\partial t}g = h$ is $\Box^* = -\frac{\partial}{\partial t} - \triangle - \frac{1}{2}\operatorname{tr}_g h$. Indeed, for functions $v, w\colon M \times [0, T] \to \mathbb{R}$, a straightforward computation yields

$$\int_0^T \int_M (\Box v) w \, dV \, dt = \left[\int_M vw \, dV \right]_0^T + \int_0^T \int_M v(\Box^* w) \, dV \, dt.$$

In our case where $h_{ij} = -2R_{ij} + 2\alpha\nabla_i\phi\nabla_j\phi$, this is $\Box^* = -\frac{\partial}{\partial t} - \triangle + R - \alpha|\nabla\phi|^2$ and thus the evolution equation for f is equivalent to e^{-f} solving the adjoint heat equation $\Box^* e^{-f} = 0$. The system now reads

$$\begin{cases} \frac{\partial}{\partial t}g = -2\mathrm{Rc} + 2\alpha\nabla\phi \otimes \nabla\phi, \\ \frac{\partial}{\partial t}\phi = \tau_g\phi, \\ \quad 0 = \Box^* e^{-f}. \end{cases} \tag{1.7}$$

This means that our system $(RH)_\alpha$ can be interpreted as the gradient flow of $\mathcal{F}_\alpha^\mu$ for any fixed background measure μ. Moreover, using (1.4), (1.5) and the diffeomorphism invariance of $\mathcal{F}_\alpha$, we get the following.

Proposition 1.1
Let $(g(t), \phi(t))_{t\in[0,T)}$ be a solution of the $(RH)_\alpha$ flow with coupling constant $\alpha(t) \equiv \alpha > 0$ and let e^{-f} solve the adjoint heat equation under this flow. Then the energy functional $\mathcal{F}_\alpha(g, \phi, f)$ defined in (1.1) is non-decreasing with

$$\frac{d}{dt}\mathcal{F}_\alpha = \int_M \Big(2\left|\mathrm{Rc} - \alpha\nabla\phi \otimes \nabla\phi + \mathrm{Hess}(f)\right|^2 + 2\alpha\left|\tau_g\phi - \langle\nabla\phi, \nabla f\rangle\right|^2 \Big) e^{-f} dV \geq 0. \tag{1.8}$$

Remark. In the fourth chapter of this thesis, we will introduce soliton solutions of the $(RH)_\alpha$ flow and we will see that $\mathcal{F}_\alpha$ is constant if and only if $(g(t), \phi(t))$ is a steady soliton. An entropy functional $\mathcal{W}_\alpha$ which is constant on shrinking solitons will be defined in Chapter 5.

Allowing also time-dependent coupling constants $\alpha(t)$, we obtain the following.

Corollary 1.2
Let $(g(t),\phi(t))_{t\in[0,T)}$ solve $(RH)_\alpha$ for a positive coupling function $\alpha(t)$ and let e^{-f} solve the adjoint heat equation under this flow. Then $\mathcal{F}_{\alpha(t)}(g(t),\phi(t),f(t))$ satisfies

$$\frac{d}{dt}\mathcal{F}_\alpha = \int_M \Big(2\left|\mathrm{Rc}-\alpha\nabla\phi\otimes\nabla\phi+\mathrm{Hess}(f)\right|^2 + 2\alpha\left|\tau_g\phi-\langle\nabla\phi,\nabla f\rangle\right|^2 - \dot{\alpha}|\nabla\phi|^2\Big)e^{-f}dV,$$

in particular, it is non-decreasing if $\alpha(t)$ is a non-increasing function.

Conformal invariants on a Riemannian measure space

In the following section, we rewrite the energy functional $\mathcal{F}^\mu_\alpha$ and its time-derivative $\frac{d}{dt}\mathcal{F}^\mu_\alpha$ under the flow by means of conformal invariants and conformally covariant operators associated to the measure μ.

Generally, a Riemannian measure space (M^m,g,μ) consists of a smooth manifold M, a Riemannian metric g and a smooth measure μ on M, which is related to dV by the density function f defined by $d\mu = e^{-f}dV$. A conformal invariant is a quantity depending on g and μ which is insensitive to conformal changes of the metric g. In particular, there is a (dimension-dependent) notion of conformally invariant curvatures introduced by Chang, Gursky and Yang in [6]. The conformally invariant Ricci curvature of (M^m,g,μ) is

$$\mathrm{Rc}^\mu_m(g) = \mathrm{Rc}(g) + \frac{m-2}{m}\,\mathrm{Hess}(f) + \frac{1}{m}(\triangle_g f)g + \frac{m-2}{m^2}\big(\nabla f\otimes\nabla f - |\nabla f|^2 g\big),$$

the conformally covariant scalar curvature is the trace of $\mathrm{Rc}^\mu_m(g)$, i.e. the function

$$R^\mu_m(g) = R(g) + \frac{2(m-1)}{m}\triangle_g f - \frac{(m-1)(m-2)}{m^2}|\nabla f|^2.$$

Under a conformal change of the metric, these curvatures satisfy

$$\mathrm{Rc}^\mu_m(e^{2w}g) = \mathrm{Rc}^\mu_m(g), \qquad R^\mu_m(e^{2w}g) = e^{-2w}R^\mu_m(g).$$

Moreover, for $d\mu \equiv dV$, the curvatures reduce to the standard Ricci and scalar curvature. Letting $m\to\infty$ formally, one obtains

$$\begin{aligned} \lim_{m\to\infty}\mathrm{Rc}^\mu_m(g) &=: \mathrm{Rc}^\mu(g) = \mathrm{Rc}(g) + \mathrm{Hess}(f),\\ \lim_{m\to\infty} R^\mu_m(g) &=: R^\mu(g) = R(g) + 2\triangle_g f - |\nabla f|^2, \end{aligned} \tag{1.9}$$

i.e. the Ricci curvature defined by Bakry and Emery [2] and the scalar curvature defined by Perelman [48]. The process of letting the dimension go to infinity can be made more concrete as follows. Take $\tilde{M}^\ell = M^m\times T^{\ell-m}$, where $(T^{\ell-m},ds^2)$ denotes the flat

$(\ell - m)$-dimensional torus and (M^m, g, μ) is a Riemannian measure space. Then look at the conformally invariant curvatures of $(\tilde{M}^\ell, \tilde{g} = g + ds^2, \tilde{\mu} = \mu \times dV(ds^2))$ and let $\ell \to \infty$. The computation of the above limits (1.9) using this construction can be found in [6].

A differential operator $L = L_g$ on (M, g) is called conformally covariant of bi-degree (a, b) if under a conformal change of the metric it transforms by

$$L_{e^{2w}g}(\psi) = e^{-bw} L_g(e^{aw}\psi).$$

A well known example is the conformal Laplacian $L_g = \triangle_g - \frac{m-2}{4(m-1)} R_g$, which is conformally covariant with $a = \frac{m-2}{2}$ and $b = \frac{m+2}{2}$. In [6, Theorem 3.1], Chang, Gursky and Yang construct a large family of such operators on a Riemannian measure space of dimension $m \geq 3$. Using their construction, we find in particular the operator

$$L^\mu_m(\psi) = \triangle_g \psi - \langle \nabla\psi, \nabla f\rangle + \frac{1}{m-2}(\triangle_g f + R_g)\psi, \tag{1.10}$$

which is conformally covariant with $a = -1$ and $b = 1$; see the appendix for a proof. Letting again $m \to \infty$ in the above sense, we find the dimension-independent operator $L^\mu(\psi) = \triangle_g(\psi) - \langle \nabla\psi, \nabla f\rangle$. Motivated by this, we define an operator $L^\mu(\psi)$ for maps $\psi : M \to N$ by

$$L^\mu(\psi) = \tau_g \psi - \langle \nabla\psi, \nabla f\rangle\,. \tag{1.11}$$

Using the notation of (1.9) and (1.11), we can rewrite our energy and its derivative in the form

$$\begin{aligned} \mathcal{F}^\mu_\alpha(g,\phi) &= \int_M \Big(R^\mu - \alpha|\nabla\phi|^2\Big) d\mu, \\ \frac{d}{dt}\mathcal{F}^\mu_\alpha(g,\phi) &= 2\int_M \Big(\left|\mathrm{Rc}^\mu - \alpha\nabla\phi\otimes\nabla\phi\right|^2 + \alpha\left|L^\mu(\phi)\right|^2\Big) d\mu. \end{aligned} \tag{1.12}$$

This shows that the quantities appearing in the definition of the entropy functional and in its time derivative are natural quantities on a Riemannian measure space. In a last step, we also want to get rid of the chosen background measure μ.

Minimizing over all probability measures

Following Perelman [48], we define

$$\lambda_\alpha(g,\phi) := \inf\left\{\mathcal{F}^\mu_\alpha(g,\phi) \;\middle|\; \mu(M) = 1\right\} = \inf\left\{\mathcal{F}_\alpha(g,\phi,f) \;\middle|\; \int_M e^{-f} dV = 1\right\}. \tag{1.13}$$

The first task is to show that the infimum is always achieved. Indeed, if we define $v = e^{-f/2}$, we can write the energy as

$$\mathcal{F}_\alpha(g,\phi,v) = \int_M \Big(Rv^2 + 4|\nabla v|^2 - \alpha|\nabla\phi|^2 v^2\Big) dV = \int_M v\Big(Rv - 4\triangle v - \alpha|\nabla\phi|^2 v\Big) dV.$$

Hence

$$\lambda_\alpha(g,\phi) = \inf\Big\{\int_M v\Big(Rv - 4\triangle v - \alpha|\nabla\phi|^2 v\Big)dV \;\Big|\; \int_M v^2 dV = 1\Big\}$$

is the smallest eigenvalue of the operator $-4\triangle + R - \alpha|\nabla\phi|^2$ and v is a corresponding normalized eigenvector. Since the operator (for any time t and map $\phi(t)$) is a Schrödinger operator, there exists a unique positive and normalized eigenvector $v_{min}(t)$, see for example Reed and Simon [52] or Rothaus [53]. From eigenvalue perturbation theory, we see that if $g(t)$ and $\phi(t)$ depend smoothly on t, then so do $\lambda_\alpha(g(t),\phi(t))$ and $v_{min}(t)$.

Proposition 1.3
Let $(g(t),\phi(t))_{t\in[0,T)}$ be a smooth solution of the $(RH)_\alpha$ flow with constant $\alpha(t)\equiv\alpha>0$. Then $\lambda_\alpha(g,\phi)$ as defined in (1.13) is monotone non-decreasing in time and it is constant if and only if

$$\begin{cases} 0 = \mathrm{Rc} - \alpha\nabla\phi\otimes\nabla\phi + \mathrm{Hess}(f), \\ 0 = \tau_g\phi - \langle\nabla\phi,\nabla f\rangle, \end{cases} \tag{1.14}$$

for the minimizing function $f = -2\log v_{min}$.

Proof. Pick arbitrary times $t_1,\ t_2 \in [0,T)$ and let $v_{min}(t_2)$ be the unique positive minimizer for $\lambda_\alpha(g(t_2),\phi(t_2))$. Put $u(t_2) = v^2_{min}(t_2) > 0$ and solve the adjoint heat equation $\Box^* u = 0$ backwards on $[t_1,t_2]$. Note that $u(x',t') > 0$ for all $x'\in M$ and $t'\in[t_1,t_2]$ by the maximum principle and the constraint $\int_M u\, dV = \int_M v^2\, dV = 1$ is preserved since

$$\frac{d}{dt}\int_M u\, dV = \int_M \big(\tfrac{\partial}{\partial t}u\big)dV + \int_M u\big(\tfrac{\partial}{\partial t}dV\big) = -\int_M \triangle u\, dV = 0.$$

Here, we used $\frac{\partial}{\partial t}dV = -\frac{1}{2}\,\mathrm{tr}_g(\frac{\partial}{\partial t}g)dV = (-R+\alpha|\nabla\phi|^2)dV$ and $\frac{\partial}{\partial t}u = (-\triangle + R - \alpha|\nabla\phi|^2)u$, the latter following from $\Box^* u = 0$. Thus, with $u(t) = e^{-\bar f(t)}$ for all $t\in[t_1,t_2]$, we obtain with Proposition 1.1

$$\lambda_\alpha(g(t_1),\phi(t_1)) \le \mathcal{F}_\alpha(g(t_1),\phi(t_1),\bar f(t_1)) \le \mathcal{F}_\alpha(g(t_2),\phi(t_2),\bar f(t_2)) = \lambda_\alpha(g(t_2),\phi(t_2)).$$

The condition (1.14) in the equality case follows directly from (1.8). □

Again the monotonicity of $\lambda_\alpha(g,\phi)$ is preserved if we allow positive non-increasing families of coupling constants $\alpha(t)$ instead of a single time-independent positive α.

CHAPTER 2

Short-time existence and evolution equations

Due to diffeomorphism invariance, our flow is only weakly parabolic. In fact, the principal symbol for the first equation is the same as for the Ricci flow since the additional term is of lower order. Thus, one cannot directly apply the standard parabolic existence theory. Fortunately, shortly after Hamilton's first proof of short-time existence for the Ricci flow in [23] which was based on the Nash-Moser implicit function theorem, DeTurck [15] found a substantially simpler proof which can easily be modified to get an existence proof for our system $(RH)_\alpha$. Note that we only consider the case where M is closed, but following Shi's short-time existence proof in [55] for the Ricci flow on complete noncompact manifolds, one can also prove more general short-time existence results for our flow. We first recall some results for the Ricci flow, following the presentation of Hamilton [27] very closely.

Since some results strongly depend on the curvature of (N,γ) it is more convenient to work with $\phi : M \to N$ itself instead of $e_N \circ \phi : M \to \mathbb{R}^d$ as in the first chapter. Therefore, repeated Greek indices are summed over from 1 to $n = \dim N$ in this chapter.

Dual Ricci-Harmonic and Ricci-DeTurck flow

Let $g(t)$ be a solution of the Ricci flow and $\psi(t)\colon (M,g) \to (M,h)$ a one parameter family of smooth maps satisfying the harmonic map heat flow equation $\frac{\partial}{\partial t}\psi = \tau_g \psi$ with respect to the evolving metric g. Note that this is $(RH)_{\alpha\equiv 0}$. If $\psi(t)$ is a diffeomorphism at time $t=0$, it will stay a diffeomorphism for at least a short time. Now, we consider the push-forward $\tilde{g} := \psi_* g$ of the metric g under ψ. The evolution equation for $\tilde{g}$ reads

$$\tfrac{\partial}{\partial t}\tilde{g}_{ij} = -2\tilde{R}_{ij} + (\mathcal{L}_V \tilde{g})_{ij} = -2\tilde{R}_{ij} + \tilde{\nabla}_i V_j + \tilde{\nabla}_j V_i, \tag{2.1}$$

where $\tilde{\nabla}$ denotes the Levi-Cività connection of $\tilde{g}$ and $\frac{\partial}{\partial t}\psi = -V \circ \psi$. One calls this the dual Ricci-Harmonic flow or also the h-flow. An easy computation (see DeTurck [15] or Chow and Knopf [9, Chapter 3]) shows that V is given by

$$V^\ell = \tilde{g}^{ij}\big({}^{\tilde{g}}\Gamma_{ij}^\ell - {}^{h}\Gamma_{ij}^\ell\big), \tag{2.2}$$

the trace of the tensor which is the difference between the Christoffel symbols of the Levi-Cività connections of $\tilde{g}$ and of h, respectively. Note that the evolution equation of $\tilde{g}$ involves only the metrics $\tilde{g}$ and h and not the metric g, and since it involves ${}^h\Gamma$ for the fixed background metric h it is no longer diffeomorphism invariant. Indeed, one can show (see Hamilton [27, Section 6]) that

$$\tfrac{\partial}{\partial t}\tilde{g}_{ij} = \tilde{g}^{k\ell}\tilde{\nabla}_k\hat{\nabla}_\ell\tilde{g}_{ij}, \tag{2.3}$$

where $\hat{\nabla}$ denotes the Levi-Cività connection of the background metric h. Since $\hat{\nabla}$ is independent of $\tilde{g}$ and $\tilde{\nabla}$ only involves first derivatives of $\tilde{g}$ this is a quasilinear equation. Its principal symbol is $\sigma(\xi) = \tilde{g}^{ij}\xi_i\xi_j \cdot \mathrm{id}$, where id is the identity on tensors $\tilde{g}$. Hence this flow equation is strictly parabolic and we get short-time existence from the standard parabolic theory for quasilinear equations.

Now, one can find a solution to the Ricci flow with smooth initial metric $g(0) = g_0$ as follows. Chose any diffeomorphism $\psi(0)\colon M \to M$. Since $g(0)$ is smooth, its push-forward $\tilde{g}(0)$ is also smooth and equation (2.3) has a smooth solution for a short time. Next, one computes the vector field V with (2.2) and solves the ODE system

$$\tfrac{\partial}{\partial t}\psi = -V \circ \psi.$$

One then recovers g as the pull-back $g = \psi^*\tilde{g}$. This method also proves uniqueness of the Ricci flow. Indeed, let $g^1(t)$ and $g^2(t)$ be two solutions of the Ricci flow equation for $t \in [0,T)$ satisfying $g^1(0) = g^2(0)$. Then one can solve the harmonic map heat flows $\frac{\partial}{\partial t}\psi^i = \tau_{g^i}\psi^i$, $i \in \{1,2\}$ with $\psi^1(0) = \psi^2(0)$. This yields two solutions $\tilde{g}^i = \psi^i_*g^i$ of the dual Ricci-Harmonic map flow with the same initial values, hence they must agree. Then the corresponding vector fields V^i agree and the two ODE systems $\frac{\partial}{\partial t}\psi^i = -V^i \circ \psi^i$ with the same initial data must have the same solutions $\psi^1 \equiv \psi^2$. Hence also the pull-back metrics g^1 and g^2 must agree for all $t \in [0,T)$.

If we additionally assume that $(M,h) = (M,g_0)$ and $\psi(0) = \mathrm{id}_M$, the flow $\tilde{g}$ which has the same initial data $\tilde{g}(0) = g_0$ as g is called the Ricci-DeTurck flow. This flow was introduced by DeTurck in [15] and written term by term in Shi's paper [55] to compute the evolution equations in coordinate form using only the fixed connection $\hat{\nabla}$. For the more general case of the dual Ricci-Harmonic flow, the evolution equations in coordinate form have been derived by Simon [56]; they read

$$\begin{aligned}\tfrac{\partial}{\partial t}\tilde{g}_{ij} &= \tilde{g}^{k\ell}\hat{\nabla}_k\hat{\nabla}_\ell\tilde{g}_{ij} - \tilde{g}^{k\ell}\tilde{g}_{ip}h^{pq}\hat{R}_{jkq\ell} - \tilde{g}^{k\ell}\tilde{g}_{jp}h^{pq}\hat{R}_{ikq\ell}\\ &\quad + \tfrac{1}{2}\tilde{g}^{k\ell}\tilde{g}^{pq}\big(\hat{\nabla}_i\tilde{g}_{pk}\hat{\nabla}_j\tilde{g}_{q\ell} + 2\hat{\nabla}_k\tilde{g}_{ip}\hat{\nabla}_q\tilde{g}_{j\ell} - 2\hat{\nabla}_k\tilde{g}_{ip}\hat{\nabla}_\ell\tilde{g}_{jq} - 4\hat{\nabla}_i\tilde{g}_{pk}\hat{\nabla}_\ell\tilde{g}_{jq}\big),\end{aligned} \tag{2.4}$$

where $\hat{\nabla}$ denotes the Levi-Cività connection with respect to the fixed background metric h as above, and $\hat{R}_{ijkl} = (\mathrm{Rm}(h))_{ijkl}$ denotes the Riemannian curvature tensor of h.

Recent work of Isenberg, Guenther and Knopf [33], Schnürer, Schulze and Simon [54] and others shows that DeTurck's trick is not only useful to prove short-time existence for the Ricci flow, but is also useful for convergence results. Their results show that the h-flow itself is also interesting to study. In this thesis however, we only use it as a technical tool. More details for the results mentioned here are found in Chow and Knopf's book [9, Chapter 3].

Short-time existence and uniqueness for $(RH)_\alpha$

Let $(g(t),\phi(t))_{t\in[0,T)}$ be a solution of the $(RH)_\alpha$ flow with initial data $(g(0),\phi(0)) = (g_0,\phi_0)$. As for the Ricci-DeTurck flow above, we now let $\psi(t)\colon (M,g(t))\to(M,g_0)$ be a solution of the harmonic map heat flow $\frac{\partial}{\partial t}\psi = \tau_g\psi$ with $\psi(0)=\mathrm{id}_M$ and denote by $(\tilde g(t),\tilde\phi(t))$ the push-forward of $(g(t),\phi(t))$ with ψ. Analogous to formula (2.1) above, we find

$$\begin{aligned} \tfrac{\partial}{\partial t}\tilde g_{ij} &= \psi_*(\tfrac{\partial}{\partial t}g)_{ij} + (\mathcal{L}_V\tilde g)_{ij} = -2\tilde R_{ij} + 2\alpha\nabla_i\tilde\phi\nabla_j\tilde\phi + \tilde\nabla_iV_j + \tilde\nabla_jV_i,\\ \tfrac{\partial}{\partial t}\tilde\phi &= \psi_*(\tfrac{\partial}{\partial t}\phi) + \mathcal{L}_V\tilde\phi = \tau_{\tilde g}\tilde\phi + \langle\nabla\tilde\phi,V\rangle, \end{aligned} \tag{2.5}$$

where $V^\ell = \tilde g^{ij}({}^{\tilde g}\Gamma^\ell_{ij} - {}^{g_0}\Gamma^\ell_{ij})$ and $\tilde\nabla$ denotes the covariant derivative with respect to $\tilde g$. Note that $\nabla_i\tilde\phi\nabla_j\tilde\phi = (d\tilde\phi\otimes d\tilde\phi)_{ij}$ as well as $\langle\nabla\tilde\phi,V\rangle = d\tilde\phi(V)$ are independent of the choice of the metric. Using (2.3), we find

$$\tfrac{\partial}{\partial t}\tilde g_{ij} = \tilde g^{k\ell}\tilde\nabla_k\hat\nabla_\ell\tilde g_{ij} + 2\alpha(\nabla\tilde\phi\otimes\nabla\tilde\phi)_{ij}, \tag{2.6}$$

which is again quasilinear strictly parabolic. The explicit evolution equation involving only the fixed Levi-Cività connection $\hat\nabla$ of g_0 can be found from (2.4) by adding $2\alpha\hat\nabla_i\tilde\phi\hat\nabla_j\tilde\phi$ on the right and replacing h by g_0.

The evolution equation for $\tilde\phi(t)$ in terms of $\hat\nabla$ can be computed as follows. Using normal coordinates on (N,γ) we have ${}^N\Gamma^\lambda_{\mu\nu}=0$ at the base point and thus $\tau_{\tilde g}\tilde\phi = \triangle_{\tilde g}\tilde\phi$. We find

$$\begin{aligned} \tfrac{\partial}{\partial t}\tilde\phi^\lambda &= \triangle_{\tilde g}\tilde\phi^\lambda + \langle\nabla\tilde\phi^\lambda,V\rangle = \tilde g^{k\ell}\big(\partial_k\partial_\ell\tilde\phi^\lambda - {}^{\tilde g}\Gamma^j_{k\ell}\nabla_j\tilde\phi^\lambda\big) + \nabla_j\tilde\phi^\lambda V^j\\ &= \tilde g^{k\ell}\big(\hat\nabla_k\hat\nabla_\ell\tilde\phi^\lambda + {}^{g_0}\Gamma^j_{k\ell}\nabla_j\tilde\phi^\lambda - {}^{\tilde g}\Gamma^j_{k\ell}\nabla_j\tilde\phi^\lambda\big) + \nabla_j\tilde\phi^\lambda\cdot\tilde g^{k\ell}\big({}^{\tilde g}\Gamma^j_{k\ell} - {}^{g_0}\Gamma^j_{k\ell}\big)\\ &= \tilde g^{k\ell}\hat\nabla_k\hat\nabla_\ell\tilde\phi^\lambda. \end{aligned} \tag{2.7}$$

Putting these results together, we have proved the following.

Proposition 2.1
Let $(g(t),\phi(t))$ be a solution of the $(RH)_\alpha$ flow with initial data $(g(0),\phi(0)) = (g_0,\phi_0)$. Let $\psi(t)\colon(M,g(t))\to(M,g_0)$ solve the harmonic map heat flow $\frac{\partial}{\partial t}\psi=\tau_g\psi$ with $\psi(0)=$ id_M *and let $(\tilde g(t),\tilde\phi(t))$ denote the push-forward of $(g(t),\phi(t))$ with ψ. Let $\hat\nabla$ be the (fixed) Levi-Cività connection with respect to g_0. Then the dual $(RH)_\alpha$ flow $(\tilde g(t),\tilde\phi(t))$ satisfies*

$$\begin{aligned} \tfrac{\partial}{\partial t}\tilde g_{ij} &= \tilde g^{k\ell}\hat\nabla_k\hat\nabla_\ell\tilde g_{ij} - \tilde g^{k\ell}\tilde g_{ip}g_0^{pq}\hat R_{jkq\ell} - \tilde g^{k\ell}\tilde g_{jp}g_0^{pq}\hat R_{ikq\ell} + 2\alpha\hat\nabla_i\tilde\phi\hat\nabla_j\tilde\phi\\ &\quad + \tfrac12\tilde g^{k\ell}\tilde g^{pq}\big(\hat\nabla_i\tilde g_{pk}\hat\nabla_j\tilde g_{q\ell} + 2\hat\nabla_k\tilde g_{ip}\hat\nabla_q\tilde g_{j\ell} - 2\hat\nabla_k\tilde g_{ip}\hat\nabla_\ell\tilde g_{jq} - 4\hat\nabla_i\tilde g_{pk}\hat\nabla_\ell\tilde g_{jq}\big),\\ \tfrac{\partial}{\partial t}\tilde\phi^\lambda &= \tilde g^{k\ell}\hat\nabla_k\hat\nabla_\ell\tilde\phi^\lambda + \tilde g^{k\ell}\big({}^N\Gamma^\lambda_{\mu\nu}\circ\phi\big)\hat\nabla_k\tilde\phi^\mu\hat\nabla_\ell\tilde\phi^\nu. \end{aligned}$$

In particular, the principal symbol for both equations is $\sigma(\xi) = \tilde{g}^{ij}\xi_i\xi_j \cdot \mathrm{id}$, *i.e. the push-forward flow is a solution to a system of strictly parabolic equations.*

Short-time existence and uniqueness for the dual flow (and hence also for the $(RH)_\alpha$ flow itself) now follow exactly as in the simpler case of the Ricci and the dual Ricci-Harmonic flow described in the previous section.

The Bochner identity

We will now reprove a well-known identity (cf. Jost [35, Section 8.7]) for the energy density $e(\phi) := \frac{1}{2}|\nabla\phi|_g^2$ of a harmonic map ϕ. This provides a good exercise for computations with connections and commutator identities on more complicated bundles like $T^*M \otimes \phi^*TN$. Readers who are not familiar with this should consult the first section of the appendix.

For the remainder of this chapter, we denote the Riemannian curvature tensor on (N,γ) by ${}^N\mathrm{Rm}$ and let Rm, Rc and R be the Riemannian, Ricci and scalar curvature on (M,g). Finally, we also use the fact that $\tau_g\phi = \nabla_p\nabla_p\phi$ for the covariant derivative ∇ on $T^*M \otimes \phi^*TN$ (cf. Jost [35, Section 8.1]).

Proposition 2.2
If $\phi \in C^\infty(M,N)$ *is a harmonic map, then*

$$-\triangle_g e(\phi) + |\nabla^2\phi|^2 + \langle \mathrm{Rc}, \nabla\phi\otimes\nabla\phi\rangle = \left\langle {}^N\mathrm{Rm}(\nabla_i\phi, \nabla_j\phi)\nabla_j\phi, \nabla_i\phi\right\rangle, \tag{2.8}$$

where the curvature terms are given in coordinates by

$$\begin{aligned}\langle \mathrm{Rc}, \nabla\phi\otimes\nabla\phi\rangle &:= R_{ij}\nabla_i\phi^\kappa\nabla_j\phi^\kappa,\\ \left\langle {}^N\mathrm{Rm}(\nabla_i\phi, \nabla_j\phi)\nabla_j\phi, \nabla_i\phi\right\rangle &:= {}^N R_{\kappa\mu\lambda\nu}\nabla_i\phi^\kappa\nabla_j\phi^\mu\nabla_i\phi^\lambda\nabla_j\phi^\nu.\end{aligned}$$

Proof. We compute, using (A.10),

$$\begin{aligned}\triangle_g e(\phi) &= \nabla_p\nabla_p\big(\tfrac{1}{2}\nabla_i\phi^\kappa\nabla_i\phi^\kappa\big) = \nabla_p\nabla_i\phi^\kappa\nabla_p\nabla_i\phi^\kappa + \nabla_p\nabla_i\nabla_p\phi^\kappa\nabla_i\phi^\kappa\\ &= \nabla_p\nabla_i\phi^\kappa\nabla_p\nabla_i\phi^\kappa + \nabla_i\nabla_p\nabla_p\phi^\kappa\nabla_i\phi^\kappa\\ &\quad + R_{pipk}\nabla_k\phi^\kappa\nabla_i\phi^\kappa + R_{pi\kappa\lambda}\nabla_p\phi^\lambda\nabla_i\phi^\kappa\\ &= \nabla_p\nabla_i\phi^\kappa\nabla_p\nabla_i\phi^\kappa + \nabla_i(\tau_g\phi)^\kappa\nabla_i\phi^\kappa\\ &\quad + R_{ik}\nabla_k\phi^\kappa\nabla_i\phi^\kappa + {}^N R_{\mu\nu\kappa\lambda}\nabla_p\phi^\mu\nabla_i\phi^\nu\nabla_p\phi^\lambda\nabla_i\phi^\kappa\\ &= |\nabla^2\phi|^2 + \langle \mathrm{Rc}, \nabla\phi\otimes\nabla\phi\rangle - \left\langle {}^N\mathrm{Rm}(\nabla_i\phi, \nabla_p\phi)\nabla_p\phi, \nabla_i\phi\right\rangle.\end{aligned}$$

Here, we used the fact that ϕ is harmonic, i.e. $\tau_g\phi = 0$, in the last step. □

Evolution equations for R, Rc, $|\nabla\phi|^2$ and $\nabla\phi\otimes\nabla\phi$

We start with the evolution equations for the scalar and Ricci curvature on M.

Proposition 2.3
Let $(g(t),\phi(t))$ be a solution to the $(RH)_\alpha$ flow equation. Then the scalar curvature evolves according to

$$\begin{aligned}\tfrac{\partial}{\partial t}R &= \triangle R + 2|\mathrm{Rc}|^2 - 4\alpha\langle \mathrm{Rc}, \nabla\phi\otimes\nabla\phi\rangle + 2\alpha|\tau_g\phi|^2 - 2\alpha|\nabla^2\phi|^2 \\ &\quad + 2\alpha\left\langle {}^N\mathrm{Rm}(\nabla_i\phi,\nabla_j\phi)\nabla_j\phi, \nabla_i\phi\right\rangle\end{aligned} \tag{2.9}$$

and the Ricci curvature evolves by

$$\begin{aligned}\tfrac{\partial}{\partial t}R_{ij} &= \triangle R_{ij} - 2R_{iq}R_{jq} + 2R_{ipjq}R_{pq} + 2\alpha\,\tau_g\phi\nabla_i\nabla_j\phi - 2\alpha\nabla_p\nabla_i\phi\nabla_p\nabla_j\phi \\ &\quad + 2\alpha R_{pijq}\nabla_p\phi\nabla_q\phi + 2\alpha\left\langle {}^N\mathrm{Rm}(\nabla_i\phi,\nabla_p\phi)\nabla_p\phi, \nabla_j\phi\right\rangle.\end{aligned} \tag{2.10}$$

Proof. We know that for $\frac{\partial}{\partial t}g_{ij} = h_{ij}$ the evolution equation for the Ricci tensor is given by

$$\tfrac{\partial}{\partial t}R_{ij} = \underbrace{\tfrac12\nabla_p\nabla_j h_{ip}}_{=:\,I}\ \underbrace{+\tfrac12\nabla_p\nabla_i h_{jp}}_{=:\,II}\ \underbrace{-\tfrac12\triangle h_{ij}}_{=:\,III}\ \underbrace{-\tfrac12\nabla_i\nabla_j(\mathrm{tr}_g\,h)}_{=:\,IV}, \tag{2.11}$$

see for example Müller [45, Proposition 1.4] for a proof of this general variation formula. We compute the four terms on the right hand side separately for our case where h_{ij} is given by $h_{ij} = -2R_{ij} + 2\alpha\nabla_i\phi\nabla_j\phi$.

$$\begin{aligned}I &= \tfrac12\nabla_p\nabla_j h_{ip} = \tfrac12\nabla_p\nabla_j(-2R_{ip} + 2\alpha\nabla_i\phi^\mu\nabla_p\phi^\mu) \\ &= -\nabla_p\nabla_j R_{ip} + \alpha\nabla_p(\nabla_j\nabla_i\phi^\mu\nabla_p\phi^\mu + \nabla_i\phi^\mu\nabla_j\nabla_p\phi^\mu) \\ &= -\tfrac12\nabla_j\nabla_i R + R_{jpiq}R_{pq} - R_{iq}R_{jq} + \alpha(\nabla_p\nabla_i\nabla_j\phi^\mu\nabla_p\phi^\mu + \nabla_i\nabla_j\phi^\mu(\tau_g\phi)^\mu) \\ &\quad + \alpha(\nabla_p\nabla_i\phi^\mu\nabla_p\nabla_j\phi^\mu + \nabla_i\phi^\mu\nabla_p\nabla_p\nabla_j\phi^\mu),\end{aligned}$$

where we used

$$\begin{aligned}\nabla_p\nabla_j R_{ip} &= \nabla_j\nabla_p R_{ip} - R_{jpiq}R_{qp} - R_{jppq}R_{iq} \\ &= \tfrac12\nabla_j\nabla_i R - R_{jpiq}R_{pq} + R_{jq}R_{iq}\end{aligned}$$

for the last step. Here, the first identity is the commutator rule for 2-tensors, the second one follows from the twice contracted second Bianchi identity $\nabla_p R_{ip} = \frac12\nabla_i R$. Finally, dropping the indices for the summation over ϕ if clear from the context, we can write I in the form

$$\begin{aligned}I &= -\tfrac12\nabla_j\nabla_i R + R_{jpiq}R_{pq} - R_{iq}R_{jq} \\ &\quad + \alpha(\nabla_p\nabla_i\nabla_j\phi\nabla_p\phi + \nabla_i\nabla_j\phi\tau_g\phi + \nabla_p\nabla_i\phi\nabla_p\nabla_j\phi + \nabla_i\phi\nabla_p\nabla_p\nabla_j\phi).\end{aligned}$$

Interchanging the indices i and j, we get

$$\begin{aligned}II &= -\tfrac12\nabla_j\nabla_i R + R_{ipjq}R_{pq} - R_{iq}R_{jq} \\ &\quad + \alpha(\nabla_p\nabla_i\nabla_j\phi\nabla_p\phi + \nabla_i\nabla_j\phi\tau_g\phi + \nabla_p\nabla_i\phi\nabla_p\nabla_j\phi + \nabla_j\phi\nabla_p\nabla_p\nabla_i\phi).\end{aligned}$$

For the third term in (2.11) we find

$$\begin{aligned} I\!I\!I &= -\tfrac{1}{2}\triangle h_{ij} = -\tfrac{1}{2}\nabla_p\nabla_p(-2R_{ij} + 2\alpha\nabla_i\phi^\mu\nabla_j\phi^\mu) \\ &= \triangle R_{ij} - \alpha\nabla_p(\nabla_p\nabla_i\phi^\mu\nabla_j\phi^\mu + \nabla_i\phi^\mu\nabla_p\nabla_j\phi^\mu) \\ &= \triangle R_{ij} - \alpha(\nabla_p\nabla_p\nabla_i\phi\nabla_j\phi + \nabla_i\phi\nabla_p\nabla_p\nabla_j\phi + 2\nabla_p\nabla_i\phi\nabla_p\nabla_j\phi), \end{aligned}$$

and for the last term we obtain

$$\begin{aligned} I\!V &= -\tfrac{1}{2}\nabla_i\nabla_j(\operatorname{tr}_g h) = -\tfrac{1}{2}\nabla_i\nabla_j(-2R + 2\alpha\nabla_p\phi^\mu\nabla_p\phi^\mu) \\ &= \nabla_i\nabla_j R - \alpha(2\nabla_i\nabla_j\nabla_p\phi\nabla_p\phi + 2\nabla_i\nabla_p\phi\nabla_j\nabla_p\phi). \end{aligned}$$

If we plug the formulas for I – $I\!V$ into (2.11), a lot of cancellations occur and we obtain

$$\begin{aligned} \tfrac{\partial}{\partial t}R_{ij} &= \triangle R_{ij} - 2R_{iq}R_{jq} + 2R_{ipjq}R_{pq} + 2\alpha\,\tau_g\phi\nabla_i\nabla_j\phi - 2\alpha\nabla_p\nabla_i\phi\nabla_p\nabla_j\phi \\ &\quad + 2\alpha\nabla_p\nabla_i\nabla_j\phi^\kappa\nabla_p\phi^\kappa - 2\alpha\nabla_i\nabla_j\nabla_p\phi^\kappa\nabla_p\phi^\kappa. \end{aligned}$$

Note that the last line is exactly the commutator

$$\begin{aligned} &2\alpha\nabla_p\nabla_i\nabla_j\phi^\kappa\nabla_p\phi^\kappa - 2\alpha\nabla_i\nabla_p\nabla_j\phi^\kappa\nabla_p\phi^\kappa \\ &\qquad\qquad = 2\alpha R_{pijq}\nabla_q\phi^\kappa\nabla_p\phi^\kappa + 2\alpha R_{\mu\nu\kappa\lambda}\nabla_p\phi^\mu\nabla_i\phi^\nu\nabla_j\phi^\lambda\nabla_p\phi^\kappa. \end{aligned}$$

This proves (2.10). For the evolution equation for R, we simply have to take the trace of (2.10). Indeed, we compute

$$\begin{aligned} \tfrac{\partial}{\partial t}R &= \tfrac{\partial}{\partial t}(g^{ij}R_{ij}) = g^{ij}(\tfrac{\partial}{\partial t}R_{ij}) + (2R_{ij} - 2\alpha\nabla_i\phi\nabla_j\phi)R_{ij} \\ &= \triangle R - 2|\mathrm{Rc}|^2 + 2|\mathrm{Rc}|^2 + 2\alpha|\tau_g\phi|^2 - 2\alpha|\nabla^2\phi|^2 \\ &\quad - 2\alpha R_{pq}\nabla_p\phi\nabla_q\phi + 2\alpha\,{}^N\!R_{\mu\nu\kappa\lambda}\nabla_q\phi^\mu\nabla_p\phi^\nu\nabla_q\phi^\kappa\nabla_p\phi^\lambda \\ &\quad + 2|\mathrm{Rc}|^2 - 2\alpha R_{ij}\nabla_i\phi\nabla_j\phi, \end{aligned}$$

and equation (2.9) follows. □

Next, we compute the evolution equations for $|\nabla\phi|^2$ and $\nabla\phi\otimes\nabla\phi$.

Proposition 2.4
Let $(g(t),\phi(t))$ be a solution of $(RH)_\alpha$. Then the energy density of ϕ satisfies the evolution equation

$$\tfrac{\partial}{\partial t}|\nabla\phi|^2 = \triangle|\nabla\phi|^2 - 2\alpha|\nabla\phi\otimes\nabla\phi|^2 - 2|\nabla^2\phi|^2 + 2\left\langle {}^N\mathrm{Rm}(\nabla_i\phi,\nabla_j\phi)\nabla_j\phi,\nabla_i\phi\right\rangle. \quad (2.12)$$

Furthermore, we have

$$\begin{aligned} \tfrac{\partial}{\partial t}(\nabla_i\phi\nabla_j\phi) &= \triangle(\nabla_i\phi\nabla_j\phi) - 2\nabla_p\nabla_i\phi\nabla_p\nabla_j\phi - R_{ip}\nabla_p\phi\nabla_j\phi - R_{jp}\nabla_p\phi\nabla_i\phi \\ &\quad + 2\left\langle {}^N\mathrm{Rm}(\nabla_i\phi,\nabla_p\phi)\nabla_p\phi,\nabla_j\phi\right\rangle. \end{aligned} \quad (2.13)$$

Proof. We start with the second statement. We have

$$\tfrac{\partial}{\partial t}(\nabla_i\phi\nabla_j\phi) = (\nabla_t\nabla_i\phi)\nabla_j\phi + (\nabla_t\nabla_j\phi)\nabla_i\phi,$$

where the meaning of the covariant time derivative ∇_t is explained in the first section of the appendix. For the first term on the right hand side, we compute

$$\begin{aligned}(\nabla_t\nabla_i\phi)\nabla_j\phi &= (\nabla_i\nabla_t\phi)\nabla_j\phi = \nabla_i(\tfrac{\partial}{\partial t}\phi)^\kappa\nabla_j\phi^\kappa = \nabla_i\nabla_p\nabla_p\phi^\kappa\nabla_j\phi^\kappa\\ &= \nabla_p\nabla_i\nabla_p\phi^\kappa\nabla_j\phi^\kappa + R_{ippq}\nabla_q\phi^\kappa\nabla_j\phi^\kappa + R_{ip\kappa\lambda}\nabla_p\phi^\lambda\nabla_j\phi^\kappa\\ &= \nabla_p\nabla_p\nabla_i\phi\nabla_j\phi - R_{iq}\nabla_q\phi\nabla_j\phi + {}^N\!R_{\mu\nu\kappa\lambda}\nabla_i\phi^\mu\nabla_p\phi^\nu\nabla_p\phi^\lambda\nabla_j\phi^\kappa\\ &= \triangle\nabla_i\phi\nabla_j\phi - R_{iq}\nabla_q\phi\nabla_j\phi + \left\langle {}^N\mathrm{Rm}(\nabla_i\phi,\nabla_p\phi)\nabla_p\phi,\nabla_j\phi\right\rangle.\end{aligned}$$

For the second term, we get the same expression with indices i and j interchanged. With the symmetries of the Riemann tensor and the equation

$$\triangle(\nabla_i\phi\nabla_j\phi) = \triangle\nabla_i\phi\nabla_j\phi + \triangle\nabla_j\phi\nabla_i\phi + 2\nabla_p\nabla_i\phi\nabla_p\nabla_j\phi$$

we get the desired evolution equation (2.13). Then, we obtain (2.12) from (2.13) by simply taking the trace. Indeed

$$\begin{aligned}\tfrac{\partial}{\partial t}|\nabla\phi|^2 &= (\tfrac{\partial}{\partial t}g^{ij})\nabla_i\phi\nabla_j\phi + g^{ij}\tfrac{\partial}{\partial t}(\nabla_i\phi\nabla_j\phi)\\ &= (2R_{ij} - 2\alpha\nabla_i\phi\nabla_j\phi)\nabla_i\phi\nabla_j\phi + \triangle|\nabla\phi|^2 - 2|\nabla^2\phi|^2\\ &\quad - 2R_{ip}\nabla_i\phi\nabla_p\phi + 2\,{}^N\!R_{\kappa\mu\lambda\nu}\nabla_i\phi^\kappa\nabla_i\phi^\lambda\nabla_p\phi^\mu\nabla_p\phi^\nu\\ &= -2\alpha|\nabla\phi\otimes\nabla\phi|^2 + \triangle|\nabla\phi|^2 - 2|\nabla^2\phi|^2 + 2\left\langle {}^N\mathrm{Rm}(\nabla_i\phi,\nabla_j\phi)\nabla_j\phi,\nabla_i\phi\right\rangle. \square\end{aligned}$$

Evolution of $\mathcal{S} = \mathrm{Rc} - \alpha\nabla\phi\otimes\nabla\phi$ and its trace

If we define the symmetric tensor $\mathcal{S} := \mathrm{Rc} - \alpha\nabla\phi\otimes\nabla\phi$ with components

$$S_{ij} = R_{ij} - \alpha\nabla_i\phi\nabla_j\phi \tag{2.14}$$

and let $S = \mathrm{tr}_g\,\mathcal{S} = g^{ij}S_{ij} = R - \alpha|\nabla\phi|^2$ be its trace, we can write the $(RH)_\alpha$ flow as

$$\begin{cases} \tfrac{\partial}{\partial t}g_{ij} = -2S_{ij},\\ \tfrac{\partial}{\partial t}\phi = \tau_g\phi,\end{cases}$$

and the energy from Chapter 1 as

$$\mathcal{F}_\alpha(g,\phi,f) := \int_M \Big(S + |\nabla f|_g^2\Big)e^{-f}dV_g. \tag{2.15}$$

It is thus convenient to study the evolution equations for $\mathcal{S}$ and S. Indeed many terms cancel and we get much nicer equations than in the previous section.

Theorem 2.5
Let $(g(t),\phi(t))$ solve $(RH)_\alpha$ with $\alpha(t)\equiv\alpha>0$. Then $\mathcal{S}$ and S defined as above satisfy the following evolution equations

$$\begin{aligned}\tfrac{\partial}{\partial t}S &= \triangle S + 2|S_{ij}|^2 + 2\alpha|\tau_g\phi|^2,\\ \tfrac{\partial}{\partial t}S_{ij} &= \triangle_L S_{ij} + 2\alpha\,\tau_g\phi\nabla_i\nabla_j\phi,\end{aligned} \tag{2.16}$$

where $\triangle_L$ denotes the Lichnerowicz Laplacian, introduced in [40], which is defined on symmetric two-tensors t_{ij} by

$$\triangle_L t_{ij} := \triangle t_{ij} + 2R_{ipjq}t_{pq} - R_{ip}t_{pj} - R_{jp}t_{pi}.$$

Proof. This follows directly by combining the evolution equations from Proposition 2.3 with those from Proposition 2.4. □

Remark. Note that in contrast to the evolution of Rc, R, $\nabla\phi\otimes\nabla\phi$ and $|\nabla\phi|^2$ the evolution equations in Theorem 2.5 for the combinations $\mathrm{Rc}-\alpha\nabla\phi\otimes\nabla\phi$ and $R-\alpha|\nabla\phi|^2$ do *not* depend on the intrinsic curvature of N.

Corollary 2.6
For a solution $(g(t),\phi(t))$ of $(RH)_\alpha$ with a time-dependent coupling function $\alpha(t)$, we get

$$\begin{aligned}\tfrac{\partial}{\partial t}S &= \triangle S + 2|S_{ij}|^2 + 2\alpha|\tau_g\phi|^2 - \dot\alpha|\nabla\phi|^2,\\ \tfrac{\partial}{\partial t}S_{ij} &= \triangle_L S_{ij} + 2\alpha\,\tau_g\phi\nabla_i\nabla_j\phi - \dot\alpha\nabla_i\phi\nabla_j\phi.\end{aligned} \tag{2.17}$$

First results about singularities

An immediate consequence of Corollary 2.6 is the following.

Corollary 2.7
Let $(g(t),\phi(t))$ be a solution to the $(RH)_\alpha$ flow with a nonnegative, non-increasing coupling function $\alpha(t)$. Let $S(t)=R(g(t))-\alpha(t)|\nabla\phi(t)|^2_{g(t)}$ as above, with initial data $S(0)>0$ on M. Then $R_{min}(t):=\min_{x\in M}R(x,t)\to\infty$ in finite time and thus $g(t)$ must become singular in finite time $T_{sing}\le\frac{m}{2S_{min}(0)}<\infty$.

Proof. Since $\alpha(t)\ge 0$ and $\dot\alpha(t)\le 0$ for all $t\ge 0$, Corollary 2.6 yields

$$\tfrac{\partial}{\partial t}S \ge \triangle S + 2|S_{ij}|^2 \ge \triangle S + \tfrac{2}{m}S^2 \tag{2.18}$$

and thus by comparing with solutions of the ODE $\frac{d}{dt}a(t)=\frac{2}{m}a(t)^2$ which are

$$a(t)=\frac{a(0)}{1-\frac{2t}{m}a(0)},$$

the maximum principle from the appendix, Proposition A.1, yields

$$S_{min}(t) \geq \frac{S_{min}(0)}{1 - \frac{2t}{m} S_{min}(0)} \tag{2.19}$$

for all $t \geq 0$ as long as the flow exists. In particular, if $S_{min}(0) > 0$ this implies that $S_{min}(t) \to \infty$ in finite time $T_0 \leq \frac{m}{2S_{min}(0)} < \infty$. Since $R = S + \alpha|\nabla\phi|^2 \geq S$, we find that also $R_{min}(t) \to \infty$ before T_0 and thus $g(t)$ has to become singular in finite time $T_{sing} \leq T_0 \leq \frac{m}{2S_{min}(0)} < \infty$. □

As a second consequence, we see that if the energy density $e(\phi) = \frac{1}{2}|\nabla\phi|^2$ blows up at some point in space-time while $\alpha(t)$ is bounded away from zero, then also $g(t)$ must become singular at this point.

Corollary 2.8
Let $(g(t), \phi(t))_{t\in[0,T)}$ be a smooth solution of $(RH)_\alpha$ with a non-increasing coupling function $\alpha(t)$ satisfying $\alpha(t) \geq \underline{\alpha} > 0$ for all $t \in [0,T)$. Suppose that $|\nabla\phi|^2(x_k, t_k) \to \infty$ for a sequence $(x_k, t_k)_{k\in\mathbb{N}}$ with $t_k \nearrow T$. Then also $R(x_k, t_k) \to \infty$ for this sequence and thus $g(t_k)$ must become singular as t_k approaches T.

Proof. From (2.18) we obtain $S \geq S_{min}(0)$ and thus

$$R = \alpha|\nabla\phi|^2 + S \geq \underline{\alpha}|\nabla\phi|^2 + S_{min}(0), \quad \forall (x,t) \in M \times [0,T). \tag{2.20}$$

Hence, if $|\nabla\phi|^2(x_k, t_k) \to \infty$ for a sequence $(x_k, t_k)_{k\in\mathbb{N}} \subset M \times [0,T)$ with $t_k \nearrow T$ then also $R(x_k, t_k) \to \infty$ for this sequence and $g(t_k)$ must become singular as $t_k \nearrow T$. □

Remark. The proof shows that Corollary 2.8 stays true if $\alpha(t) \searrow 0$ as $t \nearrow T$ as long as $|\nabla\phi|^2(x_k, t_k) \to \infty$ fast enough such that $\alpha(t_k)|\nabla\phi|^2(x_k, t_k) \to \infty$ still holds true.

Now, we derive for $t > 0$ an improved version of (2.20) which does not depend on the initial data $S(0)$. Using (2.18) and the maximum principle, we see that if $S_{min}(0) \geq C \in \mathbb{R}$ we obtain

$$S_{min}(t) \geq \frac{C}{1 - \frac{2t}{m}C} \longrightarrow -\frac{m}{2t} \qquad (C \to -\infty)$$

and thus $S(t) \geq -\frac{m}{2t}$ for all $t > 0$ as long as the flow exists, independent of $S(0)$. More rigorously, this is obtained as follows. The inequality (2.18) implies

$$\tfrac{\partial}{\partial t}(tS) = S + t\big(\tfrac{\partial}{\partial t}S\big) \geq \triangle(tS) + S\big(1 + \tfrac{2t}{m}S\big).$$

If (x_0, t_0) is a point where tS first reaches its minimum over $M \times [0, T-\delta]$, $\delta > 0$ arbitrarily small, we get $S(x_0, t_0)\big(1 + \frac{2t_0}{m}S(x_0, t_0)\big) \leq 0$, which is only possible for $t_0 S(x_0, t_0) \geq -\frac{m}{2}$. Hence $tS \geq -\frac{m}{2}$ on all of $M \times [0, T-\delta]$. Since δ was arbitrary, we obtain the desired inequality $S(t) \geq -\frac{m}{2t}$ everywhere on $M \times (0,T)$. This yields

$$R \geq \alpha|\nabla\phi|^2 - \tfrac{m}{2t} \geq \underline{\alpha}|\nabla\phi|^2 - \tfrac{m}{2t}, \quad \forall (x,t) \in M \times (0,T),$$

which immediately implies the converse of Corollary 2.8.

Corollary 2.9
Let $(g(t),\phi(t))_{t\in[0,T)}$ be a smooth solution of $(RH)_\alpha$ with a non-increasing coupling function $\alpha(t)\geq \underline{\alpha}>0$ for all $t\in[0,T)$. Assume that $R\leq R_0$ on $M\times[0,T)$. Then

$$|\nabla\phi|^2\leq \frac{R_0}{\underline{\alpha}}+\frac{m}{2\underline{\alpha}t},\quad \forall(x,t)\in M\times(0,T). \tag{2.21}$$

Singularities of the type as in Corollary 2.8, where the energy density of ϕ blows up, can not only be ruled out if the curvature of M stays bounded. There is also a way to rule them out a-priori. Namely, such singularities cannot form if either N has non-positive sectional curvatures or if we choose the coupling constants $\alpha(t)$ large enough such that

$$\max_{y\in N}{}^N K(y)\leq\frac{\alpha}{m},$$

where ${}^N K$ denotes the sectional curvature of N. In fact, we have the following estimates for the energy density $|\nabla\phi|^2$.

Proposition 2.10
Let $(g(t),\phi(t))_{t\in[0,T)}$ be a solution of $(RH)_\alpha$ with a non-increasing coupling function $\alpha(t)\geq 0$ and let the sectional curvature of N be bounded above by ${}^N K\leq c_0$. Then

i) if N has non-positive sectional curvatures or more generally if $c_0-\frac{\alpha(t)}{m}\leq 0$, the energy density of ϕ is bounded by its initial data,

$$|\nabla\phi(x,t)|^2\leq\max_{y\in M}|\nabla\phi(y,0)|^2,\quad\forall(x,t)\in M\times[0,T). \tag{2.22}$$

ii) if N has non-positive sectional curvatures and $\alpha(t)\geq\underline{\alpha}>0$, we have in addition to (2.22) the estimate

$$|\nabla\phi(x,t)|^2\leq\frac{m}{2\underline{\alpha}t},\quad\forall(x,t)\in M\times(0,T). \tag{2.23}$$

iii) in general, the energy density satisfies

$$|\nabla\phi(x,t)|^2\leq 2\max_{y\in M}|\nabla\phi(y,0)|^2,\quad\forall(x,t)\in M\times[0,T^*), \tag{2.24}$$

where $T^*:=\min\big\{T,\big(4c_0\max_{y\in M}|\nabla\phi(y,0)|^2\big)^{-1}\big\}$.

Proof. This is a consequence of the evolution equation (2.12) and the Cauchy-Schwarz inequality

$$\tfrac{1}{m}|\nabla\phi|^4=\tfrac{1}{m}|g^{ij}\nabla_i\phi\nabla_j\phi|^2\leq|\nabla_i\phi\nabla_j\phi|^2\leq|\nabla\phi|^4. \tag{2.25}$$

i) If N has non-positive sectional curvatures, $\big\langle{}^N\mathrm{Rm}(\nabla_i\phi,\nabla_j\phi)\nabla_j\phi,\nabla_i\phi\big\rangle\leq 0$, the evolution equation (2.12) implies

$$\tfrac{\partial}{\partial t}|\nabla\phi|^2\leq\triangle|\nabla\phi|^2. \tag{2.26}$$

If $c_0 - \frac{\alpha(t)}{m} \leq 0$, we have $2\left\langle {}^N\mathrm{Rm}(\nabla_i\phi, \nabla_j\phi)\nabla_j\phi, \nabla_i\phi\right\rangle \leq 2c_0|\nabla\phi|^4 \leq 2\frac{\alpha}{m}|\nabla\phi|^4 \leq 2\alpha|\nabla_i\phi\nabla_j\phi|^2$, and we get again (2.26) from (2.12). The claim now follows from the maximum principle applied to (2.26).

ii) If $\left\langle {}^N\mathrm{Rm}(\nabla_i\phi, \nabla_j\phi)\nabla_j\phi, \nabla_i\phi\right\rangle \leq 0$, (2.12) and (2.25) imply

$$\tfrac{\partial}{\partial t}|\nabla\phi|^2 \leq \triangle|\nabla\phi|^2 - 2\alpha|\nabla_i\phi\nabla_j\phi|^2 \leq \triangle|\nabla\phi|^2 - 2\tfrac{\alpha}{m}|\nabla\phi|^4.$$

We obtain

$$\tfrac{\partial}{\partial t}\big(t|\nabla\phi|^2\big) = |\nabla\phi|^2 + t\big(\tfrac{\partial}{\partial t}|\nabla\phi|^2\big) \leq \triangle\big(t|\nabla\phi|^2\big) + |\nabla\phi|^2\big(1 - 2t\tfrac{\alpha}{m}|\nabla\phi|^2\big).$$

At the first point (x_0, t_0) where $t|\nabla\phi|^2$ reaches its maximum over $M \times [0, T-\delta]$, $\delta > 0$ arbitrary, we find $1 - 2t_0\frac{\alpha}{m}|\nabla\phi|^2(x_0, t_0) \geq 0$, i.e.

$$t_0|\nabla\phi|^2(x_0, t_0) \leq \frac{m}{2\alpha} \leq \frac{m}{2\underline{\alpha}},$$

which implies that $t|\nabla\phi|^2 \leq \frac{m}{2\underline{\alpha}}$ for every $(x, t) \in M \times [0, T-\delta]$. The claim follows.

iii) From (2.12), we get

$$\tfrac{\partial}{\partial t}|\nabla\phi|^2 \leq \triangle|\nabla\phi|^2 + 2c_0|\nabla\phi|^4.$$

By comparing with solutions of the ODE $\frac{d}{dt}a(t) = 2c_0 a(t)^2$, which are

$$a(t) = \frac{a(0)}{1 - 2c_0 a(0)t}, \quad t \leq \frac{1}{2c_0 a(0)},$$

the maximum principle (Proposition A.1) implies

$$|\nabla\phi(x,t)|^2 \leq \frac{\max_{y\in M}|\nabla\phi(y,0)|^2}{1 - 2c_0\max_{y\in M}|\nabla\phi(y,0)|^2\, t}, \tag{2.27}$$

for all $x \in M$ and $t \leq \min\big\{T, \big(2c_0\max_{y\in M}|\nabla\phi(y,0)|^2\big)^{-1}\big\}$. In particular, this proves the doubling-time estimate that we claimed. □

CHAPTER 3

Gradient estimates and long-time existence

For solutions $(g(t), \phi(t))$ of the $(RH)_\alpha$ flow with non-increasing $\alpha(t) \geq \underline{\alpha} > 0$, we have seen in Corollary 2.9 that a uniform bound on the curvature of $(M, g(t))$ implies a uniform bound on $|\nabla\phi|^2$. Therefore, one expects that a uniform curvature bound suffices to show long-time existence for our flow. The proof of this result is the main goal of this chapter.

Evolution equations for Rm and $\nabla^2\phi$

We start this section with the computation of the evolution equation for the Christoffel symbols. With $\frac{\partial}{\partial t} g_{ij} = h_{ij} := -2R_{ij} + 2\alpha\nabla_i\phi^\mu\nabla_j\phi^\mu$, we find

$$\begin{aligned}
\tfrac{\partial}{\partial t}\Gamma_{ij}^p &= \tfrac{1}{2} g^{pq}(\nabla_i h_{jq} + \nabla_j h_{iq} - \nabla_q h_{ij}) \\
&= g^{pq}(-\nabla_i R_{jq} - \nabla_j R_{iq} + \nabla_q R_{ij}) \\
&\quad + \alpha\big(\nabla_i(\nabla_j\phi^\mu\nabla^p\phi^\mu) + \nabla_j(\nabla_i\phi^\mu\nabla^p\phi^\mu) - \nabla^p(\nabla_i\phi^\mu\nabla_j\phi^\mu)\big) \\
&= g^{pq}(-\nabla_i R_{jq} - \nabla_j R_{iq} + \nabla_q R_{ij}) \\
&\quad + \alpha\big(\nabla_i\nabla_j\phi\nabla^p\phi + \nabla_j\phi\nabla_i\nabla^p\phi + \nabla_j\nabla_i\phi\nabla^p\phi\big) \\
&\quad + \alpha\big(\nabla_i\phi\nabla_j\nabla^p\phi - \nabla^p\nabla_i\phi\nabla_j\phi - \nabla_i\phi\nabla^p\nabla_j\phi\big) \\
&= g^{pq}(-\nabla_i R_{jq} - \nabla_j R_{iq} + \nabla_q R_{ij}) + 2\alpha\nabla_i\nabla_j\phi\nabla^p\phi
\end{aligned} \tag{3.1}$$

Again, where clear from the context, the summation indices for ϕ have been dropped. Lowering the index p, equation (3.1) becomes

$$g_{pq}\big(\tfrac{\partial}{\partial t}\Gamma_{ij}^q\big) = -\nabla_i R_{jp} - \nabla_j R_{ip} + \nabla_p R_{ij} + 2\alpha\nabla_i\nabla_j\phi\nabla_p\phi. \tag{3.2}$$

There is also another way of writing such equations, using the following useful convention.

Definition 3.1
For two quantities A and B, we denote by $A * B$ any quantity obtained from $A \otimes B$ by summation over pairs of matching (Latin and Greek) indices, contractions with the metrics g and γ and their inverses, and multiplication with constants depending only

on $m = \dim M$, $n = \dim N$ and the ranks of A and B. We also write $(A)^{*1} := 1 * A$, $(A)^{*2} = A * A$, etc.

This notation allows us to write (3.1) in the short form

$$\tfrac{\partial}{\partial t}\Gamma = (\nabla \mathrm{Rm})^{*1} + \alpha \nabla^2\phi * \nabla\phi. \tag{3.3}$$

We are now ready to compute the evolution of the Riemann curvature tensor. Knowing the formula $\frac{\partial}{\partial t}\mathrm{Rm} = \triangle \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm}$ for the Riemannian curvature under Ricci flow, one can guess the formula (3.7) below. However, in order to obtain a clean proof of this formula we cannot use the $*$-notation directly.

Proposition 3.2
Let $(g(t), \phi(t))_{t\in[0,T)}$ be a solution of $(RH)_\alpha$. Then the Riemann tensor satisfies

$$\begin{aligned}\tfrac{\partial}{\partial t}R_{ijk\ell} &= \nabla_i\nabla_k R_{j\ell} - \nabla_i\nabla_\ell R_{jk} - \nabla_j\nabla_k R_{i\ell} + \nabla_j\nabla_\ell R_{ik} - R_{ijq\ell}R_{kq} - R_{ijkq}R_{\ell q}\\ &\quad + 2\alpha\big(\nabla_i\nabla_k\phi\nabla_j\nabla_\ell\phi - \nabla_i\nabla_\ell\phi\nabla_j\nabla_k\phi - \big\langle {}^N\mathrm{Rm}(\nabla_i\phi,\nabla_j\phi)\nabla_k\phi,\nabla_\ell\phi\big\rangle\big).\end{aligned} \tag{3.4}$$

Proof. Remember that the Riemannian $(3,1)$-tensor is given by

$$R_{ij\ell}^p = \big(\partial_i\Gamma_{j\ell}^p + \Gamma_{j\ell}^q\Gamma_{iq}^p\big) - \big(\partial_j\Gamma_{i\ell}^p + \Gamma_{i\ell}^q\Gamma_{jq}^p\big).$$

From (3.1), we obtain

$$\begin{aligned}\tfrac{\partial}{\partial t}R_{ij\ell}^p &= \nabla_i\big(\tfrac{\partial}{\partial t}\Gamma_{j\ell}^p\big) - \nabla_j\big(\tfrac{\partial}{\partial t}\Gamma_{i\ell}^p\big)\\ &= g^{pq}\nabla_i(-\nabla_j R_{\ell q} - \nabla_\ell R_{jq} + \nabla_q R_{j\ell}) - g^{pq}\nabla_j(-\nabla_i R_{\ell q} - \nabla_\ell R_{iq} + \nabla_q R_{i\ell})\\ &\quad + 2\alpha\big(\nabla_i(\nabla_j\nabla_\ell\phi\nabla^p\phi) - \nabla_j(\nabla_i\nabla_\ell\phi\nabla^p\phi)\big)\\ &= g^{pq}(-\nabla_i\nabla_j R_{\ell q} - \nabla_i\nabla_\ell R_{jq} + \nabla_i\nabla_q R_{j\ell} + \nabla_j\nabla_i R_{\ell q} + \nabla_j\nabla_\ell R_{iq} - \nabla_j\nabla_q R_{i\ell})\\ &\quad + 2\alpha(\nabla_i\nabla_j\nabla_\ell\phi\nabla^p\phi + \nabla_j\nabla_\ell\phi\nabla_i\nabla^p\phi - \nabla_j\nabla_i\nabla_\ell\phi\nabla^p\phi - \nabla_i\nabla_\ell\phi\nabla_j\nabla^p\phi).\end{aligned}$$

The first equality is easily justified using normal coordinates. From the two equations $\nabla_i\nabla_j R_{\ell q} - \nabla_j\nabla_i R_{\ell q} = R_{ij\ell r}R_{rq} + R_{ijqr}R_{\ell r}$ and

$$\begin{aligned}&\nabla_i\nabla_j\nabla_\ell\phi\nabla^p\phi - \nabla_j\nabla_i\nabla_\ell\phi\nabla^p\phi\\ &\qquad = R_{ij\ell r}\nabla_r\phi^\kappa\nabla^p\phi^\kappa + R_{ij\kappa\lambda}\nabla_\ell\phi^\lambda\nabla^p\phi^\kappa\\ &\qquad = R_{ij\ell r}\nabla_r\phi\nabla^p\phi + {}^N R_{\mu\nu\kappa\lambda}\nabla_\ell\phi^\lambda\nabla^p\phi^\kappa\nabla_i\phi^\mu\nabla_j\phi^\nu\\ &\qquad = R_{ij\ell r}\nabla_r\phi\nabla^p\phi - \big\langle {}^N\mathrm{Rm}(\nabla_i\phi,\nabla_j\phi)\nabla^p\phi,\nabla_\ell\phi\big\rangle,\end{aligned}$$

which follow from (A.7), (A.10), we find

$$\begin{aligned}\tfrac{\partial}{\partial t}R_{ij\ell}^p &= g^{pq}(\nabla_i\nabla_q R_{j\ell} - \nabla_i\nabla_\ell R_{jq} - \nabla_j\nabla_q R_{i\ell} + \nabla_j\nabla_\ell R_{iq} - R_{ij\ell r}R_{rq} - R_{ijqr}R_{\ell r})\\ &\quad + 2\alpha(\nabla_j\nabla_\ell\phi\nabla_i\nabla^p\phi - \nabla_i\nabla_\ell\phi\nabla_j\nabla^p\phi + R_{ij\ell r}\nabla_r\phi\nabla^p\phi)\\ &\quad - 2\alpha\big\langle {}^N\mathrm{Rm}(\nabla_i\phi,\nabla_j\phi)\nabla^p\phi,\nabla_\ell\phi\big\rangle.\end{aligned}$$

Finally, we lower the upper index to get the evolution equation of the $(4,0)$-Riemannian curvature tensor

$$\begin{aligned}
\tfrac{\partial}{\partial t}R_{ijk\ell} &= \tfrac{\partial}{\partial t}\big(g_{kp}R^p_{ij\ell}\big) = \big(\tfrac{\partial}{\partial t}g_{kp}\big)R^p_{ij\ell} + g_{kp}\big(\tfrac{\partial}{\partial t}R^p_{ij\ell}\big)\\
&= -2R_{ijp\ell}R_{kp} + 2\alpha R_{ijp\ell}\nabla_p\phi\nabla_k\phi\\
&\quad + (\nabla_i\nabla_k R_{j\ell} - \nabla_i\nabla_\ell R_{jk} - \nabla_j\nabla_k R_{i\ell} + \nabla_j\nabla_\ell R_{ik} - R_{ij\ell r}R_{kr} - R_{ijkr}R_{\ell r})\\
&\quad + 2\alpha(\nabla_i\nabla_k\phi\nabla_j\nabla_\ell\phi - \nabla_i\nabla_\ell\phi\nabla_j\nabla_k\phi + R_{ij\ell r}\nabla_r\phi\nabla_k\phi)\\
&\quad - 2\alpha\left\langle {}^N\mathrm{Rm}(\nabla_i\phi,\nabla_j\phi)\nabla_k\phi,\nabla_\ell\phi\right\rangle.
\end{aligned}$$

After relabeling some indices, a few terms cancel and we find (3.4). □

Remark. Taking the trace of (3.4), we get

$$\begin{aligned}
\tfrac{\partial}{\partial t}R_{j\ell} &= \big(\tfrac{\partial}{\partial t}g^{ik}\big)R_{ijk\ell} + g^{ik}\big(\tfrac{\partial}{\partial t}R_{ijk\ell}\big)\\
&= 2R_{ik}R_{ijk\ell} - 2\alpha R_{ijk\ell}\nabla_i\phi\nabla_k\phi\\
&\quad + \triangle R_{j\ell} - \nabla_k\nabla_\ell R_{jk} - \nabla_j\nabla_k R_{k\ell} + \nabla_j\nabla_\ell R - R_{pjk\ell}R_{kp} - R_{jp}R_{\ell p}\\
&\quad + 2\alpha\big(\tau_g\phi\nabla_j\nabla_\ell\phi - \nabla_k\nabla_j\phi\nabla_k\nabla_\ell\phi - \left\langle {}^N\mathrm{Rm}(\nabla_k\phi,\nabla_j\phi)\nabla_k\phi,\nabla_\ell\phi\right\rangle\big),
\end{aligned}$$

from which we obtain (2.10), using the twice traced second Bianchi identity (A.5), which yields

$$\begin{aligned}
-\nabla_k\nabla_\ell R_{jk} &- \nabla_j\nabla_k R_{k\ell} + \nabla_j\nabla_\ell R\\
&= -\nabla_\ell\big(\tfrac{1}{2}\nabla_j R\big) - R_{k\ell jp}R_{kp} - R_{\ell p}R_{jp} - \nabla_j\big(\tfrac{1}{2}\nabla_\ell R\big) + \nabla_j\nabla_\ell R\\
&= -R_{k\ell jp}R_{kp} - R_{\ell p}R_{jp}.
\end{aligned}$$

This gives an alternative proof of Proposition 2.3.

If we set $\alpha = 0$ in (3.4), we obtain the evolution equation for the curvature tensor under the Ricci flow. It is well-known that this evolution equation can be written in a nicer form, in which its parabolic nature is more apparent. In [23, Lemma 7.2], Hamilton proved

$$\begin{aligned}
\nabla_i\nabla_k R_{j\ell} &- \nabla_i\nabla_\ell R_{jk} - \nabla_j\nabla_k R_{i\ell} + \nabla_j\nabla_\ell R_{ik}\\
&= \triangle R_{ijk\ell} + 2(B_{ijk\ell} - B_{ij\ell k} - B_{i\ell jk} + B_{ikj\ell})\\
&\quad - R_{pjk\ell}R_{pi} - R_{ipk\ell}R_{pj},
\end{aligned}\tag{3.5}$$

where $B_{ijk\ell} := R_{ipjq}R_{kp\ell q}$. Plugging this into (3.4) yields the following corollary.

Corollary 3.3
Along the $(RH)_\alpha$ flow, the Riemannian curvature tensor evolves by

$$\begin{aligned}
\tfrac{\partial}{\partial t}R_{ijk\ell} &= \triangle R_{ijk\ell} + 2(B_{ijk\ell} - B_{ij\ell k} - B_{i\ell jk} + B_{ikj\ell})\\
&\quad - (R_{pjk\ell}R_{pi} + R_{ipk\ell}R_{pj} + R_{ijp\ell}R_{pk} + R_{ijkp}R_{p\ell})\\
&\quad + 2\alpha\big(\nabla_i\nabla_k\phi\nabla_j\nabla_\ell\phi - \nabla_i\nabla_\ell\phi\nabla_j\nabla_k\phi - \left\langle {}^N\mathrm{Rm}(\nabla_i\phi,\nabla_j\phi)\nabla_k\phi,\nabla_\ell\phi\right\rangle\big).
\end{aligned}\tag{3.6}$$

The notation introduced in Definition 3.1 allows us to write (3.6) in the short form

$$\tfrac{\partial}{\partial t}\mathrm{Rm} = \triangle \mathrm{Rm} + (\mathrm{Rm})^{*2} + \alpha(\nabla^2\phi)^{*2} + \alpha\, {}^N\mathrm{Rm} * (\nabla\phi)^{*4}. \tag{3.7}$$

It is now easy to compute the evolution of the length of the Riemann tensor. Together with $\frac{\partial}{\partial t} g^{-1} = (\mathrm{Rm})^{*1} + \alpha(\nabla\phi)^{*2}$, the above equation yields

$$\begin{aligned}
\tfrac{\partial}{\partial t}|\mathrm{Rm}|^2 &= \tfrac{\partial}{\partial t}\big(g^{ir}g^{js}g^{kp}g^{\ell q}R_{ijk\ell}R_{rspq}\big)\\
&= \big(\tfrac{\partial}{\partial t}g^{-1}\big) * \mathrm{Rm} * \mathrm{Rm} + 2R_{ijk\ell}\big(\tfrac{\partial}{\partial t}R_{ijk\ell}\big)\\
&= (\mathrm{Rm})^{*3} + \alpha(\mathrm{Rm})^{*2} * (\nabla\phi)^{*2}\\
&\quad + 2R_{ijk\ell}(\triangle R_{ijk\ell}) + \alpha\mathrm{Rm} * (\nabla^2\phi)^{*2} + \alpha\mathrm{Rm} * {}^N\mathrm{Rm} * (\nabla\phi)^{*4}\\
&= \triangle|\mathrm{Rm}|^2 - 2|\nabla\mathrm{Rm}|^2 + (\mathrm{Rm})^{*3} + \alpha(\mathrm{Rm})^{*2} * (\nabla\phi)^{*2}\\
&\quad + \alpha\mathrm{Rm} * (\nabla^2\phi)^{*2} + \alpha\mathrm{Rm} * {}^N\mathrm{Rm} * (\nabla\phi)^{*4}.
\end{aligned} \tag{3.8}$$

We get the following corollary.

Corollary 3.4
Along the $(RH)_\alpha$ flow, the Riemannian curvature tensor satisfies

$$\begin{aligned}
\tfrac{\partial}{\partial t}|\mathrm{Rm}|^2 \leq \triangle|\mathrm{Rm}|^2 - 2|\nabla\mathrm{Rm}|^2 + C|\mathrm{Rm}|^3 + \alpha C|\nabla\phi|^2|\mathrm{Rm}|^2\\
+ \alpha C|\nabla^2\phi|^2|\mathrm{Rm}| + \alpha C c_0|\nabla\phi|^4|\mathrm{Rm}|,
\end{aligned} \tag{3.9}$$

for constants $C \geq 0$ depending only on the dimension of M and $c_0 = c_0(N) \geq 0$ depending only on the curvature of N. If N is flat, we can choose $c_0 = 0$.

Proof. Follows directly from (3.8) and the fact that $|{}^N\mathrm{Rm}|$ is bounded on the compact manifold N. □

For the evolution equation for the Hessian of ϕ, it would again be easiest to work with the notation introduced in Definition 3.1. But again, it is important that we do *not* use it directly. Indeed, we will see that all the terms containing derivatives of the curvature of M cancel each other (using the second Bianchi identity), a phenomenon which cannot be seen when working with the $*$-notation.

We start by computing the commutator $[\nabla_i\nabla_j, \triangle]\phi^\lambda = \nabla_i\nabla_j\tau_g\phi^\lambda - \triangle\nabla_i\nabla_j\phi^\lambda$. Using

(A.7), (A.8) we get

$$\begin{aligned}
\nabla_i\nabla_j\tau_g\phi^\lambda &= \nabla_i(\nabla_k\nabla_j\nabla_k\phi^\lambda + R_{jkkp}\nabla_p\phi^\lambda + R^\lambda_{jk\kappa}\nabla_k\phi^\kappa)\\
&= \nabla_i\nabla_k\nabla_k\nabla_j\phi^\lambda - \nabla_i R_{jp}\nabla_p\phi^\lambda - R_{jp}\nabla_i\nabla_p\phi^\lambda\\
&\quad + \nabla_i R^\lambda_{jk\kappa}\nabla_k\phi^\kappa + R^\lambda_{jk\kappa}\nabla_i\nabla_k\phi^\kappa\\
&= \nabla_k\nabla_i\nabla_k\nabla_j\phi^\lambda + R_{ikkp}\nabla_p\nabla_j\phi^\lambda + R_{ikjp}\nabla_k\nabla_p\phi^\lambda + R^\lambda_{ik\kappa}\nabla_k\nabla_j\phi^\kappa\\
&\quad - \nabla_i R_{jp}\nabla_p\phi^\lambda - R_{jp}\nabla_i\nabla_p\phi^\lambda + \nabla_i R^\lambda_{jk\kappa}\nabla_k\phi^\kappa + R^\lambda_{jk\kappa}\nabla_i\nabla_k\phi^\kappa\\
&= \nabla_k(\nabla_k\nabla_i\nabla_j\phi^\lambda + R_{ikjp}\nabla_p\phi^\lambda + R^\lambda_{ik\kappa}\nabla_j\phi^\kappa)\\
&\quad + R_{ikjp}\nabla_k\nabla_p\phi^\lambda - R_{ip}\nabla_j\nabla_p\phi^\lambda - \nabla_i R_{jp}\nabla_p\phi^\lambda - R_{jp}\nabla_i\nabla_p\phi^\lambda\\
&\quad + \nabla_i R^\lambda_{jk\kappa}\nabla_k\phi^\kappa + R^\lambda_{jk\kappa}\nabla_i\nabla_k\phi^\kappa + R^\lambda_{ik\kappa}\nabla_k\nabla_j\phi^\kappa\\
&= \triangle\nabla_i\nabla_j\phi^\lambda + \nabla_k R_{jpik}\nabla_p\phi^\lambda + 2R_{ikjp}\nabla_k\nabla_p\phi^\lambda - R_{ip}\nabla_j\nabla_p\phi^\lambda\\
&\quad - \nabla_i R_{jp}\nabla_p\phi^\lambda - R_{jp}\nabla_i\nabla_p\phi^\lambda + \nabla_k R^\lambda_{ik\kappa}\nabla_j\phi^\kappa + R^\lambda_{ik\kappa}\nabla_k\nabla_j\phi^\kappa\\
&\quad + \nabla_i R^\lambda_{jk\kappa}\nabla_k\phi^\kappa + R^\lambda_{jk\kappa}\nabla_i\nabla_k\phi^\kappa + R^\lambda_{ik\kappa}\nabla_k\nabla_j\phi^\kappa.
\end{aligned}$$

and hence the commutator satisfies

$$\begin{aligned}
[\nabla_i\nabla_j,\triangle]\phi^\lambda &= \nabla_k R_{jpik}\nabla_p\phi^\lambda + 2R_{ikjp}\nabla_k\nabla_p\phi^\lambda\\
&\quad - R_{ip}\nabla_j\nabla_p\phi^\lambda - \nabla_i R_{jp}\nabla_p\phi^\lambda - R_{jp}\nabla_i\nabla_p\phi^\lambda\\
&\quad + \big({}^N\mathrm{Rm} * \nabla^2\phi * (\nabla\phi)^{*2} + (\partial\,{}^N\mathrm{Rm}) * (\nabla\phi)^{*4}\big)_{ij}.
\end{aligned} \tag{3.10}$$

Since the $\nabla^2\phi$ live in different bundles for different times, we need to work again with the interpretation of $\nabla^2\phi$ as a 2-linear TN-valued map along $\tilde\phi$ and with the covariant time derivative ∇_t, as we already did in Chapter 2. See appendix for details. At the base point of coordinates satisfying (A.13), we find with (A.14) and the remark following it

$$\begin{aligned}
\nabla_t(\nabla_i\nabla_j\phi^\lambda) &= \nabla_i\nabla_j\tfrac{\partial}{\partial t}\phi^\lambda - \big(\tfrac{\partial}{\partial t}\Gamma^k_{ij}\big)\nabla_k\phi^\lambda + {}^N\mathrm{Rm}(\tfrac{\partial}{\partial t}\phi,\nabla_i\phi)\nabla_j\phi^\lambda\\
&= \triangle\nabla_i\nabla_j\phi^\lambda + [\nabla_i\nabla_j,\triangle]\phi^\lambda - \big(\tfrac{\partial}{\partial t}\Gamma^k_{ij}\big)\nabla_k\phi^\lambda + \big({}^N\mathrm{Rm} * \nabla^2\phi * (\nabla\phi)^{*2}\big)_{ij}.
\end{aligned}$$

With (3.1) and (3.10) we continue

$$\begin{aligned}
[\nabla_i\nabla_j,\triangle]\phi^\lambda - \big(\tfrac{\partial}{\partial t}\Gamma^k_{ij}\big)\nabla_k\phi^\lambda &= \nabla_k R_{jpik}\nabla_p\phi^\lambda + 2R_{ikjp}\nabla_k\nabla_p\phi^\lambda - R_{ip}\nabla_j\nabla_p\phi^\lambda\\
&\quad - \nabla_i R_{jp}\nabla_p\phi^\lambda - R_{jp}\nabla_i\nabla_p\phi^\lambda + \nabla_i R_{jk}\nabla_k\phi^\lambda\\
&\quad + \nabla_j R_{ik}\nabla_k\phi^\lambda - \nabla_k R_{ij}\nabla_k\phi^\lambda - 2\alpha\nabla_i\nabla_j\phi\nabla_k\phi\nabla_k\phi^\lambda\\
&\quad + \big({}^N\mathrm{Rm} * \nabla^2\phi * (\nabla\phi)^{*2} + (\partial\,{}^N\mathrm{Rm}) * (\nabla\phi)^{*4}\big)_{ij}\\
&= \big(\mathrm{Rm} * \nabla^2\phi^\lambda\big)_{ij} - 2\alpha\nabla_i\nabla_j\phi\nabla_k\phi\nabla_k\phi^\lambda\\
&\quad + \big({}^N\mathrm{Rm} * \nabla^2\phi * (\nabla\phi)^{*2} + (\partial\,{}^N\mathrm{Rm}) * (\nabla\phi)^{*4}\big)_{ij},
\end{aligned}$$

where we used the second Bianchi identity $(\nabla_k R_{jpik} + \nabla_j R_{ip} - \nabla_p R_{ij})\nabla_p \phi^\lambda = 0$ to cancel all terms containing derivatives of the curvature of (M, g). Thus

$$\begin{aligned}\nabla_t(\nabla_i\nabla_j\phi^\lambda) &= \triangle\nabla_i\nabla_j\phi^\lambda + (\mathrm{Rm} * \nabla^2\phi^\lambda)_{ij} + \alpha\nabla_i\nabla_j\phi * \nabla\phi * \nabla\phi^\lambda \\ &\quad + \big({}^N\mathrm{Rm} * \nabla^2\phi * (\nabla\phi)^{*2} + (\partial\,{}^N\mathrm{Rm}) * (\nabla\phi)^{*4}\big)_{ij}.\end{aligned} \tag{3.11}$$

Furthermore, we have

$$\triangle|\nabla^2\phi|^2 = \triangle(\nabla_i\nabla_j\phi^\lambda\nabla_i\nabla_j\phi^\lambda) = 2\triangle(\nabla_i\nabla_j\phi^\lambda)\nabla_i\nabla_j\phi^\lambda + 2|\nabla^3\phi|^2.$$

Together, we finally obtain

$$\begin{aligned}\tfrac{\partial}{\partial t}|\nabla^2\phi|^2 &= \tfrac{\partial}{\partial t}(g^{ik}g^{j\ell}\nabla_i\nabla_j\phi^\lambda\nabla_k\nabla_\ell\phi^\lambda) \\ &= (\tfrac{\partial}{\partial t}g^{-1}) * (\nabla^2\phi)^{*2} + 2\nabla_t(\nabla_i\nabla_j\phi^\lambda)\nabla_i\nabla_j\phi^\lambda \\ &= \mathrm{Rm} * (\nabla^2\phi)^{*2} + \alpha(\nabla\phi)^{*2} * (\nabla^2\phi)^{*2} + \triangle|\nabla^2\phi|^2 - 2|\nabla^3\phi|^2 \\ &\quad + {}^N\mathrm{Rm} * (\nabla^2\phi)^{*2} * (\nabla\phi)^{*2} + (\partial\,{}^N\mathrm{Rm}) * (\nabla^2\phi) * (\nabla\phi)^{*4}\end{aligned} \tag{3.12}$$

Since $|{}^N\mathrm{Rm}|$ and $|\partial\,{}^N\mathrm{Rm}|$ are bounded on compact manifolds N, say by a constant c_1, this proves the following proposition.

Proposition 3.5
Let $(g(t), \phi(t))_{t\in[0,T)}$ be a solution of $(RH)_\alpha$. Then the norm of the Hessian of ϕ satisfies the estimate

$$\begin{aligned}\tfrac{\partial}{\partial t}|\nabla^2\phi|^2 &\le \triangle|\nabla^2\phi|^2 - 2|\nabla^3\phi|^2 + C|\mathrm{Rm}||\nabla^2\phi|^2 \\ &\quad + \alpha C|\nabla\phi|^2|\nabla^2\phi|^2 + Cc_1|\nabla\phi|^4|\nabla^2\phi| + Cc_1|\nabla\phi|^2|\nabla^2\phi|^2\end{aligned} \tag{3.13}$$

along the flow for some constants $C = C(m) \ge 0$ and $c_1 = c_1(N) \ge 0$ depending on the dimension m of M and the curvature of N, respectively. If N is flat, we may choose $c_1 = 0$.

Remark. If we set $\alpha \equiv 0$, Corollary 3.4 and Proposition 3.5 yield the formulas for the Ricci-DeTurck flow $(RH)_0$, in particular (3.9) reduces to the well-known evolution inequality

$$\tfrac{\partial}{\partial t}|\mathrm{Rm}|^2 \le \triangle|\mathrm{Rm}|^2 - 2|\nabla\mathrm{Rm}|^2 + C|\mathrm{Rm}|^3$$

for the Ricci flow, first derived by Hamilton [23, Corollary 13.3]. Moreover, if $\alpha \equiv 2$ and $N = \mathbb{R}$ (and thus $c_0 = c_1 = 0$), the estimates (3.9) and (3.13) reduce to the estimates found by List (cf. [41, Lemma 2.15 and 2.16]) for his flow (0.6).

Interior-in-time higher order gradient estimates

Using the evolution equations for the curvature tensor and the Hessian of ϕ from the last section, we get evolution equations for higher order derivatives by induction.

Definition 3.6
To keep the notation short, we define for $k \geq 0$

$$\begin{aligned} I_k := \sum_{i+j=k} \nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm} + \alpha \sum_{A_k} (\partial^i\, {}^N\mathrm{Rm} + 1) * \nabla^{j_1}\phi * \ldots * \nabla^{j_\ell}\phi \\ + \alpha \sum_{B_k} \nabla^{j_1}\phi * \ldots * \nabla^{j_{\ell-1}}\phi * \nabla^{j_\ell}\mathrm{Rm}, \end{aligned} \tag{3.14}$$

where the last two sums are taken over all elements of the index sets defined by

$$A_k := \{(i, j_1, \ldots, j_\ell) \mid 0 \leq i \leq k+1,\ 1 \leq j_s \leq k+2\ \forall s \text{ and } j_1 + \ldots + j_\ell = k+4\},$$
$$B_k := \{(j_1, \ldots, j_\ell) \mid 1 \leq j_s < k+2\ \forall s < \ell,\ 0 \leq j_\ell \leq k \text{ and } j_1 + \ldots + j_\ell = k+2\},$$

respectively.

Lemma 3.7
Let $(g(t), \phi(t))_{t\in[0,T)}$ be a solution to the $(RH)_\alpha$ flow. Then for $k \geq 0$, with I_k defined as in (3.14), we obtain

$$\tfrac{\partial}{\partial t}\nabla^k \mathrm{Rm} = \triangle \nabla^k \mathrm{Rm} + I_k. \tag{3.15}$$

Remark. If $(N, \gamma) = (\mathbb{R}, \delta)$, all terms in I_k containing $\partial^i\, {}^N\mathrm{Rm}$ vanish, and the result reduces to (a slightly weaker version of) List's result [41, Lemma 2.19]. Note that we do not need all the elements of A_k here, but defining A_k this way allows us to use the same index set again in Definition 3.8.

Proof. From (3.7), we see that (3.15) holds for $k = 0$. For the induction step, assume that (3.15) holds for some $k \geq 0$ and compute

$$\begin{aligned} \tfrac{\partial}{\partial t}\nabla^{k+1}\mathrm{Rm} &= \tfrac{\partial}{\partial t}(\partial \nabla^k \mathrm{Rm} + \Gamma * \nabla^k \mathrm{Rm}) \\ &= \nabla(\tfrac{\partial}{\partial t}\nabla^k \mathrm{Rm}) + \tfrac{\partial}{\partial t}\Gamma * \nabla^k \mathrm{Rm} \\ &= \nabla(\triangle \nabla^k \mathrm{Rm}) + \nabla I_k + \tfrac{\partial}{\partial t}\Gamma * \nabla^k \mathrm{Rm}. \end{aligned}$$

Since ∇I_k is of the form I_{k+1} and also the last term,

$$\tfrac{\partial}{\partial t}\Gamma * \nabla^k \mathrm{Rm} = \big(\nabla \mathrm{Rm} + \alpha \nabla^2\phi * \nabla\phi\big) * \nabla^k \mathrm{Rm},$$

appears in I_{k+1}, it remains to compute the very first term. With the commutator rule (A.7), we get

$$\begin{aligned} \nabla(\triangle \nabla^k \mathrm{Rm}) &= \triangle \nabla^{k+1}\mathrm{Rm} + \nabla \mathrm{Rm} * \nabla^k \mathrm{Rm} + \mathrm{Rm} * \nabla^{k+1}\mathrm{Rm} \\ &= \triangle \nabla^{k+1}\mathrm{Rm} + I_{k+1}. \end{aligned}$$ □

Similar to (3.8), we obtain

$$\begin{aligned} \tfrac{\partial}{\partial t}|\nabla^k \mathrm{Rm}|^2 &= \big(\tfrac{\partial}{\partial t}g^{-1}\big) * \nabla^k \mathrm{Rm} * \nabla^k \mathrm{Rm} + 2\nabla^k \mathrm{Rm}\big(\tfrac{\partial}{\partial t}\nabla^k \mathrm{Rm}\big) \\ &= \mathrm{Rm} * (\nabla^k \mathrm{Rm})^{*2} + \alpha\, (\nabla\phi)^{*2} * (\nabla^k \mathrm{Rm})^{*2} \\ &\quad + 2\nabla^k \mathrm{Rm}(\triangle \nabla^k \mathrm{Rm}) + \nabla^k \mathrm{Rm} * I_k. \end{aligned}$$

Hence, using the fact that $\mathrm{Rm}*\nabla^k\mathrm{Rm}$ as well as $\alpha\,(\nabla\phi)^{*2}*\nabla^k\mathrm{Rm}$ are already contained in I_k, we find

$$\tfrac{\partial}{\partial t}|\nabla^k\mathrm{Rm}|^2 = \triangle|\nabla^k\mathrm{Rm}|^2 - 2|\nabla^{k+1}\mathrm{Rm}|^2 + \nabla^k\mathrm{Rm} * I_k. \tag{3.16}$$

Definition 3.8
To compute the higher order derivatives of ϕ, we define

$$\begin{aligned} J_k := & \sum_{i+j=k} \nabla^i\mathrm{Rm} * \nabla^{j+2}\phi + \sum_{A_k}(\partial^i\,{}^N\mathrm{Rm} + 1) * \nabla^{j_1}\phi * \ldots * \nabla^{j_\ell}\phi \\ & + \alpha \sum_{B_k} \nabla^{j_1}\phi * \ldots * \nabla^{j_{\ell-1}}\phi * \nabla^{j_\ell+2}\phi, \end{aligned} \tag{3.17}$$

with A_k and B_k defined as in Definition 3.6.

Lemma 3.9
Let $(g(t),\phi(t))_{t\in[0,T)}$ be a solution to the $(RH)_\alpha$ flow. Then for $k\geq 0$, with J_k defined as in (3.17), we have

$$\nabla_t(\nabla^{k+2}\phi) = \triangle\nabla^{k+2}\phi + J_k. \tag{3.18}$$

Proof. For $k=0$, the statement holds by (3.11). For the induction step, we use again the interpretation of $\nabla^k\phi$ as a k-linear TN-valued map along $\tilde{\phi}$ and compute analogously to (A.14) and the remark following it

$$\begin{aligned} \nabla_t(\nabla^{k+3}\phi) &= \nabla\nabla_t(\nabla^{k+2}\phi) + \tfrac{\partial}{\partial t}\Gamma * \nabla^{k+2}\phi + {}^N\mathrm{Rm}(\tfrac{\partial}{\partial t}\phi, \nabla\phi)\nabla^{k+2}\phi \\ &= \nabla(\triangle\nabla^{k+2}\phi) + \nabla J_k + \tfrac{\partial}{\partial t}\Gamma * \nabla^{k+2}\phi + {}^N\mathrm{Rm} * \nabla^2\phi * \nabla\phi * \nabla^{k+2}\phi. \end{aligned}$$

Again, we only have to look at the first term, since ∇J_k, ${}^N\mathrm{Rm}*\nabla^2\phi*\nabla\phi*\nabla^{k+2}\phi$ and

$$\tfrac{\partial}{\partial t}\Gamma * \nabla^{k+2}\phi = \big(\nabla\mathrm{Rm} + \alpha\nabla^2\phi*\nabla\phi\big) * \nabla^{k+2}\phi$$

are obviously of the form J_{k+1}. With a higher order analog to (A.10), we obtain

$$\begin{aligned} \nabla(\triangle\nabla^{k+2}\phi) &= \nabla_p\nabla\nabla_p\nabla^{k+2}\phi + \mathrm{Rm}*\nabla^{k+3}\phi + {}^N\mathrm{Rm}*\nabla^{k+3}\phi*\nabla\phi*\nabla\phi \\ &= \nabla_p(\nabla_p\nabla\nabla^{k+2}\phi + \mathrm{Rm}*\nabla^{k+2}\phi + {}^N\mathrm{Rm}*\nabla^{k+2}\phi*\nabla\phi*\nabla\phi) \\ &\quad + \mathrm{Rm}*\nabla^{k+3}\phi + {}^N\mathrm{Rm}*\nabla^{k+3}\phi*\nabla\phi*\nabla\phi \\ &= \nabla_p\nabla_p\nabla^{k+3}\phi + \nabla\mathrm{Rm}*\nabla^{k+2}\phi + \mathrm{Rm}*\nabla^{k+3}\phi \\ &\quad + (\partial\,{}^N\mathrm{Rm})*\nabla^{k+2}\phi*(\nabla\phi)^{*3} + {}^N\mathrm{Rm}*\nabla^{k+3}\phi*(\nabla\phi)^{*2} \\ &\quad + {}^N\mathrm{Rm}*\nabla^{k+3}\phi*\nabla^2\phi*\nabla\phi \\ &= \triangle\nabla^{k+3}\phi + J_{k+1}, \end{aligned}$$

the claim follows. ☐

As in (3.12), we compute

$$\begin{aligned}\tfrac{\partial}{\partial t}|\nabla^{k+2}\phi|^2 &= \big(\tfrac{\partial}{\partial t}g^{-1}\big) * \nabla^{k+2}\phi * \nabla^{k+2}\phi + 2\nabla^{k+2}\phi^\lambda\nabla_t\big(\nabla^{k+2}\phi^\lambda\big)\\ &= \mathrm{Rm} * (\nabla^{k+2}\phi)^{*2} + \alpha\,(\nabla\phi)^{*2} * (\nabla^{k+2}\phi)^{*2}\\ &\quad + 2\nabla^{k+2}\phi^\lambda(\triangle\nabla^{k+2}\phi^\lambda) + \nabla^{k+2}\phi * J_k\\ &= 2\nabla^{k+2}\phi^\lambda(\triangle\nabla^{k+2}\phi^\lambda) + \nabla^{k+2}\phi * J_k.\end{aligned}$$

For the last step, we used that $\mathrm{Rm} * \nabla^{k+2}\phi$ and $\alpha\,(\nabla\phi)^{*2} * \nabla^{k+2}\phi$ are contained in J_k. With

$$\triangle|\nabla^{k+2}\phi|^2 = 2\nabla^{k+2}\phi^\lambda(\triangle\nabla^{k+2}\phi^\lambda) + 2|\nabla^{k+3}\phi|^2,$$

we finally find

$$\tfrac{\partial}{\partial t}|\nabla^{k+2}\phi|^2 = \triangle|\nabla^{k+2}\phi|^2 - 2|\nabla^{k+3}\phi|^2 + \nabla^{k+2}\phi * J_k. \tag{3.19}$$

The next trick is to combine the two equations (3.16) and (3.19) to a single equation. Remember that we already used a similar trick in Chapter 2, where we combined the evolution equations of Rc and $\nabla\phi\otimes\nabla\phi$ (respectively R and $|\nabla\phi|^2$) to a single equation for a combined quantity S_{ij} (respectively S), which was much more convenient to deal with. Here, we define the "vector"

$$\mathfrak{T} = (\mathrm{Rm}, \nabla^2\phi) \in \Gamma\big((T^*M)^{\otimes 4}\big) \times \Gamma\big((T^*M)^{\otimes 2}\otimes\phi^*TN\big) \tag{3.20}$$

with norm $|\mathfrak{T}|^2 = |\mathrm{Rm}|^2 + |\nabla^2\phi|^2$ and derivatives $\nabla^k\mathfrak{T} = (\nabla^k\mathrm{Rm}, \nabla^{k+2}\phi)$. Combining the evolution equations (3.16) and (3.19), we get

$$\tfrac{\partial}{\partial t}|\nabla^k\mathfrak{T}|^2 = \triangle|\nabla^k\mathfrak{T}|^2 - 2|\nabla^{k+1}\mathfrak{T}|^2 + L_k, \tag{3.21}$$

where $L_k := \nabla^k\mathrm{Rm} * I_k + \nabla^{k+2}\phi * J_k$. We can now apply Bernstein's ideas [4] to obtain interior-in-time estimates for all derivatives $|\nabla^k\mathfrak{T}|^2$ via an induction argument. For the Ricci flow, this was independently done by Bando [3] and Shi [55].

Theorem 3.10
Let $(g(t),\phi(t))_{t\in[0,T)}$ solve $(RH)_\alpha$ with non-increasing $\alpha(t)\in[\underline{\alpha},\bar{\alpha}]$, $0<\underline{\alpha}\le\bar{\alpha}<\infty$ and $T<\infty$. Let the Riemannian curvature tensor of M be uniformly bounded along the flow, $|\mathrm{Rm}|\le R_0$. Then there exists a constant $K=K(\underline{\alpha},\bar{\alpha},R_0,T,m,N)<\infty$ such that the following two estimates hold

$$|\nabla\phi|^2 \le \tfrac{K}{t}, \quad \forall (x,t)\in M\times(0,T), \tag{3.22}$$

$$|\mathfrak{T}|^2 = |\mathrm{Rm}|^2 + |\nabla^2\phi|^2 \le \tfrac{K^2}{t^2}, \quad \forall (x,t)\in M\times(0,T). \tag{3.23}$$

Moreover, there exist constants C_k depending on k, $\bar{\alpha}$, m and N, such that

$$|\nabla^k\mathfrak{T}|^2 = |\nabla^k\mathrm{Rm}|^2 + |\nabla^{k+2}\phi|^2 \le C_k\big(\tfrac{K}{t}\big)^{k+2}, \quad \forall (x,t)\in M\times(0,T). \tag{3.24}$$

Proof. Using Corollary 2.9, we have

$$|\nabla\phi|^2 \leq \frac{m^2 R_0}{\underline{\alpha}} + \frac{m}{2\underline{\alpha} t} \leq \frac{K_1}{t} \quad \text{and} \quad |\mathrm{Rm}| \leq \frac{K_1}{t}, \quad \forall (x,t) \in M \times (0,T),$$

where $1 \leq K_1 := \max\big\{\frac{2m^2 R_0 T + m}{2\underline{\alpha}}, R_0 T, 1\big\} < \infty$. For the following computations, C denotes a constant depending only on K_1, $\bar{\alpha}$, m and the geometry of N, possibly changing from line to line. With the estimates for $|\mathrm{Rm}|$ and $|\nabla\phi|^2$, we obtain from (3.13)

$$\big(\tfrac{\partial}{\partial t} - \triangle\big)|\nabla^2\phi|^2 \leq -2|\nabla^3\phi|^2 + \tfrac{C}{t}|\nabla^2\phi|^2 + \tfrac{C}{t^2}|\nabla^2\phi|,$$

for all $(x,t) \in M \times (0,T)$. Using $|\nabla^2\phi| \leq \frac{1}{t} + t|\nabla^2\phi|^2$ for the last term, we get

$$\big(\tfrac{\partial}{\partial t} - \triangle\big)\big(t^2|\nabla^2\phi|^2\big) \leq -2t^2|\nabla^3\phi|^2 + \tfrac{C}{t}\big(t^2|\nabla^2\phi|^2\big) + \tfrac{C}{t}, \quad \forall (x,t) \in M \times (0,T). \tag{3.25}$$

Similarly, (2.12) yields

$$\big(\tfrac{\partial}{\partial t} - \triangle\big)\big(t|\nabla\phi|^2\big) \leq -2t|\nabla^2\phi|^2 + \tfrac{C}{t}, \quad \forall (x,t) \in M \times (0,T). \tag{3.26}$$

To get the desired estimate for the Hessian of ϕ, we apply (3.25) and (3.26) to the function $f(x,t) := t^2|\nabla^2\phi|^2\big(8K_1 + t|\nabla\phi|^2\big)$. This implies

$$\begin{aligned}
\big(\tfrac{\partial}{\partial t} - \triangle\big)f &= \big(\tfrac{\partial}{\partial t} - \triangle\big)\big(t^2|\nabla^2\phi|^2\big) \cdot \big(8K_1 + t|\nabla\phi|^2\big) + \big(\tfrac{\partial}{\partial t} - \triangle\big)\big(t|\nabla\phi|^2\big) \cdot t^2|\nabla^2\phi|^2 \\
&\quad - 2t^3\nabla|\nabla^2\phi|^2 \cdot \nabla|\nabla\phi|^2 \\
&\leq -2t^2|\nabla^3\phi|^2\big(8K_1 + t|\nabla\phi|^2\big) + \tfrac{C}{t}f + \tfrac{C}{t} \cdot 9K_1 - 2t^3|\nabla^2\phi|^4 + \tfrac{C}{t}f \\
&\quad + 8t^3|\nabla^3\phi||\nabla^2\phi| \cdot |\nabla^2\phi||\nabla\phi|
\end{aligned}$$

on $M \times (0,T)$. The very last term in this estimate can be absorbed by the two negative terms as follows

$$\begin{aligned}
8t^3|\nabla^3\phi||\nabla^2\phi|^2|\nabla\phi| &= (8K_1)^{1/2}2t|\nabla^3\phi| \cdot (8K_1)^{-1/2}4t^2|\nabla^2\phi|^2|\nabla\phi| \\
&\leq \tfrac{1}{2}(8K_1)\big(4t^2|\nabla^3\phi|^2\big) + \tfrac{1}{2}(8K_1)^{-1}\big(16t^4|\nabla^2\phi|^4|\nabla\phi|^2\big) \\
&= 2t^2|\nabla^3\phi|^2 \cdot 8K_1 + \tfrac{8t|\nabla\phi|^2}{8K_1} \cdot t^3|\nabla^2\phi|^4 \\
&\leq 2t^2|\nabla^3\phi|^2\big(8K_1 + t|\nabla\phi|^2\big) + t^3|\nabla^2\phi|^4.
\end{aligned}$$

Here, we used $\frac{8t|\nabla\phi|^2}{8K_1} \leq 1$ which motivates our choice of the constant $8K_1$ in the definition of f. We continue

$$\big(\tfrac{\partial}{\partial t} - \triangle\big)f \leq \tfrac{C}{t}f + \tfrac{C}{t} - t^3|\nabla^2\phi|^4 \leq \tfrac{1}{(9K_1)^2 t}\big(Cf + C - f^2\big) \tag{3.27}$$

for all $(x,t) \in M \times (0,T)$. Let (x_0,t_0) be a point where f first reaches its maximum over $M \times [0, T-\delta]$, $\delta > 0$ arbitrary. Since $f(\cdot,0) = 0$, we have $t_0 > 0$ unless $f \equiv 0$ on $M \times [0, T-\delta]$, in which case we are done. At (x_0,t_0) we have $\frac{\partial}{\partial t}f \geq 0$, $\nabla f = 0$, $\triangle f \leq 0$ and from (3.27) we get $-f^2 + Cf + C \geq 0$, which is only possible for $f \leq$

$\frac{1}{2}\big(C+\sqrt{C^2+4C}\big) =: D$. Since δ was arbitrary, we conclude that $f \leq D$ on $M \times [0,T)$. For positive t, this implies

$$|\nabla^2\phi|^2 = \frac{f}{t^2(8K_1 + t|\nabla\phi|^2)} \leq \frac{D}{8K_1 t^2} \leq \Big(\frac{K_2}{t}\Big)^2, \tag{3.28}$$

where $K_2 := \sqrt{D/8K_1} < \infty$. Setting $K := K_1 + K_2$, we get (3.22) and (3.23).

Using a similar argument, we now prove (3.24) inductively. For $k = 0$ this is simply (3.23) with $C_0 = 1$. Hence we only need to prove the induction step. For the rest of the proof, C denotes a constant depending only on $\bar{\alpha}$, m, N and k (but not on K or T), again possibly changing its value from line to line. We will need (3.21) together with the estimate

$$L_k \leq C\Bigg(\sum_{i+j=k}|\nabla^i\mathfrak{T}||\nabla^j\mathfrak{T}| + \sum_{A'_k}|\nabla^{j_1}\phi|\cdots|\nabla^{j_\ell}\phi| + \sum_{B_k}|\nabla^{j_1}\phi|\cdots|\nabla^{j_{\ell-1}}\phi||\nabla^{j_\ell}\mathfrak{T}|\Bigg)|\nabla^k\mathfrak{T}|,$$

where the index set A'_k is defined by

$$A'_k := \{(j_1,\ldots,j_\ell) \mid 1 \leq j_s \leq k+2 \;\forall s \text{ and } j_1 + \ldots + j_\ell = k+4\}$$

and B_k is the index set defined in Definition 3.6. This estimate follows immediately from Definition 3.6 and Definition 3.8 together with the fact that all $|\partial^i\, {}^N\mathrm{Rm}|$ are bounded since N is compact. Now, assume that (3.24) holds for some fixed $k \geq 0$. Then, we obtain for positive times (using the convention $C_{-1} = C_0 = 1$)

$$\begin{aligned} L_k \leq C\Bigg(&\sum_{i+j=k} C_i\big(\tfrac{K}{t}\big)^{\frac{i+2}{2}} C_j\big(\tfrac{K}{t}\big)^{\frac{j+2}{2}} + \sum_{A'_k} C_{j_1-2}\big(\tfrac{K}{t}\big)^{\frac{j_1}{2}}\cdots C_{j_\ell-2}\big(\tfrac{K}{t}\big)^{\frac{j_\ell}{2}} \\ &+ \sum_{B_k} C_{j_1-2}\big(\tfrac{K}{t}\big)^{\frac{j_1}{2}}\cdots C_{j_{(\ell-1)}-2}\big(\tfrac{K}{t}\big)^{\frac{j_{(\ell-1)}}{2}}\cdot C_{j_\ell}\big(\tfrac{K}{t}\big)^{\frac{j_\ell+2}{2}}\Bigg) C_k\big(\tfrac{K}{t}\big)^{\frac{k+2}{2}} \\ \leq C\Bigg(&\sum_{i+j=k} C_iC_j + \sum_{A'_k} C_{j_1-2}\cdots C_{j_\ell-2} + \sum_{B_k} C_{j_1-2}\cdots C_{j_{(\ell-1)}-2}C_{j_\ell}\Bigg) C_k\big(\tfrac{K}{t}\big)^{k+3} \end{aligned}$$

and hence $t^{k+2}L_k \leq \frac{C}{t}K^{k+3}$. From (3.21) we get for $t > 0$

$$\big(\tfrac{\partial}{\partial t} - \triangle\big)\big(t^{k+2}|\nabla^k\mathfrak{T}|^2\big) \leq -2t^{k+2}|\nabla^{k+1}\mathfrak{T}|^2 + \tfrac{C}{t}K^{k+3} + \tfrac{C}{t}\big(t^{k+2}|\nabla^k\mathfrak{T}|^2\big). \tag{3.29}$$

To obtain an analogous estimate for L_{k+1}, we divide L_{k+1} into the part where $|\nabla^{k+1}\mathfrak{T}|$ appears only linearly and the part which contains quadratic terms in $|\nabla^{k+1}\mathfrak{T}|$. Then, we use

$$|\nabla^{k+1}\mathfrak{T}| \leq \big(\tfrac{K}{t}\big)^{\frac{k+3}{2}} + \big(\tfrac{t}{K}\big)^{\frac{k+3}{2}}|\nabla^{k+1}\mathfrak{T}|^2.$$

We obtain

$$\begin{aligned}
L_{k+1} &\le C|\nabla^{k+1}\mathcal{T}| \sum_{\substack{i+j=k+1\\ i>0,\, j>0}} |\nabla^i\mathcal{T}||\nabla^j\mathcal{T}| + C|\nabla^{k+1}\mathcal{T}| \sum_{\substack{A'_{k+1}\\ j_s\le k+2}} |\nabla^{j_1}\phi|\cdots|\nabla^{j_\ell}\phi| \\
&\quad + C|\nabla^{k+1}\mathcal{T}| \sum_{\substack{B_{k+1}\\ j_\ell\le k}} |\nabla^{j_1}\phi|\cdots|\nabla^{j_{\ell-1}}\phi||\nabla^{j_\ell}\mathcal{T}| + C|\nabla^{k+1}\mathcal{T}|^2\big(|\mathcal{T}|+|\nabla\phi|^2+|\nabla^2\phi|\big) \\
&\le C\big(\tfrac{K}{t}\big)^{\frac{k+5}{2}}|\nabla^{k+1}\mathcal{T}| + C\tfrac{K}{t}|\nabla^{k+1}\mathcal{T}|^2 \\
&\le C\big(\tfrac{K}{t}\big)^{k+4} + C\tfrac{K}{t}|\nabla^{k+1}\mathcal{T}|^2,
\end{aligned}$$

and the evolution equation (3.21) yields for $(x,t)\in M\times(0,T)$

$$\big(\tfrac{\partial}{\partial t}-\triangle\big)\big(t^{k+3}|\nabla^{k+1}\mathcal{T}|^2\big) \le -2t^{k+3}|\nabla^{k+2}\mathcal{T}|^2 + \tfrac{CK}{t}t^{k+3}|\nabla^{k+1}\mathcal{T}|^2 + \tfrac{C}{t}K^{k+4}. \tag{3.30}$$

Similar to the function f above, we now define $h(x,t) := t^{k+3}|\nabla^{k+1}\mathcal{T}|^2\big(\lambda + t^{k+2}|\nabla^k\mathcal{T}|^2\big)$ with $\lambda = 8C_kK^{k+2}$. Applying (3.29) and (3.30), we get

$$\begin{aligned}
\big(\tfrac{\partial}{\partial t}-\triangle\big)h &= \big(\tfrac{\partial}{\partial t}-\triangle\big)\big(t^{k+3}|\nabla^{k+1}\mathcal{T}|^2\big)\cdot\big(\lambda+t^{k+2}|\nabla^k\mathcal{T}|^2\big) \\
&\quad + \big(\tfrac{\partial}{\partial t}-\triangle\big)\big(t^{k+2}|\nabla^k\mathcal{T}|^2\big)\cdot t^{k+3}|\nabla^{k+1}\mathcal{T}|^2 \\
&\quad - 2t^{2k+5}\nabla|\nabla^{k+1}\mathcal{T}|^2\cdot\nabla|\nabla^k\mathcal{T}|^2 \\
&\le -2t^{k+3}|\nabla^{k+2}\mathcal{T}|^2\big(\lambda+t^{k+2}|\nabla^k\mathcal{T}|^2\big) + \tfrac{CK}{t}h + \tfrac{C}{t}K^{k+4}\big(\lambda+t^{k+2}|\nabla^k\mathcal{T}|^2\big) \\
&\quad - 2t^{2k+5}|\nabla^{k+1}\mathcal{T}|^4 + \tfrac{C}{t}h + \tfrac{C}{t}K^{k+3}t^{k+3}|\nabla^{k+1}\mathcal{T}|^2 \\
&\quad + 8t^{2k+5}|\nabla^{k+2}\mathcal{T}||\nabla^{k+1}\mathcal{T}|\cdot|\nabla^{k+1}\mathcal{T}||\nabla^k\mathcal{T}|.
\end{aligned}$$

Since $K\ge 1$, we have $\frac{C}{t}h \le \frac{CK}{t}h$. By the inductive assumption, we can estimate

$$\tfrac{C}{t}K^{k+4}\big(\lambda+t^{k+2}|\nabla^k\mathcal{T}|^2\big) \le \tfrac{C}{t}K^{k+4}\big(9C_kK^{k+2}\big) = \tfrac{C}{t}K^{2k+6}.$$

Moreover, we have

$$\tfrac{C}{t}K^{k+3}t^{k+3}|\nabla^{k+1}\mathcal{T}|^2 \le \tfrac{1}{2}t^{2k+5}|\nabla^{k+1}\mathcal{T}|^4 + \tfrac{C}{t}K^{2k+6}.$$

Plugging these three estimates into the evolution inequality for h, we find

$$\begin{aligned}
\big(\tfrac{\partial}{\partial t}-\triangle\big)h &\le -2t^{k+3}|\nabla^{k+2}\mathcal{T}|^2\big(\lambda+t^{k+2}|\nabla^k\mathcal{T}|^2\big) - \tfrac{3}{2}t^{2k+5}|\nabla^{k+1}\mathcal{T}|^4 \\
&\quad + \tfrac{CK}{t}h + \tfrac{C}{t}K^{2k+6} + 8t^{2k+5}|\nabla^{k+2}\mathcal{T}||\nabla^{k+1}\mathcal{T}|^2|\nabla^k\mathcal{T}|.
\end{aligned}$$

As for the function f above, the very last term can be absorbed by the negative terms

$$\begin{aligned}
8t^{2k+5}|\nabla^{k+2}\mathcal{T}||\nabla^{k+1}\mathcal{T}|^2|\nabla^k\mathcal{T}| &= \lambda^{1/2}2t^{\frac{k+3}{2}}|\nabla^{k+2}\mathcal{T}|\cdot\lambda^{-1/2}4t^{\frac{3k+7}{2}}|\nabla^{k+1}\mathcal{T}|^2|\nabla^k\mathcal{T}| \\
&\le \tfrac{1}{2}\lambda\big(4t^{k+3}|\nabla^{k+2}\mathcal{T}|^2\big) + \tfrac{1}{2}\lambda^{-1}\big(16t^{3k+7}|\nabla^{k+1}\mathcal{T}|^4|\nabla^k\mathcal{T}|^2\big) \\
&= 2t^{k+3}|\nabla^{k+2}\mathcal{T}|^2\cdot\lambda + \tfrac{8t^{k+2}|\nabla^k\mathcal{T}|^2}{\lambda}\cdot t^{2k+5}|\nabla^{k+1}\mathcal{T}|^4 \\
&\le 2t^{k+3}|\nabla^{k+2}\mathcal{T}|^2\big(\lambda+t^{k+2}|\nabla^k\mathcal{T}|^2\big) + t^{2k+5}|\nabla^{k+1}\mathcal{T}|^4,
\end{aligned}$$

which explains our choice of λ. Now, we have

$$\begin{aligned}\left(\tfrac{\partial}{\partial t}-\triangle\right)h &\le \tfrac{CK}{t}h+\tfrac{C}{t}K^{2k+6}-\tfrac{1}{2}t^{2k+5}|\nabla^{k+1}\mathcal{T}|^4\\ &\le \tfrac{1}{2t(9C_kK^{k+2})^2}\Big(CK^{2k+5}h+CK^{4k+10}-h^2\Big).\end{aligned}\tag{3.31}$$

Chose $\delta>0$ small and let (x_0,t_0) be a point where h first reaches its maximum over $M\times[0,T-\delta]$. Again, we have $t_0>0$ unless $h\equiv 0$ on $M\times[0,T-\delta]$ and from $\frac{\partial}{\partial t}h\ge 0$, $\triangle h\le 0$, we obtain

$$-h^2+CK^{2k+5}h+CK^{4k+10}\ge 0$$

at (t_0,x_0). This is only possible if $h\le\frac{1}{2}(C+\sqrt{C^2+4C})K^{2k+5}=:DK^{2k+5}$. Since δ was arbitrary, we conclude that $h\le DK^{2k+5}$ on $M\times[0,T)$ and thus for $t>0$

$$|\nabla^{k+1}\mathcal{T}|^2=\frac{h}{t^{k+3}(\lambda+t^{k+2}|\nabla^k\mathcal{T}|^2)}\le\frac{DK^{2k+5}}{t^{k+3}8C_kK^{k+2}}=C_{k+1}\Big(\frac{K}{t}\Big)^{k+3},$$

where $C_{k+1}:=D/(8C_k)$. This proves the induction step and hence the theorem. □

Long-time existence for $(RH)_\alpha$

This section follows Section 6.7 about long-time existence for the Ricci flow from Chow and Knopf's book [9]. We first need a technical lemma.

Lemma 3.11
Let $(g(t),\phi(t))_{t\in[0,T)}$ solve $(RH)_\alpha$ with a non-increasing $\alpha(t)\in[\underline{\alpha},\bar{\alpha}]$, $0<\underline{\alpha}\le\bar{\alpha}<\infty$ and $T<\infty$. Let the Riemannian curvature tensor of M be uniformly bounded along the flow, $|\mathrm{Rm}|\le R_0$, and fix a background metric $\tilde{g}$. Then for each $k\ge 0$ there exists a constant C_k depending on k, m, N, T, $\underline{\alpha}$, $\bar{\alpha}$, R_0 and the initial data $(g(0),\phi(0))$ such that

$$|\tilde{\nabla}^k g(x,t)|^2_{\tilde{g}}+|\tilde{\nabla}^k\,\mathrm{Rm}(x,t)|^2_{\tilde{g}}+|\tilde{\nabla}^k\phi(x,t)|^2_{\tilde{g}}\le C_k\tag{3.32}$$

for all $(x,t)\in M\times[0,T)$. Here, $\tilde{\nabla}={}^{\tilde{g}}\nabla$ denotes the Levi-Cività connection with respect to the background metric $\tilde{g}$.

Proof. With Theorem 3.10, the proof becomes a straight forward computation, and we therefore only give a sketch. In [9, Section 6.7], all the details are carried out in the case of the Ricci flow and they can easily be adopted to our flow. Since M is closed, there exists a finite atlas for which we have uniform bounds on the derivatives of the local charts. Working is such a chart $\psi:U\to\mathbb{R}^m$, it suffices to derive the desired estimates for the Euclidean metric δ in U and the ordinary derivatives, since $\tilde{g}$ and $\tilde{\nabla}$ are fixed. In particular, we can interpret Γ as a tensor, namely $\Gamma=\Gamma-{}^{\delta}\Gamma$. On the compact interval $[0,T/2]$, all the derivatives $|\nabla^k\phi|^2_g$ and $|\nabla^k\mathrm{Rm}|^2_g$ are uniformly bounded. On the interval $[T/2,T)$, Theorem 3.10 above yields uniform bounds for these derivatives. Hence

$$|\nabla^k\phi|^2_g+|\nabla^k\mathrm{Rm}|^2_g\le\bar{C}_k\tag{3.33}$$

for some $\bar{C}_k < \infty$. In particular, $\mathcal{S} = \mathrm{Rc} - \alpha \nabla\phi \otimes \nabla\phi$ is uniformly bounded on $[0,T)$. From [9, Lemma 6.49], we infer that all $g(t)$ are uniformly equivalent on $[0,T)$, and thus for some constant C

$$C^{-1}\delta \le g(x,t) \le C\delta, \quad \forall (x,t) \in U \times [0,T). \tag{3.34}$$

With $\frac{\partial}{\partial t}(\partial g) = \partial(\frac{\partial}{\partial t} g) = -2\partial \mathcal{S} = -2(\nabla \mathcal{S} + \Gamma * \mathcal{S})$, we compute

$$|\tfrac{\partial}{\partial t}\partial g|_\delta \le C|\tfrac{\partial}{\partial t}\partial g| \le C|\nabla \mathcal{S}| + C|\Gamma||\mathcal{S}|. \tag{3.35}$$

From (3.2), we have

$$|\tfrac{\partial}{\partial t}\Gamma| \le C|\nabla \mathrm{Rc}| + 2\bar{\alpha}|\nabla\phi||\nabla^2\phi|$$

which yields a bound for $|\Gamma|$ by integration. Together with the bounds for $|\mathcal{S}|$ and $|\nabla\mathcal{S}|$ that we obtain from (3.33), we conclude that $|\frac{\partial}{\partial t}\partial g|_\delta$ is uniformly bounded, and hence – again by integration – $|\partial g|_\delta$ is uniformly bounded on $U \times [0,T)$. Finally, using a partition of unity for the chosen atlas, we obtain a uniform bound for $|\tilde{\nabla} g|_{\tilde{g}}$ on $M \times [0,T)$. A short computation as for (3.35) yields

$$|\tfrac{\partial}{\partial t}\tilde{\nabla}^k g|_{\tilde{g}} \le C|\tfrac{\partial}{\partial t}\tilde{\nabla}^k g| \le \sum_{i=0}^{k} c_i|\Gamma|^i|\nabla^{k-i}\mathcal{S}| + \sum_{i=1}^{k-1} c_i'|\partial^i\Gamma||\tilde{\nabla}^{k-1-i}\mathcal{S}|, \tag{3.36}$$

where the constants c_i, c_i' only depend on m and k. From this formula, we inductively obtain the desired bounds for $|\tilde{\nabla}^k g|_{\tilde{g}}$. Similarly, the estimates for $|\tilde{\nabla}^k \mathrm{Rm}(x,t)|_{\tilde{g}}$ and $|\tilde{\nabla}^k\phi(x,t)|_{\tilde{g}}$ are obtained from (3.33) with a transformation analogous to (3.36). The lemma then follows by plugging everything together. □

Finally, we obtain our desired criterion for long-time existence.

Theorem 3.12
Let $(g(t),\phi(t))_{t\in[0,T)}$ solve $(RH)_\alpha$ with non-increasing $\alpha(t) \in [\underline{\alpha}, \bar{\alpha}]$, $0 < \underline{\alpha} \le \bar{\alpha} < \infty$ and $T < \infty$. Suppose that $T < \infty$ is maximally chosen, i.e. the solution cannot be extended beyond T in a smooth way. Then the curvature of $(M, g(t))$ has to become unbounded for $t \nearrow T$ in the sense that

$$\limsup_{t\nearrow T}\Big(\max_{x\in M}|\mathrm{Rm}(x,t)|^2\Big) = \infty. \tag{3.37}$$

Proof. The proof is by contradiction. Suppose that the curvature stays bounded on $[0,T)$, say $|\mathrm{Rm}| \le R_0$. For any point $x \in M$ and vector $X \in T_xM$, define $g(x,T)(X,X) := \lim_{t\to T} g(x,t)(X,X)$. We estimate

$$\begin{aligned}|g(x,T)(X,X) - g(x,t)(X,X)| &\le \int_t^T |\tfrac{\partial}{\partial t} g(x,\tau)(X,X)|d\tau \\ &\le \int_t^T 2|\mathcal{S}(x,\tau)(X,X)|d\tau \le C|X|^2(T-t),\end{aligned}$$

where we used again the fact that $\mathcal{S}$ is uniformly bounded on $M \times [0,T)$. This shows that the limit $g(x,T)(X,X)$ is well defined and continuous in x. Hence, we obtain a continuous limit $g(\cdot,T) \in \Gamma(\mathrm{Sym}^2(T^*M))$ by polarization. From [9, Lemma 6.49], all metrics $g(\cdot,t)$ are uniformly equivalent – as in (3.34) – which implies that this limit must be a (continuous) Riemannian metric. Moreover, we define $\phi(x,T) := \lim_{t\to T}\phi(x,t)$, where we use again the embedding $e_N : N \hookrightarrow \mathbb{R}^d$ to interpret ϕ as a map into $\mathbb{R}^d$. We estimate

$$|\phi(x,T) - \phi(x,t)| \leq \int_t^T |\tfrac{\partial}{\partial t}\phi(x,\tau)|d\tau \leq C(T-t),$$

since the bound on $|\nabla^2\phi|$ yields a bound on $|\frac{\partial}{\partial t}\phi| = |\tau_g\phi|$. This implies that $\phi(\cdot,T)$ is well defined and continuous in x. The uniform bounds (3.32) from Lemma 3.11 then also hold for the limit $(g(T),\phi(T))$ and hence $g(T)$ and $\phi(T)$ are smooth. Indeed, for an arbitrary background metric $\tilde{g}$, we have

$$|\tilde{\nabla}^k g(T) - \tilde{\nabla}^k g(t)|_{\tilde{g}} \leq \int_t^T |\tfrac{\partial}{\partial t}\tilde{\nabla}^k g(\tau)|_{\tilde{g}} d\tau \leq C(T-t),$$

which follows from the uniform bound for $|\frac{\partial}{\partial t}\tilde{\nabla} g|_{\tilde{g}}$ that we have derived in the proof above. To wit, the convergence $g(t) \to g(T)$ is smooth. With

$$|\tilde{\nabla}^k \phi(T) - \tilde{\nabla}^k \phi(t)|_{\tilde{g}} \leq \int_t^T |\tfrac{\partial}{\partial t}\tilde{\nabla}^k \phi(\tau)|_{\tilde{g}} d\tau \leq C(T-t)$$

we see that also $\phi(t) \to \phi(T)$ uniformly in any C^k-norm. Finally, restarting the flow with $(g(T),\phi(T))$ as new initial data, we obtain a solution $(g(t),\phi(t))_{t\in[T,T+\varepsilon)}$ by the short-time existence result from Chapter 2. This yields an extension of our solution beyond time T which is smooth in space for each time. From the flow equations and the uniform bounds on $|\nabla^k \mathrm{Rm}|$ as well as $|\nabla^k\phi|$, the time derivatives (and hence also the mixed derivatives) are smooth too, in particular near $t = T$. To wit, the extension of the flow is smooth in space *end* time, contradicting the maximality of T. □

CHAPTER 4

Examples and special solutions

In this chapter, we only consider time-independent coupling constants $\alpha(t) \equiv \alpha$. In the first section, we study two very simple homogeneous examples for the $(RH)_\alpha$ flow system. These examples will illustrate the different behavior of the flow for different coupling constants α, in particular the existence or non-existence of singularities will depend on the choice of α. In the second section, we introduce gradient solitons and derive equations which are satisfied by the potential of these solitons. The next section is devoted to the study of the volume-preserving version of the flow. We say that $(g(t), \phi(t))$ is a solution of the normalized $(RH)_\alpha$ flow, if it satisfies

$$\tfrac{\partial}{\partial t} g = -2\mathcal{S} + \tfrac{2}{m}\, g \fint_M S\; dV, \qquad \tfrac{\partial}{\partial t}\phi = \tau_g \phi. \tag{4.1}$$

Here, $\fint_M S\; dV := \int_M S\; dV / \int_M dV$ denotes the average of $S = R - \alpha|\nabla\phi|^2$. Finally, the last section deals with the case where ϕ is a co-rotational map into a sphere. This is the only example where a finite time blow-up was explicitly proved for the harmonic map heat flow in dimension 2.

Two homogeneous examples with $\phi = \mathrm{id}$

First, assume that $(M, g(0))$ is a round two-sphere of constant scalar curvature 2. Under the Ricci flow, the sphere shrinks to a point in finite time. Let us now consider the $(RH)_\alpha$ flow, assuming that $(N, \gamma) = (M, g(0))$ and $\phi(0)$ is the identity map. We make the ansatz $g(t) = c(t)g(0)$, $c(0) = 1$. Since $\phi(0)$ is harmonic for all $g(t)$, the map $\phi(t) = \phi(0)$ will not change. Hence, the $(RH)_\alpha$ flow reduces to

$$\tfrac{\partial}{\partial t} c(t) = -2 + 2\alpha.$$

For $\alpha < 1$, $c(t)$ goes to zero in finite time. This means that $(M, g(t))$ shrinks to a point, while the scalar curvature R and the energy density $|\nabla\phi|^2$ both go to infinity. For $\alpha = 1$, the solution is stationary. For $\alpha > 1$, $c(t)$ grows linearly and the flow exists forever, while both the scalar curvature R and the energy density $|\nabla\phi|^2$ vanish asymptotically. Instead of changing α, we can also scale the metric γ on the target manifold. Mapping into a larger sphere has the same consequences as choosing a larger α. On the other hand, scaling the initial metric $g(0)$ on M does not have any consequences. This example shows that for large α, the coupled flow system might have a completely different singularity behavior than the Ricci flow. However, in this example the volume-preserving version (4.1) of the flow is always stationary, as it is for the normalized Ricci flow, too.

A more interesting example is obtained if we let $(M^4, g(t)) = (S^2 \times L, c(t)g_{S^2} \oplus d(t)g_L)$, where (S^2, g_{S^2}) is again a round sphere with scalar curvature 2 and (L, g_L) is a surface (of genus ≥ 2) with constant scalar curvature -2. Under the Ricci flow, $\frac{\partial}{\partial t}c(t) = -2$ and $\frac{\partial}{\partial t}d(t) = +2$. In particular, $c(t)$ goes to zero in finite time while $d(t)$ always expands. Under the normalized Ricci flow $\frac{\partial}{\partial t}g_{ij} = -2R_{ij} + \frac{1}{2}(\fint R\, dV)g_{ij}$, we have

$$\tfrac{\partial}{\partial t}c = -2 + \tfrac{d-c}{d} = -1 - c^2, \qquad \tfrac{\partial}{\partial t}d = +2 + \tfrac{d-c}{c} = +1 + d^2.$$

Again, $c(t)$ goes to zero in finite time. At the same time, $d(t)$ goes to infinity.

Now, let us consider the $(RH)_\alpha$ flow for this example, setting $(N, \gamma) = (M, g(0))$ and $\phi(0) = \mathrm{id}$. Making the same ansatz for $g(t)$ as in the standard Ricci flow case above, we first note that $\phi(0)$ is always harmonic and thus $\phi(t) = \phi(0)$ is unchanged. Note that the identity map between the same manifold with two different metrics is *not* harmonic in general, but in this case it is. This can be seen by choosing normal coordinates for γ around some point: In these coordinates both the Christoffel symbols for the metric γ and for the metrics $g(t)$ vanish. The flow equations reduce to $\frac{\partial}{\partial t}c(t) = -2 + 2\alpha$ and $\frac{\partial}{\partial t}d(t) = +2 + 2\alpha$. While $d(t)$ always grows, the behavior of $c(t)$ is exactly the same as in the first example above, where we only had a two-sphere. On the other hand, if we consider the normalized flow $\frac{\partial}{\partial t}g_{ij} = -2S_{ij} + \frac{1}{2}(\fint S\, dV)g_{ij}$, we obtain with a short computation

$$\tfrac{\partial}{\partial t}c = (\alpha - 1) - (\alpha + 1)c^2, \qquad \tfrac{\partial}{\partial t}d = (\alpha + 1) - (\alpha - 1)d^2.$$

In the case where $\alpha < 1$, $c(t)$ goes to zero in finite time, while $d(t)$ blows up at the same time, similar to the normalized Ricci flow above. For $\alpha = 1$, $c(t) = (1 + 2t)^{-1}$ goes to zero in infinite time while $d(t) = (1 + 2t)$ grows linearly, i.e. we have long-time existence but no natural convergence (to a manifold with the same topology). In the third case, where $\alpha > 1$, both $c(t)$ and $d(t)$ converge with

$$c(t) \to \sqrt{\tfrac{\alpha-1}{\alpha+1}}, \qquad d(t) \to \sqrt{\tfrac{\alpha+1}{\alpha-1}}, \qquad \text{as } t \to \infty.$$

This example shows that also the normalized version of our flow can behave very differently from the normalized Ricci flow, if α is chosen large.

The normalized (i.e. volume-preserving) version of the flow will be studied in more detail in one of the sections below.

Gradient solitons

A solution to $(RH)_\alpha$ which changes under a one-parameter family of diffeomorphisms on M and scaling is called a soliton (or a self-similar solution). These solutions correspond to fixed points modulo diffeomorphisms and scaling. The more general class of periodic solutions modulo diffeomorphisms and scaling, the so-called breathers, will be defined (but also ruled out) in the next Chapter.

Definition 4.1
A solution $(g(t),\phi(t))_{t\in[0,T)}$ of $(RH)_\alpha$ is called a soliton *if there exists a one-parameter family of diffeomorphisms $\psi_t : M \to M$, satisfying $\psi_0 = \mathrm{id}_M$, and a scaling function $c : [0,T) \to \mathbb{R}_+$ such that*

$$\begin{cases} g(t) = c(t)\psi_t^* g(0), \\ \phi(t) = \psi_t^* \phi(0). \end{cases} \tag{4.2}$$

The cases $\frac{\partial}{\partial t}c = \dot{c} < 0$, $\dot{c} = 0$ and $\dot{c} > 0$ correspond to shrinking, steady *and* expanding *solitons, respectively. If the diffeomorphisms ψ_t are generated by a (possibly time-dependent) vector field $X(t)$ that is the gradient of some function $f(t)$ on M, then the soliton is called* gradient *soliton and f is called the* potential *of the soliton.*

Lemma 4.2
Let $(g(t),\phi(t))_{t\in[0,T)}$ be a gradient soliton with potential f. Then for any $t_0 \in [0,T)$, the solution satisfies the coupled elliptic system

$$\begin{cases} 0 = \mathrm{Rc} - \alpha\nabla\phi\otimes\nabla\phi + \mathrm{Hess}(f) + \sigma g, \\ 0 = \tau_g\phi - \langle\nabla\phi,\nabla f\rangle, \end{cases} \tag{4.3}$$

for some constant $\sigma(t_0)$. Conversely, if we are given a function f on M and a solution of (4.3) at $t = 0$, there exist one-parameter families of scalars $c(t)$ and diffeomorphisms $\psi_t : M \to M$ such that defining $(g(t),\phi(t))$ as in (4.2) yields a solution of $(RH)_\alpha$. Moreover, $c(t)$ can be chosen linear in t.

Proof. Suppose we have a soliton solution to $(RH)_\alpha$. Without loss of generality, assume $c(0) = 1$ and $\psi_0 = \mathrm{id}_M$. Thus, the solution satisfies

$$\begin{aligned} -2\,\mathrm{Rc}(g(0)) + 2\alpha(\nabla\phi\otimes\nabla\phi)(0) &= \tfrac{\partial}{\partial t}g(t)|_{t=0} = \tfrac{\partial}{\partial t}(c(t)\psi_t^* g(0))|_{t=0} \\ &= \dot{c}(0)g(0) + \mathcal{L}_{X(0)}g(0) = \dot{c}(0)g(0) + 2\,\mathrm{Hess}(f(\cdot,0)), \end{aligned}$$

where $X(t)$ is the family of vector fields generating ψ_t. Moreover, we compute

$$(\tau_g\phi)(0) = \tfrac{\partial}{\partial t}\phi(t)|_{t=0} = \mathcal{L}_{X(0)}\phi(0) = \langle\nabla\phi,\nabla f\rangle.$$

Together, this proves (4.3) with $\sigma = \frac{1}{2}\dot{c}(0)$ for $t_0 = 0$. Hence, by a time-shifting argument, (4.3) must hold for any $t_0 \in [0,T)$. On can easily see that $\sigma(t_0) = \dot{c}(t_0)/2c(t_0)$.

Conversely, let $(g(0),\phi(0))$ solve (4.3) for some function f on M. Define $c(t) := 1+2\sigma t$ and $X(t) := \nabla f/c(t)$. Let ψ_t be the diffeomorphisms generated by the family of vector fields $X(t)$ (with $\psi_0 = \mathrm{id}_M$) and define $(g(t),\phi(t))$ as in (4.2). For $\sigma < 0$ this is possible on the time interval $t \in [0, \frac{-1}{2\sigma})$, in the case $\sigma \geq 0$ it is possible for $t \in [0,\infty)$. Then

$$\begin{aligned}\tfrac{\partial}{\partial t} g(t) &= \dot c(t)\psi_t^*(g(0)) + c(t)\psi_t^*(\mathcal{L}_{X(t)}g(0)) = \psi_t^*(2\sigma g(0) + \mathcal{L}_{\nabla f} g(0))\\ &= \psi_t^*(2\sigma g(0) + 2\,\mathrm{Hess}(f)) = \psi_t^*\big(-2\,\mathrm{Rc}(g(0)) + 2\alpha(\nabla\phi\otimes\nabla\phi)(0)\big)\\ &= -2\,\mathrm{Rc}(g(t)) + 2\alpha(\nabla\phi\otimes\nabla\phi)(t),\end{aligned}$$

as well as

$$\begin{aligned}\tfrac{\partial}{\partial t}\phi(t) &= \psi_t^*(\mathcal{L}_{X(t)}\phi(0)) = \psi_t^*\langle\nabla\phi(0),\nabla f/c(t)\rangle = c(t)^{-1}\psi_t^*(\tau_{g(0)}\phi(0))\\ &= c(t)^{-1}\tau_{\psi_t^* g(0)}\phi(t) = \tau_{g(t)}\phi(t).\end{aligned}$$

This means that $(g(t),\phi(t))$ is a solution of $(RH)_\alpha$ and thus a soliton solution. □

By Lemma 4.2 and rescaling, we may assume that $c(t) = T - t$ for shrinking solitons (here T is the maximal time of existence for the flow), $c(t) = 1$ for steady solitons or $c(t) = t - T$ for expanding solitons (where T defines a *birth* time). An example for a soliton solution is the first example from the last section, where $(M, g(0)) = (N,\gamma) = (S^2, g_{S^2})$ and $\phi(0) = \mathrm{id}$. For $\alpha < 1$, the soliton is shrinking, for $\alpha = 1$ steady and for $\alpha > 1$ expanding. Since $\phi_t = \mathrm{id}_M$ for all t in all three cases, these are gradient solitons with potential $f = 0$.

Let us derive more equations for gradient solitons! Taking the trace of the first equation in (4.3), we see that a soliton must satisfy

$$R - \alpha|\nabla\phi|^2 + \triangle f + \sigma m = 0. \tag{4.4}$$

Taking a covariant derivative of the first equation in (4.3), we obtain

$$\nabla_k R_{ij} - \alpha\nabla_k(\nabla_i\phi\nabla_j\phi) + \nabla_k\nabla_i\nabla_j f = 0.$$

Subtracting the same equation with indices i and k interchanged, we get

$$\nabla_k R_{ij} - \nabla_i R_{kj} - \alpha(\nabla_i\phi\nabla_k\nabla_j\phi - \nabla_k\phi\nabla_i\nabla_j\phi) + R_{kijp}\nabla_p f = 0.$$

We trace this with g^{kj} which yields

$$\nabla_j R_{ij} - \nabla_i R - \alpha\big(\nabla_i\phi\tau_g\phi - \tfrac12\nabla_i|\nabla\phi|^2\big) + R_{ip}\nabla_p f = 0.$$

Finally, we use the twice traced second Bianchi identity $\nabla_j R_{ij} = \frac12\nabla_i R$ from (A.5) and plug in both equations from (4.3) for R_{ip} and for $\tau_g\phi$ to obtain

$$-\tfrac12\nabla_i\big(R - \alpha|\nabla\phi|^2 + |\nabla f|^2 + 2\sigma f\big) = 0.$$

This means that

$$R - \alpha|\nabla\phi|^2 + |\nabla f|^2 + 2\sigma f = const. \tag{4.5}$$

Finally, with $f(\cdot,t)=\psi_t^*(f(\cdot,0))$, we get

$$\tfrac{\partial}{\partial t}f=\mathcal{L}_X f=|\nabla f|^2. \tag{4.6}$$

Combining this with (4.4), we obtain the evolution equation

$$(\tfrac{\partial}{\partial t}+\triangle)f=|\nabla f|^2-R+\alpha|\nabla\phi|^2-\sigma m. \tag{4.7}$$

For steady solitons, this means that the formulas (4.3)–(4.7) hold with $\sigma=0$ and (4.7) is equivalent to $u=e^{-f}$ solving the adjoint heat equation

$$\Box^* u=-\tfrac{\partial}{\partial t}u-\triangle u+Ru-\alpha|\nabla\phi|^2u=0. \tag{4.8}$$

For shrinking solitons, (4.3)–(4.7) hold with $\sigma(t)=-\frac{1}{2}(T-t)^{-1}$ and (4.7) is equivalent to $u=(4\pi(T-t))^{-m/2}e^{-f}$ solving the adjoint heat equation. Finally, for expanding solitons, $\sigma(t)=+\frac{1}{2}(t-T)^{-1}$ and (4.7) is equivalent to the fact that the function $u=(4\pi(t-T))^{-m/2}e^{-f}$ solves the adjoint heat equation (4.8).

Volume-preserving version of $(RH)_\alpha$

Here, we show that the unnormalized $(RH)_\alpha$ flow and the normalized version are related by rescaling the metric g and the time, while keeping the map ϕ unchanged. Indeed, assume that $(g(t),\phi(t))_{t\in[0,T)}$ is a solution of $(RH)_\alpha$. Define a family of rescaling factors $\lambda(t)$ by

$$\lambda(t):=\Big(\int_M dV_{g(t)}\Big)^{-2/m},\quad t\in[0,T), \tag{4.9}$$

and let $\bar g(t)$ be the family of rescaled metrics $\bar g(t)=\lambda(t)g(t)$ having constant unit volume

$$\int_M dV_{\bar g(t)}=\int_M \lambda^{m/2}(t)\,dV_{g(t)}=1,\quad \forall t\in[0,T).$$

It is an immediate consequence of the scaling behavior of Rc, R, $\nabla\phi\otimes\nabla\phi$ and $|\nabla\phi|^2$ that $\bar{\mathcal{S}}=\mathcal{S}$ and $\bar S=\lambda^{-1}S$ for $\bar g=\lambda g$. Note that $\lambda(t)$ is a smooth function of time, with

$$\frac{d}{dt}\lambda(t)=-\tfrac{2}{m}\Big(\int_M dV_g\Big)^{-\frac{2+m}{m}}\int_M(-S)dV_g=\tfrac{2}{m}\,\lambda\,⨍_M S\,dV_g=\tfrac{2}{m}\,\lambda^2⨍_M \bar S\,dV_{\bar g}.$$

Now, we rescale the time. Put $\bar t(t):=\int_0^t\lambda(s)ds$, so that $\frac{d\bar t}{dt}=\lambda(t)$. Then, we obtain

$$\begin{aligned}&\tfrac{\partial}{\partial\bar t}\bar g=\lambda^{-1}\tfrac{\partial}{\partial t}(\lambda g)=\tfrac{\partial}{\partial t}g+\big(\lambda^{-2}\tfrac{\partial}{\partial t}\lambda\big)\bar g=-2\bar{\mathcal{S}}+\tfrac{2}{m}\,\bar g⨍_M\bar S\,dV_{\bar g},\\&\tfrac{\partial}{\partial\bar t}\phi=\lambda^{-1}\tfrac{\partial}{\partial t}\phi=\lambda^{-1}\tau_g\phi=\tau_{\bar g}\phi.\end{aligned}$$

This means that $(\bar g(\bar t),\phi(\bar t))$ solves the volume-preserving $(RH)_\alpha$ flow (4.1) on $[0,\bar T)$, where $\bar T=\int_0^T\lambda(s)ds$.

Co-rotational maps $\phi : D^2 \to S^2$

In this section, we discuss the flow for the case where $M = D^2$ is a two-dimensional disc of radius R and the target manifold $N = S^2$ is the round unit two-sphere embedded in $\mathbb{R}^3$. Moreover, we assume that the initial metric g_0 on D^2 is rotationally symmetric and the initial map ϕ_0 is a co-rotational map, i.e. in polar coordinates (r, ϑ) we can write them as

$$g_0(r) = a_0^2(r)dr^2 + b_0^2(r)d\vartheta^2,$$

where $a_0, b_0 : [0, R] \to \mathbb{R}_+$ are functions with $a_0(0) > 0$, $b_0(0) = 0$ and

$$\phi_0(x) = \big(\tfrac{x}{|x|}\sin h_0(r), \cos h_0(r)\big) = \big(\cos\vartheta \sin h_0(r), \sin\vartheta \sin h_0(r), \cos h_0(r)\big),$$

where $h_0 : [0, R] \to \mathbb{R}_+$ is a function with $h_0(0) = 0$. Furthermore, we fix the values of g and ϕ on the boundary ∂M of the domain manifold M. Note that so far we always worked with closed M, but the results (in particular short-time existence and uniqueness) hold for this setting, too. By uniqueness, the rotational symmetry of g and the equivariance of ϕ are preserved and a solution $(g(t), \phi(t))_{t\in[0,T)}$ can be written in the form

$$\begin{cases} g(r,t) = a^2(r,t)dr^2 + b^2(r,t)d\vartheta^2 \\ \phi(x,t) = \big(\tfrac{x}{|x|}\sin h(r,t), \cos h(r,t)\big) \end{cases} \tag{4.10}$$

with initial and boundary conditions

$$\begin{array}{llll} a(r,0) = a_0(r), & b(r,0) = b_0(r), & h(r,0) = h_0(r), & \forall\, 0 \le r \le R, \\ a(R,t) = a_0(R), & b(R,t) = b_0(R), & h(R,t) = h_0(R) =: d, & \forall\, 0 \le t < T, \\ a(0,t) > 0, & b(0,t) = 0, & h(0,t) = 0, & \forall\, 0 \le t < T. \end{array}$$

In the following, we denote $\frac{\partial}{\partial r}a$ by a_r etc. A short computation shows that for $r \in (0, R)$ the Christoffel symbols of g are given by

$$\Gamma^r_{rr} = \frac{a_r}{a}, \quad \Gamma^r_{\vartheta\vartheta} = -\frac{bb_r}{a^2}, \quad \Gamma^\vartheta_{r\vartheta} = \Gamma^\vartheta_{\vartheta r} = \frac{b_r}{b},$$

the other four Christoffel symbols vanish. Thus, the Ricci tensor is given by

$$R_{rr} = \frac{1}{b}\Big(\frac{a_r b_r}{a} - b_{rr}\Big), \quad R_{\vartheta\vartheta} = \frac{b}{a^2}\Big(\frac{a_r b_r}{a} - b_{rr}\Big), \quad R_{\vartheta r} = R_{r\vartheta} = 0,$$

the scalar curvature is

$$R = \frac{2}{a^2 b}\Big(\frac{a_r b_r}{a} - b_{rr}\Big). \tag{4.11}$$

Furthermore, we have

$$\nabla_r\phi\nabla_r\phi = h_r^2, \quad \nabla_\vartheta\phi\nabla_\vartheta\phi = \sin^2 h, \quad \nabla_r\phi\nabla_\vartheta\phi = 0, \tag{4.12}$$

from which we obtain

$$|\nabla\phi|_g^2 = \frac{1}{a^2}h_r^2 + \frac{1}{b^2}\sin^2 h. \tag{4.13}$$

The Laplace-Beltrami operator applied to ϕ is given by

$$\triangle_g \phi = \frac{1}{a^2}\Big(\phi_{rr} - \frac{a_r}{a}\phi_r\Big) + \frac{1}{b^2}\Big(\phi_{\vartheta\vartheta} + \frac{bb_r}{a^2}\phi_r\Big).$$

Thus, the harmonic map flow equation $\frac{\partial}{\partial t}\phi - \triangle_g \phi = |\nabla\phi|_g^2 \phi$ with respect to the evolving metric is equivalent to the equation

$$h_t = \frac{1}{a^2}h_{rr} + \Big(\frac{b_r}{a^2 b} - \frac{a_r}{a^3}\Big)h_r - \frac{\sin(2h)}{2b^2}, \tag{4.14}$$

while the flow equation $\frac{\partial}{\partial t}g = -2\mathrm{Rc} + 2\alpha\nabla\phi\otimes\nabla\phi$ for the metric on D^2 is equivalent to the system

$$\begin{cases} \dfrac{a_t}{a} = \dfrac{1}{a^2}\Big(-R_{rr} + \alpha h_r^2\Big) = -\dfrac{R}{2} + \alpha\Big(\dfrac{1}{a^2}h_r^2\Big) \\ \dfrac{b_t}{b} = \dfrac{1}{b^2}\Big(-R_{\vartheta\vartheta} + \alpha\sin^2 h\Big) = -\dfrac{R}{2} + \alpha\Big(\dfrac{1}{b^2}\sin^2 h\Big). \end{cases} \tag{4.15}$$

Remark. Let us verify these formulas for an example which we have already studied: Let $R = \pi$ and let $g_0(r) = dr^2 + \sin^2 r\, d\vartheta^2$, (i.e. $a_0(r) \equiv 1$ and $b_0(r) = \sin r$) such that (D^2, g) has the geometry of a round sphere. Moreover, let ϕ_0 be the identity map between these two spheres, i.e. $h_0(r) = r$. We make the ansatz $g(t,r) = c(t)g_0(r)$, which means that $a(r,t) = c(t)^{1/2}a_0(r)$ and $b(r,t) = c(t)^{1/2}b_0(r)$. Equation (4.14) yields $h_t \equiv 0$, i.e. $h(r,t) = h_0(r) = r$. With $R = 2/c$, which follows from (4.11), the evolution equation (4.15) reduces to

$$\frac{a_t}{a} = \frac{b_t}{b} = \frac{c_t}{2c} = -\frac{1}{c} + \frac{\alpha}{c},$$

which justifies the ansatz. To wit, we found again $\frac{\partial}{\partial t}c(t) = -2 + 2\alpha$, as in the first homogeneous example at the beginning of this chapter.

The more interesting example which we want to study in this section is the example where (D^2, g) is a *flat* disc with radius $R = 1$ at time $t = 0$. This means that $a_0(r) = 1$, $b_0(r) = r$. For $\alpha = 0$, the first equation of the $(RH)_\alpha$ flow system is simply the Ricci flow. But since $\mathrm{Rc} \equiv 0$, the metric g is unchanged and the second equation of the $(RH)_\alpha$ flow system is simply the standard harmonic map heat flow between the fixed manifolds $M = D^2$ and $N = S^2$. Equation (4.14) simplifies to

$$h_t = h_{rr} + \frac{1}{r}h_r - \frac{\sin(2h)}{2r^2}. \tag{4.16}$$

We have the following results, due to Chang, Ding and Ye.

Proposition 4.3
Let $h(r,t)$ be a solution of (4.16) with initial data $h_0(r)$ satisfying $h_0(0) = 0$ and $h_0(1) = d$ as above. Then

i) if $|h_0| \leq \pi$, the solution exists for all time.

ii) if $|d| > \pi$, the solution blows up in finite time.

Proof. The first statement is proved by Chang and Ding in [7], the second by Chang, Ding and Ye in [8]. A short proof of both statements can be found in Struwe's survey [59]. □

For the $(RH)_\alpha$ flow with a positive coupling constant $\alpha(t) \equiv \alpha > 0$ one has to investigate the cases of small α and large α separately. We conjecture that in the case of small α the behavior of the flow is similar to the case $\alpha = 0$ above, in particular in the smooth case where $|h_0| \leq \pi$. For the case $|d| > \pi$, we suspect to find a singularity. However, we also know from Corollary 2.8 that if $|\nabla\phi|$ blows up, then g has to become singular as well. On the other hand, the behavior is completely different for large α, since Proposition 2.10 rules out concentration of ϕ a-priori, and we suspect to obtain a long-time existence result independent of the initial data h_0.

Proving these conjectures is work in progress.

CHAPTER 5

Monotonicity formula and no breathers theorem

The entropy functional $\mathcal{W}_\alpha$ introduced in this chapter is the analogue of Perelman's shrinker entropy for the Ricci flow from [48, Section 3]. It is obtained from the energy functional $\mathcal{F}_\alpha$ from (1.1), (2.15) by introducing a positive scale factor τ (later interpreted as a backwards time) and some correction terms. For detailed explanations of Perelman's result, we again refer to Chow et al. [10, Chapter 6] and Müller [45, Chapter 3]. Moreover for the special case (0.6) of the $(RH)_\alpha$ flow, the entropy functional $\mathcal{W}_\alpha$ can be found in List's dissertation [41].

The entropy functional and its first variation

Let again $g = g_{ij} \in \Gamma\big(\operatorname{Sym}^2_+(T^*M)\big)$, $\phi \in C^\infty(M,N)$, $f : M \to \mathbb{R}$ and $\tau > 0$. For a time-independent coupling constant $\alpha(t) \equiv \alpha > 0$, we set

$$\mathcal{W}_\alpha(g,\phi,f,\tau) := \int_M \Big(\tau\big(R_g + |\nabla f|_g^2 - \alpha|\nabla\phi|_g^2\big) + f - m\Big)(4\pi\tau)^{-m/2}e^{-f}dV_g. \tag{5.1}$$

As in Chapter 1, we take variations

$$\begin{aligned}
g_{ij}^\varepsilon &= g_{ij} + \varepsilon h_{ij}, && h_{ij} \in \Gamma\big(\operatorname{Sym}^2(T^*M)\big),\\
f^\varepsilon &= f + \varepsilon\ell, && \ell \in C^\infty(M),\\
\phi^\varepsilon &= \pi_N(\phi + \varepsilon\vartheta), && \vartheta \in C^\infty(M,\mathbb{R}^n) \text{ with } \vartheta(x) \in T_{\phi(x)}N,
\end{aligned}$$

such that $\delta g = h$, $\delta f = \ell$ and $\delta\phi = \vartheta$. Here, π_N denotes again the nearest-neighbour projection on a tubular neighbourhood of N as in Chapter 1. Additionally, set $\tau^\varepsilon = \tau + \varepsilon\sigma$ for some $\sigma \in \mathbb{R}$, i.e. $\delta\tau = \sigma$. The variation

$$\delta\mathcal{W}_\alpha := \delta\mathcal{W}_{\alpha,g,\phi,f,\tau}(h,\vartheta,\ell,\sigma) := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{W}_\alpha(g+\varepsilon h, \pi_N(\phi+\varepsilon\vartheta), f+\varepsilon\ell, \tau+\varepsilon\sigma)$$

is easiest computed using the variation of $\mathcal{F}_\alpha$ and

$$\mathcal{W}_\alpha(g,\phi,f,\tau) = (4\pi\tau)^{-m/2}\Big(\tau\,\mathcal{F}_\alpha(g,\phi,f) + \int_M (f-m)e^{-f}dV\Big). \tag{5.2}$$

Using

$$\delta \int_M (f-m)e^{-f}dV = \int_M \Big(\ell + (f-m)\big(\tfrac{1}{2}\operatorname{tr}_g h - \ell\big)\Big)e^{-f}dV$$

and equation (1.2) for the variation of $\mathcal{F}_\alpha(g,\phi,f)$, we get from (5.2)

$$\begin{aligned}\delta\mathcal{W}_\alpha = &\int_M -\tau h_{ij}\big(R_{ij} + \nabla_i\nabla_j f - \alpha\nabla_i\phi\nabla_j\phi\big)d\mu \\ &+ \int_M \tau\big(\tfrac{1}{2}\operatorname{tr}_g h - \ell\big)\big(2\triangle f - |\nabla f|^2 + R - \alpha|\nabla\phi|^2 + \tfrac{f-m}{\tau}\big)d\mu \\ &+ \int_M \big(\ell + \sigma(1-\tfrac{m}{2})(R + |\nabla f|^2 - \alpha|\nabla\phi|^2) - \tfrac{m\sigma}{2\tau}(f-m)\big)d\mu \\ &+ \int_M 2\tau\alpha\vartheta\big(\tau_g\phi - \langle\nabla\phi,\nabla f\rangle\big)d\mu,\end{aligned}$$

where we used the abbreviation $d\mu := (4\pi\tau)^{-m/2}e^{-f}dV$. Rearranging the terms and using $\int_M(\triangle f - |\nabla f|^2)d\mu = -(4\pi\tau)^{-m/2}\int_M \triangle(e^{-f})dV = 0$, we get

$$\begin{aligned}\delta\mathcal{W}_\alpha = &\int_M (-\tau h_{ij} + \sigma g_{ij})\big(R_{ij} + \nabla_i\nabla_j f - \alpha\nabla_i\phi\nabla_j\phi\big)d\mu \\ &+ \int_M \tau\big(\tfrac{1}{2}\operatorname{tr}_g h - \ell - \tfrac{m\sigma}{2\tau}\big)\big(2\triangle f - |\nabla f|^2 + R - \alpha|\nabla\phi|^2 + \tfrac{f-m}{\tau}\big)d\mu \\ &+ \int_M \ell\, d\mu + \int_M 2\tau\alpha\vartheta\big(\tau_g\phi - \langle\nabla\phi,\nabla f\rangle\big)d\mu.\end{aligned}$$

Finally, writing

$$\ell = (-\tau h_{ij} + \sigma g_{ij})\big(\tfrac{-1}{2\tau}g_{ij}\big) + \tau\big(\tfrac{1}{2}\operatorname{tr}_g h - \ell - \tfrac{m\sigma}{2\tau}\big)\big(\tfrac{-1}{\tau}\big),$$

we obtain

$$\begin{aligned}\delta\mathcal{W}_\alpha = &\int_M (-\tau h_{ij} + \sigma g_{ij})\big(R_{ij} + \nabla_i\nabla_j f - \alpha\nabla_i\phi\nabla_j\phi - \tfrac{1}{2\tau}g_{ij}\big)d\mu \\ &+ \int_M \tau\big(\tfrac{1}{2}\operatorname{tr}_g h - \ell - \tfrac{m\sigma}{2\tau}\big)\big(2\triangle f - |\nabla f|^2 + R - \alpha|\nabla\phi|^2 + \tfrac{f-m-1}{\tau}\big)d\mu \\ &+ \int_M 2\tau\alpha\vartheta\big(\tau_g\phi - \langle\nabla\phi,\nabla f\rangle\big)d\mu.\end{aligned} \tag{5.3}$$

Gradient flow for fixed background measure

Similar to Chapter 1, we now fix the measure $d\mu = (4\pi\tau)^{-m/2}e^{-f}dV$. To wit, we have $f = -\log((4\pi\tau)^{m/2}\frac{d\mu}{dV})$ and from $0 = \delta d\mu = (\frac{1}{2}\operatorname{tr}_g h - \ell - \frac{m\sigma}{2\tau})d\mu$, we deduce $\ell = \frac{1}{2}\operatorname{tr}_g h - \frac{m\sigma}{2\tau}$. Moreover, we require the variation of τ to satisfy $\delta\tau = \sigma = -1$. This allows us to interpret τ as backwards time later. We then write

$$\mathcal{W}^\mu_\alpha(g,\phi,\tau) := \mathcal{W}_\alpha\big(g,\phi,-\log\big((4\pi\tau)^{m/2}\tfrac{d\mu}{dV}\big),\tau\big) \tag{5.4}$$

and

$$\delta\mathcal{W}^{\mu}_{\alpha,g,\phi,\tau}(h,\vartheta) := \delta\mathcal{W}_{\alpha,g,\phi,-\log((4\pi\tau)^{m/2}\frac{d\mu}{dV}),\tau}\left(h,\vartheta,\tfrac{1}{2}\operatorname{tr}_g h + \tfrac{m}{2\tau},-1\right).$$

The variation formula (5.3) reduces to

$$\begin{aligned}\delta\mathcal{W}^{\mu}_{\alpha,g,\phi,\tau}(h,\vartheta) = &\int_M (-\tau h_{ij} - g_{ij})\big(R_{ij} + \nabla_i\nabla_j f - \alpha\nabla_i\phi\nabla_j\phi - \tfrac{1}{2\tau}g_{ij}\big)d\mu \\ &+ \int_M 2\tau\alpha\vartheta\big(\tau_g\phi - \langle\nabla\phi,\nabla f\rangle\big)d\mu,\end{aligned} \tag{5.5}$$

which is monotone under the gradient-like system of evolution equations given by

$$\begin{cases}\frac{\partial}{\partial t}g_{ij} = -2(R_{ij} + \nabla_i\nabla_j f - \alpha\nabla_i\phi\nabla_j\phi),\\ \frac{\partial}{\partial t}\phi = \tau_g\phi - \langle\nabla\phi,\nabla f\rangle,\\ \frac{\partial}{\partial t}f = -R - \triangle f + \alpha|\nabla\phi|^2 + \frac{m}{2\tau},\\ \frac{\partial}{\partial t}\tau = -1.\end{cases} \tag{5.6}$$

As in Chapter 1, pulling back the solutions of (5.6) with the family of diffeomorphisms generated by ∇f, we get a solution of

$$\begin{cases}\frac{\partial}{\partial t}g = -2\mathrm{Rc} + 2\alpha\nabla\phi\otimes\nabla\phi,\\ \frac{\partial}{\partial t}\phi = \tau_g\phi,\\ \frac{\partial}{\partial t}\tau = -1,\\ \quad 0 = \Box^*((4\pi\tau)^{-m/2}e^{-f}).\end{cases} \tag{5.7}$$

Since $\mathcal{W}_\alpha$ is diffeomorphism invariant, we find the analogue to Proposition 1.1.

Proposition 5.1
Let $(g(t),\phi(t))_{t\in[0,T)}$ be a solution of $(RH)_\alpha$ with $\alpha(t)\equiv\alpha>0$, τ a backwards time with $\frac{\partial}{\partial t}\tau = -1$ and $(4\pi\tau)^{-m/2}e^{-f}$ a solution of the adjoint heat equation under the flow. Then the entropy functional $\mathcal{W}_\alpha(g,\phi,f,\tau)$ is non-decreasing with

$$\begin{aligned}\frac{d}{dt}\mathcal{W}_\alpha = &\int_M 2\tau\left|\mathrm{Rc} - \alpha\nabla\phi\otimes\nabla\phi + \mathrm{Hess}(f) - \tfrac{g}{2\tau}\right|^2(4\pi\tau)^{-m/2}e^{-f}dV \\ &+ \int_M 2\tau\alpha\left|\tau_g\phi - \langle\nabla\phi,\nabla f\rangle\right|^2(4\pi\tau)^{-m/2}e^{-f}dV.\end{aligned} \tag{5.8}$$

Remark. As seen in Corollary 1.2 for the energy $\mathcal{F}_\alpha$, the monotonicity of the entropy $\mathcal{W}_\alpha$ also holds true for non-increasing families of positive coupling constants $\alpha(t)$ instead of a single positive α.

Minimizing over all probability measures

Similar to $\lambda_\alpha(g,\phi)$ defined in (1.13), we set

$$\begin{aligned}\mu_\alpha(g,\phi,\tau) &:= \inf\left\{\mathcal{W}_\alpha^\mu(g,\phi,\tau) \mid \mu(M)=1\right\}\\ &= \inf\left\{\mathcal{W}_\alpha(g,\phi,f,\tau) \;\middle|\; \int_M (4\pi\tau)^{-m/2}e^{-f}dV = 1\right\}.\end{aligned} \tag{5.9}$$

Our goal is again to show that the infimum is always achieved. Note that for $\tilde{g}=\tau g$ we have $R_{\tilde{g}} = \frac{1}{\tau}R_g$, $|\nabla f|^2_{\tilde{g}} = \frac{1}{\tau}|\nabla f|^2_g$, $|\nabla\phi|^2_{\tilde{g}} = \frac{1}{\tau}|\nabla\phi|^2_g$, $dV_{\tilde{g}} = \tau^{m/2}dV_g$ and thus

$$\mu_\alpha(\tau g,\phi,\tau) = \mu_\alpha(g,\phi,1).$$

We can hence reduce the problem to the special case where $\tau=1$. Set $v=(4\pi)^{-m/4}e^{-f/2}$. This yields

$$\begin{aligned}\mathcal{W}_\alpha(g,\phi,v,1) &= \int_M \Big(Rv^2 + 4|\nabla v|^2 - \alpha|\nabla\phi|^2v^2 - \big(2\log v + \tfrac{m}{2}\log(4\pi) + m\big)v^2\Big)dV\\ &= \int_M v\Big(Rv - 4\triangle v - \alpha|\nabla\phi|^2 v - 2v\log v - \tfrac{mv}{2}\log(4\pi) - mv\Big)dV,\end{aligned}$$

and hence

$$\mu_\alpha(g,\phi,1) = \inf\left\{\mathcal{W}_\alpha(g,\phi,v,1) \;\middle|\; \int_M v^2 dV = 1\right\}$$

is the smallest eigenvalue of the operator $L(v) = -4\triangle v + (R - \alpha|\nabla\phi|^2 - \frac{m}{2}\log(4\pi) - m)v - 2v\log v$ and v is a corresponding normalized eigenvector. As in Chapter 1, a unique smooth positive normalized eigenvector v_{min} exists (cf. Rothaus [53] or List [41]) and we get the following.

Proposition 5.2
Let $(g(t),\phi(t))$ solve $(RH)_\alpha$ for a constant $\alpha>0$ and let $\frac{\partial}{\partial t}\tau = -1$. Then $\mu_\alpha(g,\phi,\tau)$ is monotone non-decreasing in time. Moreover, it is constant if and only if

$$\begin{cases} 0 = \mathrm{Rc} - \alpha\nabla\phi\otimes\nabla\phi + \mathrm{Hess}(f) - \frac{g}{2\tau},\\ 0 = \tau_g\phi - \langle\nabla\phi,\nabla f\rangle.\end{cases} \tag{5.10}$$

for the minimizer f (that corresponds to v_{min}). As always, the monotonicity result stays true if we allow $\alpha(t)$ to be a positive non-increasing function instead of a constant.

Proof. The proof is completely analogous to the proof of Proposition 1.3, using the monotonicity of $\mathcal{W}_\alpha$ instead on the monotonicity of $\mathcal{F}_\alpha$. □

Non-existence of nontrivial breathers

Breathers correspond to generalised periodic solutions modulo diffeomorphisms and scaling, similar to the notion of solitons defined in Definition 4.1. A-priori, this seems like a more general concept than the one of solitons. The goal of this section is to prove that it is not.

Definition 5.3
A solution $(g(t),\phi(t))_{t\in[0,T)}$ *of* $(RH)_\alpha$ *is called a* breather *if there exists* $t_1,t_2\in[0,T)$, $t_1<t_2$, *a diffeomorphism* $\psi: M\to M$ *and a constant* $c\in\mathbb{R}_+$ *such that*

$$\begin{cases} g(t_2)=c\,\psi^* g(t_1), \\ \phi(t_2)=\psi^*\phi(t_1). \end{cases} \tag{5.11}$$

The cases $c<1$, $c=1$ *and* $c>1$ *correspond to* shrinking, steady *and* expanding *breathers.*

Theorem 5.4
Let M *and* N *be closed and let* $(g(t),\phi(t))_{t\in[0,T)}$ *be a solution of* $(RH)_\alpha$ *with* $\alpha(t)\equiv\alpha$.

i) If this solution is a steady breather, then it necessarily is a steady gradient soliton solution. Moreover, $\phi(t)$ *is harmonic and* $\mathrm{Rc}=\alpha\nabla\phi\otimes\nabla\phi$, *i.e. the solution is stationary.*

ii) If the solution is an expanding breather, then it necessarily is an expanding gradient soliton. Again $\phi(t)$ *must be harmonic (and thus stationary,* $\phi(t)=\phi(0)$*), while* $g(t)$ *changes only by scaling.*

iii) If the solution is a shrinking breather, then it has to be a shrinking gradient soliton.

If we assume in addition that $\dim M=2$ *or that* $(M,g(0))$ *is Einstein, then in the first two cases above,* $\phi(t)$ *is not only harmonic but also conformal, hence a minimal branched immersion, provided that it is non-constant.*

Proof. This is an application of the monotonicity results for $\lambda_\alpha(g,\phi)$ from Proposition 1.3 and for $\mu_\alpha(g,\phi,\tau)$ from Proposition 5.2. Since the proof is very similar to the Ricci flow case solved by Perelman in [48], we closely follow the notes from Kleiner and Lott [36] on Perelman's paper.

i) Assume $(g(t),\phi(t))_{t\in[0,T)}$ is a steady breather. Then there exist two times t_1, t_2 such that (5.11) holds with $c=1$. From diffeomorphism invariance of $\lambda_\alpha(g,\phi)$ defined in (1.13), we obtain $\lambda_\alpha(g,\phi)(t_1)=\lambda_\alpha(g,\phi)(t_2)$. From Proposition 1.3, we get condition (1.14) on $[t_1,t_2]$, which means that $(g(t),\phi(t))$ must be a gradient steady soliton according to Lemma 4.2 and uniqueness of solutions. Moreover, the minimizer $f=-2\log v_{min}$ which realizes $\lambda_\alpha(g,\phi)$ is the soliton potential. From $(-4\triangle+R-\alpha|\nabla\phi|^2)v_{min}=\lambda_\alpha(g,\phi)v_{min}=:\lambda_\alpha v_{min}$, we obtain

$$2\triangle f-|\nabla f|^2+R-\alpha|\nabla\phi|^2=\lambda_\alpha. \tag{5.12}$$

Since $(g(t),\phi(t))$ is a steady soliton, (4.4) holds with $\sigma = 0$. Plugging this into (5.12) yields $\triangle f - |\nabla f|^2 = \lambda_\alpha$, and we obtain from $\int_M e^{-f} dV = 1$

$$\lambda_\alpha = \int_M \lambda_\alpha e^{-f} dV = \int_M (\triangle f - |\nabla f|^2) e^{-f} dV = -\int_M \triangle(e^{-f}) dV = 0,$$

i.e. $\triangle f = |\nabla f|^2$. Another integration yields

$$\int_M |\nabla f|^2 dV = \int_M \triangle f \; dV = 0,$$

and thus $\nabla f \equiv 0$, $\mathrm{Hess}(f) \equiv 0$ on $M \times [0,T)$ and (1.14) becomes

$$\begin{cases} 0 = \mathrm{Rc} - \alpha \nabla\phi \otimes \nabla\phi, \\ 0 = \tau_g \phi. \end{cases}$$

In particular, $\phi(t)$ is harmonic and $(g(t),\phi(t))$ is stationary.

ii) The proof here is analogous to the case of steady breathers, but we first need to construct a scaling invariant version of $\lambda_\alpha(g,\phi)$. We define

$$\bar\lambda_\alpha(g,\phi) := \lambda_\alpha(g,\phi)\Big(\int_M dV_g\Big)^{2/m}. \tag{5.13}$$

This quantity is invariant under rescaling $\tilde g = cg$. A proof of this fact is given in the appendix.

Claim 1: At times where $\bar\lambda_\alpha(t) := \bar\lambda_\alpha(g,\phi)(t) \le 0$, we have $\frac{\partial}{\partial t}\bar\lambda_\alpha(t) \ge 0$.

Proof. We compute, using the abbreviations $d\mu := e^{-f} dV$ and $V(t) := \int_M dV_{g(t)}$ as well as the fact $\frac{\partial}{\partial t} V(t) = -\int_M S \, dV_{g(t)}$

$$\begin{aligned} \tfrac{\partial}{\partial t}\bar\lambda_\alpha(t) &= (\tfrac{\partial}{\partial t}\lambda_\alpha(t)) V(t)^{2/m} - \tfrac{2}{m}\lambda_\alpha(t) V(t)^{(2-m)/m} \int_M S \, dV \\ &\ge V(t)^{2/m} \int_M \Big(2|\mathcal{S} + \mathrm{Hess}(f)|^2 + 2\alpha |\tau_g\phi - \langle \nabla\phi, \nabla f\rangle|^2 \Big) d\mu \\ &\quad - \tfrac{2}{m} V(t)^{2/m} \lambda_\alpha(t) V(t)^{-1} \int_M S \, dV_{g(t)}, \end{aligned} \tag{5.14}$$

where f is the minimizer of $\mathcal{F}_\alpha$ that realizes $\lambda_\alpha(t)$. Note that $\bar f = \log V(t)$ satisfies $\int_M e^{-\bar f} dV = 1$ and is thus an admissible test function in the definition of λ_α. Hence,

$$\lambda_\alpha(g,\phi) \le \mathcal{F}_\alpha(g,\phi,\log V(t)) = \int_M S \, e^{-\log V(t)} dV = V(t)^{-1} \int_M S \, dV \tag{5.15}$$

and since $\lambda_\alpha(t) \leq 0$ by assumption, $-V(t)^{-1}\lambda_\alpha(t)\int_M S\,dV \geq -\lambda_\alpha(t)^2$. Plugging this into (5.14), we find

$$\tfrac{\partial}{\partial t}\bar{\lambda}_\alpha(t) \geq V(t)^{2/m}\Big(\int_M \Big(2|\mathcal{S}+\mathrm{Hess}(f)|^2 + 2\alpha|\tau_g\phi - \langle\nabla\phi,\nabla f\rangle|^2\Big)d\mu - \tfrac{2}{m}\lambda_\alpha(t)^2\Big).$$

where $f = -2\log v_{min}$ is again the minimizer realizing $\lambda_\alpha(g,\phi)$. This can be rewritten using

$$|\mathcal{S}+\mathrm{Hess}(f)|^2 = |\mathcal{S}+\mathrm{Hess}(f) - \tfrac{1}{m}(S+\triangle f)g|^2 + \tfrac{1}{m}(S+\triangle f)^2$$

as well as

$$\lambda_\alpha(t) = \int_M \big(S + |\nabla f|^2\big)d\mu = \int_M (S+\triangle f)d\mu, \tag{5.16}$$

which gives

$$\begin{aligned}\tfrac{\partial}{\partial t}\bar{\lambda}_\alpha \geq{}& 2V^{2/m}\int_M \Big(|\mathcal{S}+\mathrm{Hess}(f) - \tfrac{1}{m}(S+\triangle f)g|^2 + \alpha|\tau_g\phi - \langle\nabla\phi,\nabla f\rangle|^2\Big)d\mu\\ &+ \tfrac{2}{m}V^{2/m}\Big(\int_M (S+\triangle f)^2 d\mu - \Big(\int_M (S+\triangle f)d\mu\Big)^2\Big).\end{aligned} \tag{5.17}$$

The second line on the right hand side is nonnegative by Hölder's inequality, which proves Claim 1. □

Now, assume that $(g(t),\phi(t))$ is an expanding breather. Since $\bar{\lambda}_\alpha(g,\phi)$ is invariant under diffeomorphisms and scaling, we have $\bar{\lambda}_\alpha(t_1) = \bar{\lambda}_\alpha(t_2)$ for the two times t_1, t_2 that satisfy (5.11). Since $V(t_1) < V(t_2)$, there must be a time $t_0 \in [t_1,t_2]$ with $\frac{\partial}{\partial t}V(t_0) > 0$ and hence with (5.15)

$$\lambda_\alpha(t_0) \leq V(t_0)^{-1}\int_M S\,dV = -V(t_0)^{-1}\tfrac{\partial}{\partial t}V(t_0) < 0.$$

Claim 1 applies and we obtain $\bar{\lambda}_\alpha(t_1) \leq \bar{\lambda}_\alpha(t_0) < 0$ and since $\bar{\lambda}_\alpha(t_2) = \bar{\lambda}_\alpha(t_1)$, we see that $\bar{\lambda}_\alpha(t)$ must be a negative constant. Hence, both lines on the right hand side of (5.17) have to vanish. This means that $(S+\triangle f)$ has to be constant in space for all t and because of (5.16) this constant has to be $\lambda_\alpha(t)$. From the first line of (5.17) we obtain

$$\begin{cases} 0 = \mathrm{Rc} - \alpha\nabla\phi\otimes\nabla\phi + \mathrm{Hess}(f) - \frac{\lambda_\alpha}{m}g,\\ 0 = \tau_g\phi - \langle\nabla\phi,\nabla f\rangle.\end{cases} \tag{5.18}$$

By Lemma 4.2, $(g(t),\phi(t))_{t\in[0,T)}$ is an expanding soliton with potential $f = -2\log v_{min}$. This means that we can use (5.12), which implies

$$0 = 2\triangle f - |\nabla f| + S - \lambda_\alpha = 2\triangle f - |\nabla f| + S - (\triangle f + S) = \triangle f - |\nabla f| \tag{5.19}$$

and thus by integration $\nabla f \equiv 0$, $\mathrm{Hess}(f) \equiv 0$. Plugging this into (5.18), the second equation tells us that $\phi(t)$ is harmonic and the first equation yields

$$\tfrac{\partial}{\partial t} g = -2\mathrm{Rc} + 2\alpha \nabla\phi \otimes \nabla\phi = -2\tfrac{\lambda_\alpha}{m} g,$$

i.e. $(M, g(t))$ simply expands exponentially without changing its shape.

iii) If $(g(t), \phi(t))$ is a shrinking breather, there exist t_1, t_2 and $c < 1$ which such that (5.11) is satisfied. We define

$$\tau_0 := \frac{t_2 - c\,t_1}{1-c} > t_2, \qquad \text{and} \qquad \tau(t) = \tau_0 - t.$$

Note that $\tau(t)$ is always positive on $[t_1, t_2]$. Moreover, $c = (\tau_0 - t_2)/(\tau_0 - t_1) = \tau(t_2)/\tau(t_1)$. Then, from the scaling behavior of $\mu_\alpha(g, \phi, \tau)$ and diffeomorphism invariance we obtain

$$\begin{aligned} \mu_\alpha(g(t_2), \phi(t_2), \tau(t_2)) &= \mu_\alpha(c\,\psi^* g(t_1), \psi^*\phi(t_1), c\tau(t_1)) \\ &= \mu_\alpha(\psi^* g(t_1), \psi^*\phi(t_1), \tau(t_1)) \\ &= \mu_\alpha(g(t_1), \phi(t_1), \tau(t_1)). \end{aligned} \tag{5.20}$$

By the equality case of the monotonicity result in Proposition 5.2, $(g(t), \phi(t))$ must satisfy (5.10) and according to Lemma 4.2 thus has to be a gradient shrinking soliton.

It remains to prove the additional statement in the cases where $\dim M = 2$ or $(M, g(0))$ is Einstein. If $(g(t), \phi(t))$ is a steady or expanding breather, we have seen that $\frac{\partial}{\partial t} g = cg$. In particular, if $(M, g(t))$ is Einstein at $t = 0$ it remains Einstein under the flow. Moreover, since $\mathrm{Rc} = \frac{R}{m} g$ in these two cases, we get

$$(\phi^*\gamma)_{ij} = \nabla_i\phi\nabla_j\phi = \tfrac{1}{2\alpha}\big(\tfrac{\partial}{\partial t} g_{ij} + 2R_{ij}\big) = \tfrac{1}{2\alpha}\big(2\tfrac{R}{m} + c\big) g_{ij},$$

i.e. ϕ is conformal. It is a well-known fact that conformal harmonic maps have to be minimal branched immersions (cf. Hartman-Wintner [30]). □

CHAPTER 6

Reduced volume and non-collapsing theorem

Huisken [32] proved that if $M^m \subset \mathbb{R}^{m+1}$ evolves by mean curvature flow,

$$\tilde{V} := \int_M (-4\pi t)^{-m/2} e^{|x|^2/4t} dV$$

is monotone, where $x \in \mathbb{R}^{m+1}$. The first goal of this chapter is to prove a similar monotonicity formula in Theorem 6.4. Moreover, we prove a non-collapsing result for solutions of the $(RH)_\alpha$ flow which follows from this monotonicity theorem. Throughout this chapter, we let M be a closed manifold with a time-dependent Riemannian metric $g_{ij}(t)$ and $\mathcal{S}(t)$ a symmetric two tensor on $(M, g(t))$ with components $S_{ij}(t)$ and trace $S(t) := \operatorname{tr}_{g(t)} \mathcal{S}(t) = g^{ij}(t) S_{ij}(t)$. We assume that $g(t)$ evolves according to the flow equation

$$\tfrac{\partial}{\partial t} g_{ij}(t) = -2S_{ij}(t). \tag{6.1}$$

In the case of $(RH)_\alpha$, $S_{ij}(t)$ is given by (2.14), but the results of the fist three sections of this chapter hold true in the general case of a flow of the form (6.1). Other examples where the assumption from Theorem 6.4 below are satisfied are given in the appendix.

Reduced volume monotonicity: the statement

In analogy to Perelman's $\mathcal{L}$-distance for the Ricci flow defined in [48], we will now introduce forwards and backwards reduced distance functions for the flow (6.1), as well as a forwards and a backwards reduced volume.

Definition 6.1
Suppose that (6.1) has a solution for $t \in [0, T]$. For $0 \leq t_0 \leq t_1 \leq T$ and a curve $\eta : [t_0, t_1] \to M$ we define the $\mathcal{L}_f$-length of $\eta(t)$ by

$$\mathcal{L}_f(\eta) := \int_{t_0}^{t_1} \sqrt{t} \left(S(\eta(t)) + \left| \tfrac{d}{dt} \eta(t) \right|^2 \right) dt.$$

For a fixed point $p \in M$ and $t_0 = 0$, we define the forwards reduced distance

$$\ell_f(q, t_1) := \inf_{\eta \in \Gamma} \left\{ \frac{1}{2\sqrt{t_1}} \int_0^{t_1} \sqrt{t} \left(S + \left| \tfrac{d}{dt} \eta \right|^2 \right) dt \right\}, \tag{6.2}$$

where $\Gamma = \{\eta : [0,t_1] \to M \mid \eta(0) = p,\ \eta(t_1) = q\}$, i.e. the forwards reduced distance is the $\mathcal{L}_f$-length of a $\mathcal{L}_f$-shortest curve times $\frac{1}{2\sqrt{t_1}}$. Existence of such $\mathcal{L}_f$-shortest curves will be discussed in the next section. Finally, the forwards reduced volume is defined to be

$$\tilde{V}_f(t) := \int_M (4\pi t)^{-m/2} e^{\ell_f(q,t)} dV(q). \tag{6.3}$$

In order to define the backwards reduced distance and volume, we need a backwards time $\tau(t)$ with $\frac{\partial}{\partial t}\tau(t) = -1$. Without loss of generality, one may assume (possibly after a time shift) that $\tau = -t$.

Definition 6.2
If (6.1) has a solution for $\tau \in [0,\bar{\tau}]$ (i.e. $t \in [-\bar{\tau}, 0]$) we define the $\mathcal{L}_b$-length of a curve $\eta : [\tau_0, \tau_1] \to M$ by

$$\mathcal{L}_b(\eta) := \int_{\tau_0}^{\tau_1} \sqrt{\tau}\left(S(\eta(\tau)) + \left|\tfrac{d}{d\tau}\eta(\tau)\right|^2\right) d\tau.$$

Again, we fix the point $p \in M$ and $\tau_0 = 0$ and define the backwards reduced distance by

$$\ell_b(q,\tau_1) := \inf_{\eta\in\Gamma}\left\{\frac{1}{2\sqrt{\tau_1}}\int_0^{\tau_1}\sqrt{\tau}\left(S + \left|\tfrac{d}{d\tau}\eta\right|^2\right) d\tau\right\}, \tag{6.4}$$

where now $\Gamma = \{\eta : [0,\tau_1] \to M \mid \eta(0) = p,\ \eta(\tau_1) = q\}$. The backwards reduced volume is defined by

$$\tilde{V}_b(\tau) := \int_M (4\pi\tau)^{-m/2} e^{-\ell_b(q,\tau)} dV(q). \tag{6.5}$$

Remark. Note that $(4\pi\tau)^{-m/2} dV_g$ is scaling invariant. Moreover, in the static case, the integrand of of the reduced volume is simply the heat kernel (which can be seen from Lemma 6.7 below).

Next, we define an evolving quantity $\mathcal{D}$ associated to the tensor $\mathcal{S}$.

Definition 6.3
Let $g(t)$ evolve by $\frac{\partial}{\partial t} g_{ij} = -2S_{ij}$ and let S be the trace of $\mathcal{S}$ as above. Let $X \in \Gamma(TM)$ be a vector field on M. We set

$$\begin{aligned}\mathcal{D}(\mathcal{S}, X) := \tfrac{\partial}{\partial t} S - \triangle S - 2\left|S_{ij}\right|^2 + 4(\nabla_i S_{ij})X_j - 2(\nabla_j S)X_j \\ + 2R_{ij}X_iX_j - 2S_{ij}X_iX_j,\end{aligned} \tag{6.6}$$

Remark. The term $\frac{\partial}{\partial t} S - \triangle S - 2\left|S_{ij}\right|^2$ captures the evolution properties of $S = g^{ij}S_{ij}$ under the flow (6.1), while $4(\nabla_i S_{ij})X_j - 2(\nabla_j S)X_j$, is a multiple of the error term E that appears in the twice traced second Bianchi type identity $\nabla_i S_{ij} = \frac{1}{2}\nabla_j S + E_j$ for the symmetric tensor $\mathcal{S}$. Finally, the last line in (6.6) directly compares the tensor S_{ij} with the Ricci tensor. For the Ricci flow, we have $\mathcal{D} \equiv 0$.

We can now state the monotonicity result.

Theorem 6.4
Suppose that $g(t)$ evolves by (6.1) and the quantity $\mathcal{D}(\mathcal{S}, X)$ is nonnegative for all vector fields $X \in \Gamma(TM)$ and all times t for which the flow exists. Then the forwards *reduced volume $\tilde{V}_f(t)$ is non-increasing in t along the flow. Moreover, the* backwards *reduced volume $\tilde{V}_b(\tau)$ is non-increasing in τ, i.e. non-decreasing in t.*

In our case where S_{ij} is given by (2.14), the evolution equation (2.16) for S_{ij} together with $4(\nabla_i S_{ij})X_j - 2(\nabla_j S)X_j = -4\alpha\,\tau_g\phi\nabla_j\phi X_j$ yields

$$\mathcal{D}(S_{ij}, X) = 2\alpha|\tau_g\phi - \nabla_X\phi|^2 - \dot{\alpha}|\nabla\phi|^2$$

for all X on M. To wit, $\mathcal{D}(S_{ij}, X) \geq 0$ is satisfied for the $(RH)_\alpha$ flow with a positive non-increasing coupling function $\alpha(t)$ and the monotonicity of the reduced volumes holds.

A complete proof of the theorem (in the general case, not only for the $(RH)_\alpha$ flow) is developed in the next two sections. It relies on the fact that $\mathcal{D}(\mathcal{S}, X)$ is the difference between two differential Harnack type quantities for the tensor $\mathcal{S}$ defined as follows.

Definition 6.5
For two tangent vector fields $X, Y \in \Gamma(TM)$ on M, we define

$$\begin{aligned}\mathcal{H}(\mathcal{S}, X, Y) &:= 2(\tfrac{\partial}{\partial t}\mathcal{S})(Y,Y) + 2\,|\mathcal{S}(Y,\cdot)|^2 - \nabla_Y\nabla_Y S + \tfrac{1}{t}\mathcal{S}(Y,Y)\\ &\quad - 4(\nabla_X\mathcal{S})(Y,Y) + 4(\nabla_Y\mathcal{S})(X,Y) - 2\,\langle \mathrm{Rm}(X,Y)X, Y\rangle\,,\\ \mathcal{H}(\mathcal{S}, X) &:= \tfrac{\partial}{\partial t}S + \tfrac{1}{t}S - 2\,\langle\nabla S, X\rangle + 2\mathcal{S}(X,X).\end{aligned}$$

Remark. For the Ricci flow, these are Hamilton's matrix and trace Harnack expressions from [25]. For the $(RH)_\alpha$ flow (with constant α), $\mathcal{H}(\mathcal{S}, \nabla f, \cdot)$, $\mathcal{H}(\mathcal{S}, \nabla f)$ and $\mathcal{D}(\mathcal{S}, \nabla f)$ vanish on gradient expanders.

Lemma 6.6
The quantity $\mathcal{D}(\mathcal{S}, X)$ is the difference between the trace of $\mathcal{H}(\mathcal{S}, X, Y)$ with respect to the vector field Y and the expression $\mathcal{H}(\mathcal{S}, X)$, i.e. for an orthonormal basis $\{e_i\}$, we have

$$\mathcal{D}(\mathcal{S}, X) = \sum_i \mathcal{H}(\mathcal{S}, X, e_i) - \mathcal{H}(\mathcal{S}, X).$$

Proof. This is just a short computation. First, note that since the metric evolves by $\frac{\partial}{\partial t}g_{ij} = -2S_{ij}$ its inverse evolves by $\frac{\partial}{\partial t}g^{ij} = 2S^{ij} := 2g^{ik}g^{j\ell}S_{k\ell}$. As a consequence

$$\tfrac{\partial}{\partial t}S = \tfrac{\partial}{\partial t}(g^{ij}S_{ij}) = 2\,|S_{ij}|^2 + \sum_i(\tfrac{\partial}{\partial t}\mathcal{S})(e_i, e_i), \tag{6.7}$$

where $\{e_i\}$ is an orthonormal basis. Therefore, by tracing and rearranging the terms, we find

$$\begin{aligned}\sum_i \mathcal{H}(\mathcal{S}, X, e_i) &= \sum_i \big(2(\tfrac{\partial}{\partial t}\mathcal{S})(e_i, e_i) + 2\,|\mathcal{S}(e_i, \cdot)|^2 - \nabla_{e_i}\nabla_{e_i} S + \tfrac{1}{t}\mathcal{S}(e_i, e_i)\big) \\ &\quad + \sum_i \big(-4(\nabla_X \mathcal{S})(e_i, e_i) + 4(\nabla_{e_i}\mathcal{S})(X, e_i) - 2\,\langle \mathrm{Rm}(X, e_i)X, e_i\rangle\big) \\ &= 2\big(\tfrac{\partial}{\partial t} S - 2\,|S_{ij}|^2\big) + 2\,|S_{ij}|^2 - \triangle S + \tfrac{1}{t} S \\ &\quad - 4(\nabla_j S)X_j + 4(\nabla_i S_{ij})X_j + 2\,\mathrm{Rc}(X, X) \\ &= \tfrac{\partial}{\partial t} S - 2\,|S_{ij}|^2 - \triangle S - 2(\nabla_j S)X_j + 4(\nabla_i S_{ij})X_j + 2R_{ij}X_iX_j \\ &\quad - 2S_{ij}X_iX_j + \tfrac{\partial}{\partial t} S + \tfrac{1}{t} S - 2(\nabla_j S)X_j + 2S_{ij}X_iX_j \\ &= \mathcal{D}(\mathcal{S}, X) + \mathcal{H}(\mathcal{S}, X).\end{aligned}$$

This proves the lemma. □

Obviously, letting τ play the role of the forwards time, the backwards reduced distance as defined in Definition 6.2 corresponds to the forwards reduced distance for the flow $\frac{\partial}{\partial \tau} g_{ij} = +2S_{ij}$. Thus the computations in the forwards and the backwards case differ only by the change of some signs and we find it convenient to do them only for the forwards case. However, we mark all the signs that change in the backwards case with a hat. We illustrate this with an example. Equation (6.22) below reads

$$t^{3/2}\tfrac{d}{dt}\big(S + |X|^2\big) = \hat{+} t^{3/2}\mathcal{H}(\mathcal{S}, \hat{-}X) - \sqrt{t}\big(S + |X|^2\big),$$

with $\mathcal{H}(\mathcal{S}, \hat{-}X)$ evaluated at time t. For the forwards case, we simply neglect the hats and interpret $\mathcal{H}(\mathcal{S}, -X)$ as in Definition 6.5. For the backwards case, we change all t, $\frac{\partial}{\partial t}$ into τ, $\frac{\partial}{\partial \tau}$ etc. and change all the signs with a hat, i.e. the statement is

$$\tau^{3/2}\tfrac{d}{d\tau}\big(S + |X|^2\big) = -\tau^{3/2}\mathcal{H}(\mathcal{S}, X) - \sqrt{\tau}\big(S + |X|^2\big),$$

where $\mathcal{H}(\mathcal{S}, X)$ is now evaluated at $\tau = -t$, i.e.

$$\mathcal{H}(\mathcal{S}, X) = -\tfrac{\partial}{\partial \tau} S - \tfrac{1}{\tau} S - 2\,\langle \nabla S, X\rangle + 2\mathcal{S}(X, X). \tag{6.8}$$

Similarly, the matrix Harnack type expression $\mathcal{H}(\mathcal{S}, X, Y)$ from Definition 6.5 has to be interpreted as

$$\begin{aligned}\mathcal{H}(\mathcal{S}, X, Y) &= -2(\tfrac{\partial}{\partial \tau}\mathcal{S})(Y, Y) + 2\,|\mathcal{S}(Y, \cdot)|^2 - \nabla_Y\nabla_Y S - \tfrac{1}{\tau}\mathcal{S}(Y, Y) \\ &\quad - 4(\nabla_X\mathcal{S})(Y, Y) + 4(\nabla_Y\mathcal{S})(X, Y) - 2\,\langle \mathrm{Rm}(X, Y)X, Y\rangle\end{aligned} \tag{6.9}$$

in the backwards case.

$\mathcal{L}_f$-geodesics and $\mathcal{L}_b$-geodesics

For the Ricci flow, i.e. the case where $S_{ij} = R_{ij}$ is the Ricci tensor, the monotonicity of the backwards reduced volume was developed by Perelman in [48, Section 7]. There exist various other references where the necessary computations can be found in detail, for example Chow et al. [10], Kleiner and Lott [36] and Müller [45]. The forwards case for the Ricci flow can be found in Feldman, Ilmanen and Ni [21]. This and the following section follow these sources closely.

First, we derive the geodesic equation. Let $0 < t_0 \leq t_1 \leq T$ and let $\eta_s(t)$ be a variation of the path $\eta(t) : [t_0, t_1] \to M$. Using Perelman's notation, we set $Y(t) = \frac{\partial}{\partial s}\eta_s(t)|_{s=0}$ and $X(t) = \frac{\partial}{\partial t}\eta_s(t)|_{s=0}$. The first variation of $\mathcal{L}_f(\eta)$ in the direction of $Y(t)$ can then be computed as follows.

$$\begin{aligned}
\delta_Y \mathcal{L}_f(\eta) := \tfrac{\partial}{\partial s}\mathcal{L}_f(\eta_s)|_{s=0} &= \int_{t_0}^{t_1} \sqrt{t}\tfrac{\partial}{\partial s}\big(S(\eta_s(t)) + \big\langle \tfrac{\partial}{\partial t}\eta_s, \tfrac{\partial}{\partial t}\eta_s \big\rangle\big)|_{s=0}\, dt \\
&= \int_{t_0}^{t_1} \sqrt{t}\big(\nabla_Y S + 2\left\langle \nabla_Y X, X\right\rangle\big)dt = \int_{t_0}^{t_1} \sqrt{t}\big(\nabla_Y S + 2\left\langle \nabla_X Y, X\right\rangle\big)dt \\
&= \int_{t_0}^{t_1} \sqrt{t}\big(\left\langle Y, \nabla S\right\rangle + 2\tfrac{d}{dt}\left\langle Y, X\right\rangle - 2\left\langle Y, \nabla_X X\right\rangle \mathbin{\hat{+}} 4\mathcal{S}(Y, X)\big)dt \\
&= 2\sqrt{t}\left\langle Y, X\right\rangle\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \sqrt{t}\,\big\langle Y, \nabla S - \tfrac{1}{t}X - 2\nabla_X X \mathbin{\hat{+}} 4\mathcal{S}(X,\cdot)\big\rangle\, dt,
\end{aligned}$$

using a partial integration in the last step. An $\mathcal{L}_f$-*geodesic* is a critical point of the $\mathcal{L}_f$-length with respect to variations with fixed endpoints. Hence, the above first variation formula implies that the $\mathcal{L}_f$-geodesic equation reads

$$G_f(X) := \nabla_X X - \tfrac{1}{2}\nabla S + \tfrac{1}{2t}X \mathbin{\hat{-}} 2\mathcal{S}(X,\cdot) = 0. \tag{6.10}$$

Changing the variable $\lambda = \sqrt{t}$ in the definition of $\mathcal{L}_f$-length, we get

$$\mathcal{L}_f(\eta(\lambda)) = \int_{\lambda_0}^{\lambda_1} \big(2\lambda^2 S(\eta(\lambda)) + \tfrac{1}{2}\left|\tfrac{\partial}{\partial\lambda}\eta(\lambda)\right|^2\big)d\lambda,$$

and the Euler-Lagrange equation (6.10) becomes

$$G_f(\tilde{X}) := \nabla_{\tilde{X}}\tilde{X} - 2\lambda^2\nabla S \mathbin{\hat{-}} 4\lambda\mathcal{S}(\tilde{X},\cdot) = 0, \tag{6.11}$$

where $\tilde{X} = \frac{\partial}{\partial\lambda}\eta(\lambda) = 2\lambda X$.

The next issue is the existence of $\mathcal{L}_f$-geodesics. From standard existence theory for ordinary differential equations, we see that for $\lambda_0 = \sqrt{t_0}$, $p \in M$ and $v \in T_pM$ there is a unique solution $\eta(\lambda)$ to (6.11) on an interval $[\lambda_0, \lambda_0 + \varepsilon]$ with $\eta(\lambda_0) = p$ and

$\frac{\partial}{\partial\lambda}\eta(\lambda)|_{\lambda=\lambda_0} = v$. If C is a bound for $|\mathcal{S}|$ and $|\nabla S|$ on $M \times [0,T]$ and $\tilde{X}(\lambda) \neq 0$, we find for $\mathcal{L}_f$-geodesics

$$\begin{aligned} \tfrac{\partial}{\partial\lambda}|\tilde{X}| = \tfrac{1}{2|\tilde{X}|}\tfrac{\partial}{\partial\lambda}|\tilde{X}|^2 &= \hat{+}2\lambda|\tilde{X}|\,\mathcal{S}\Big(\tfrac{\tilde{X}}{|\tilde{X}|},\tfrac{\tilde{X}}{|\tilde{X}|}\Big) + 2\lambda^2\left\langle \nabla S, \tfrac{\tilde{X}}{|\tilde{X}|}\right\rangle \\ &\leq 2\lambda C|\tilde{X}| + 2\lambda^2 C. \end{aligned} \tag{6.12}$$

Hence, by a continuity argument, the unique $\mathcal{L}_f$-geodesic $\eta(\lambda)$ can be extended to the whole interval $[\lambda_0, \sqrt{T}]$, i.e. for any $p \in M$ and $t_1 \in [t_0, T]$ we get a globally defined smooth $\mathcal{L}_f$-exponential map, taking $v \in T_pM$ to $\eta(t_1)$, where $\lim_{t\to t_0} 2\sqrt{t}\frac{d}{dt}\eta(t) = v$. Moreover, $\tilde{X} = 2\sqrt{t}X(t)$ has a limit as $t \to 0$ for $\mathcal{L}_f$-geodesics and the definition of $\mathcal{L}_f(\eta)$ can be extended to $t_0 = 0$.

For all (q, t_1) there exists a minimizing $\mathcal{L}_f$-geodesic from $p = \eta(0)$ to $q = \eta(t_1)$. To see this, we can either show that $\mathcal{L}_f$-geodesics minimize for a short time and then use the broken geodesic argument as in the standard Riemannian case, or alternatively we can use the direct method of calculus of variations. There exists a minimizer of $\mathcal{L}_f(\eta)$ among all Sobolev curves, which then has to be a solution of (6.10) and hence a smooth $\mathcal{L}_f$-geodesic.

In the following, we fix $p \in M$ and $t_0 = 0$ and denote by $L_f(q, t_1)$ the $\mathcal{L}_f$-length of a shortest $\mathcal{L}_f$-geodesic $\eta(t)$ joining $p = \eta(0)$ with $q = \eta(t_1)$, i.e. the reduced length is

$$\ell_f(q, t_1) = \tfrac{1}{2\sqrt{t_1}} L_f(q, t_1).$$

Next, we prove lower and upper bounds for $L_f(q, t_1)$. Since M is closed, there is a positive constant C_0 such that $-C_0 g(t) \leq \mathcal{S}(t) \leq C_0 g(t)$ (and thus $-C_0 m \leq S(t) \leq C_0 m$) for all $t \in [0,T]$. We can then obtain the following estimates.

Lemma 6.7
Denote by $d(p,q)$ the standard distance between p and q at time $t = 0$, i.e. the Riemannian distance with respect to $g(0)$. Then the reduced distance $L_f(q, t_1)$ satisfies

$$\frac{d^2(p,q)}{2\sqrt{t_1}}\, e^{-2C_0 t_1} - \frac{2mC_0}{3}\, t_1^{3/2} \leq L_f(q, t_1) \leq \frac{d^2(p,q)}{2\sqrt{t_1}}\, e^{2C_0 t_1} + \frac{2mC_0}{3}\, t_1^{3/2}. \tag{6.13}$$

Proof. The bounds for $\mathcal{S}(t)$ imply $-2C_0 g(t) \leq \hat{-}2\mathcal{S}(t) = \frac{\partial}{\partial t} g(t) \leq 2C_0 g(t)$ and thus

$$e^{-2C_0 t} g(0) \leq g(t) \leq e^{2C_0 t} g(0). \tag{6.14}$$

Using $\lambda = \sqrt{t}$ as above, we can estimate

$$\begin{aligned} \mathcal{L}_f(\eta) &= \int_0^{\sqrt{t_1}} \Big(\tfrac{1}{2}\left|\tfrac{\partial}{\partial\lambda}\eta(\lambda)\right|^2 + 2\lambda^2 S(\eta(\lambda))\Big) d\lambda \\ &\geq \tfrac{1}{2} e^{-2C_0 t_1} \int_0^{\sqrt{t_1}} \left|\tfrac{\partial}{\partial\lambda}\eta(\lambda)\right|^2_{g(0)} d\lambda - \tfrac{2}{3} m C_0 \lambda^3\Big|_0^{\sqrt{t_1}} \\ &\geq \frac{d^2(p,q)}{2\sqrt{t_1}} e^{-2C_0 t_1} - \frac{2mC_0}{3}\, t_1^{3/2}. \end{aligned}$$

With $L_f(q, t_1) = \inf_{\eta\in\Gamma} \mathcal{L}_f(\eta)$ we get the lower bound in (6.13). For the upper bound, let $\sigma(\lambda) : [0, \sqrt{t_1}] \to M$ be a minimal geodesic from p to q with respect to $g(0)$. Then

$$\begin{aligned} L_f(q, t_1) \le \mathcal{L}_f(\sigma) &= \int_0^{\sqrt{t_1}} \left(\tfrac{1}{2} \left| \tfrac{\partial}{\partial\lambda}\sigma(\lambda) \right|^2 + 2\lambda^2 S(\sigma(\lambda)) \right) d\lambda \\ &\le \tfrac{1}{2} e^{2C_0 t_1} \int_0^{\sqrt{t_1}} \left| \tfrac{\partial}{\partial\lambda}\sigma(\lambda) \right|^2_{g(0)} d\lambda + \tfrac{2}{3} m C_0 \lambda^3 \Big|_0^{\sqrt{t_1}} \\ &= \frac{d^2(p, q)}{2\sqrt{t_1}} e^{2C_0 t_1} + \frac{2mC_0}{3} t_1^{3/2}, \end{aligned}$$

which proves the claim. □

Lemma 6.8
The distance $L_f : M \times (0, T) \to \mathbb{R}$ is locally Lipschitz continuous with respect to the metric $g(t) + dt^2$ on space-time and smooth outside of a set of measure zero.

Proof. For any $0 < t_* < T$, $q_* \in M$ and small $\varepsilon > 0$, let $t_1 < t_2$ be in $(t_* - \varepsilon, t_* + \varepsilon)$ and $q_1, q_2 \in B_{g(t_*)}(q_*, \varepsilon) = \{q \in M \mid d_{g(t_*)}(q_*, q) < \varepsilon\}$, where $d_{g(t_*)}(\cdot, \cdot)$ denotes the Riemannian distance with respect to the metric $g(t_*)$. Since

$$|L_f(q_1, t_1) - L_f(q_2, t_2)| \le |L_f(q_1, t_1) - L_f(q_1, t_2)| + |L_f(q_1, t_2) - L_f(q_2, t_2)|,$$

it suffices for the Lipschitz continuity with respect to $g(t)+dt^2$ to show that $L_f(q_1, \cdot)$ is locally Lipschitz in the time variable uniformly in $q_1 \in B_{g(t_*)}(q_*, \varepsilon)$ and $L_f(\cdot, t)$ is locally Lipschitz in the space variable uniformly in $t \in (t_* - \varepsilon, t_* + \varepsilon)$. Our proof is related to the proofs of [10, Lemma 7.28 and Lemma 7.30]. In the following, $C = C(C_0, m, t_*, \varepsilon)$ denotes a generic constant which might change from line to line.

Claim 1: $L_f(q_1, t_2) \le L_f(q_1, t_1) + C(t_2 - t_1)$.

Proof. Let $\eta : [0, t_1] \to M$ be a minimal $\mathcal{L}_f$-geodesic from p to q_1 and define the curve $\sigma : [0, t_2] \to M$ by

$$\sigma(t) := \begin{cases} \eta(t) & \text{if } t \in [0, t_1], \\ q_1 & \text{if } t \in [t_1, t_2]. \end{cases} \tag{6.15}$$

We compute

$$\begin{aligned} L_f(q_1, t_2) \le \mathcal{L}_f(\sigma) &= \mathcal{L}_f(\eta) + \int_{t_1}^{t_2} \sqrt{t} S(q_1, t) dt \\ &\le L_f(q_1, t_1) + \tfrac{2}{3} m C_0 \left(t_2^{3/2} - t_1^{3/2} \right) \\ &\le L_f(q_1, t_1) + C(t_2 - t_1), \end{aligned}$$

which proves Claim 1. □

Claim 2: $L_f(q_1, t_1) \leq L_f(q_1, t_2) + C(t_2 - t_1)$.

Proof. Let $\eta : [0, t_2] \to M$ be a minimal $\mathfrak{L}_f$-geodesic from p to q_1 and define the curve $\sigma : [0, t_1] \to M$ by

$$\sigma(t) := \begin{cases} \eta(t) & \text{if } t \in [0, 2t_1 - t_2], \\ \eta(\phi(t)) & \text{if } t \in [2t_1 - t_2, t_1], \end{cases} \tag{6.16}$$

where $\phi(t) := 2t + t_2 - 2t_1 \geq t$ on $[2t_1 - t_2, t_1]$ with $\frac{d}{dt}\phi(t) \equiv 2$. We compute

$$\begin{aligned} L_f(q_1, t_1) \leq \mathfrak{L}_f(\sigma) &= \mathfrak{L}_f(\eta) - \int_{2t_1-t_2}^{t_2} \sqrt{t}\Big(S(\eta(t), t) + \big|\tfrac{d}{dt}\eta(t)\big|^2\Big)dt \\ &\quad + \int_{2t_1-t_2}^{t_1} \sqrt{t}\Big(S(\eta(\phi(t)), t) + \big|\tfrac{d}{dt}\eta(\phi(t)) \cdot \tfrac{d}{dt}\phi(t)\big|^2\Big)dt \\ &\leq L(q_1, t_2) + \tfrac{2}{3}mC_0\Big(t_2^{3/2} - (2t_1 - t_2)^{3/2}\Big) + \tfrac{2}{3}mC_0\Big(t_1^{3/2} - (2t_1 - t_2)^{3/2}\Big) \\ &\quad + 2\int_{2t_1-t_2}^{t_2} \sqrt{\phi^{-1}(t)}\, \big|\tfrac{d}{dt}\eta(t)\big|^2_{g(\phi^{-1}(t))}\, dt \\ &\leq L(q_1, t_1) + C(t_2 - t_1) + 2\int_{2t_1-t_2}^{t_2} \sqrt{\phi^{-1}(t)}\, \big|\tfrac{d}{dt}\eta(t)\big|^2_{g(\phi^{-1}(t))}\, dt. \end{aligned}$$

Since $\phi^{-1}(t) \leq t$ and $t - \phi^{-1}(t) \leq 2\varepsilon$ on $[2t_1 - t_2, t_2]$, we can estimate the very last term via (6.14) by

$$\int_{2t_1-t_2}^{t_2} \sqrt{\phi^{-1}(t)}\, \big|\tfrac{d}{dt}\eta(t)\big|^2_{g(\phi^{-1}(t))}\, dt \leq e^{4C_0\varepsilon} \int_{2t_1-t_2}^{t_2} \sqrt{t}\, \big|\tfrac{d}{dt}\eta(t)\big|^2_{g(t)}\, dt.$$

As a consequence of the upper bound from Lemma 6.7 and the growth condition (6.12), $\big|\frac{d}{dt}\eta(t)\big|^2_{g(t)}$ must be uniformly bounded on $[2t_1 - t_2, t_2]$ by a constant C_1. Thus

$$\int_{2t_1-t_2}^{t_2} \sqrt{\phi^{-1}(t)}\, \big|\tfrac{d}{dt}\eta(t)\big|^2_{g(\phi^{-1}(t))}\, dt \leq e^{4C_0\varepsilon} C_1\big(t_2^{3/2} - (2t_1 - t_2)^{3/2}\big) \leq C(t_2 - t_1).$$

Together with the computation above, this proves the claim. □

Claim 3: $L_f(q_1, t_2) \leq L_f(q_2, t_2) + Cd_{g(t_2)}(q_1, q_2)$.

Proof. Let $\eta : [0, t_2] \to M$ be a minimal $\mathfrak{L}_f$-geodesic from p to q_2 and define the curve $\sigma : [0, t_2 + d_{g(t_2)}(q_1, q_2)] \to M$ by

$$\sigma(t) := \begin{cases} \eta(t) & \text{if } t \in [0, t_2], \\ \beta(t) & \text{if } t \in [t_2, t_2 + d_{g(t_2)}(q_1, q_2)], \end{cases} \tag{6.17}$$

where $\beta : [t_2, t_2 + d_{g(t_2)}(q_1, q_2)] \to M$ is a minimal geodesic of constant unit speed with respect to $g(t_2)$, joining q_2 to q_1. Then, using $\big|\frac{d}{dt}\beta(t)\big|^2_{g(t)} \leq e^{4C_0\varepsilon} \big|\frac{d}{dt}\beta(t)\big|^2_{g(t_2)} = e^{4C_0\varepsilon}$,

we obtain

$$\begin{aligned} L_f(q_1, t_2 + d_{g(t_2)}(q_1,q_2)) &\leq \mathcal{L}_f(\sigma) \\ &= L_f(q_2,t_2) + \int_{t_2}^{t_2+d_{g(t_2)}(q_1,q_2)} \sqrt{t}\Big(S(\beta(t),t) + \big|\tfrac{d}{dt}\beta(t)\big|^2\Big)dt \\ &\leq L_f(q_2,t_2) + \tfrac{2}{3}\Big(C_0 m + e^{4C_0\varepsilon}\Big)\Big((t_2 + d_{g(t_2)}(q_1,q_2))^{3/2} - t_2^{3/2}\Big) \\ &\leq L_f(q_2,t_2) + C d_{g(t_2)}(q_1,q_2). \end{aligned}$$

Finally, using Claim 2 from above, we find

$$\begin{aligned} L_f(q_1,t_2) &\leq L_f(q_1, t_2 + d_{g(t_2)}(q_1,q_2)) + C d_{g(t_2)}(q_1,q_2) \\ &\leq L_f(q_2,t_2) + C d_{g(t_2)}(q_1,q_2), \end{aligned}$$

which proves Claim 3. □

The Lipschitz continuity in the time variable follows from Claim 1 and Claim 2. The Lipschitz continuity in the space variable follows from Claim 3 and the symmetry between q_1 and q_2.

From the definition of $L_f : M \times (0,T) \to \mathbb{R}$, we see that it is smooth outside of the set $\bigcup_t (C(t) \times \{t\})$, where for a fixed time t_1 the *cut locus* $C(t_1)$ is defined to be the set of points $q \in M$ such that either there is more than one minimal $\mathcal{L}_f$-geodesic $\eta : [0,t_1] \to M$ from $p = \eta(0)$ to $q = \eta(t_1)$ or q is conjugate to p along η. A point q is called conjugate to p along η if there exists a nontrivial $\mathcal{L}_f$-Jacobi field J along η with $J(0) = J(t_1) = 0$.

As in the standard Riemannian geometry, the set $C_1(t_1)$ of conjugate points to $(p,0)$ is contained in the set of critical values for the $\mathcal{L}_f$-exponential map from $(p,0)$ defined above. Hence it has measure zero by Sard's theorem. If there exist more than one minimal $\mathcal{L}_f$-geodesic from p to q, then $L(q,t_1)$ is not differentiable at q. But since $L_f(q,t_1)$ is Lipschitz, it has to be differentiable almost everywhere by Rademacher's theorem and thus the set $C_2(t_1)$ consisting of points for which there exist more than one minimal $\mathcal{L}_f$-geodesic also has to have measure zero. Combining this, $C(t_1) = C_1(t_1) \cup C_2(t_1)$ has measure zero for all $t_1 \in (0,T)$ and so $\bigcup_t (C(t) \times \{t\})$ is of measure zero, too. This finishes the proof of the lemma. □

Reduced volume monotonicity: the proof

Making use of Lemma 6.8, we first pretend that $L_f(q,t_1)$ is smooth everywhere and derive formulas for $|\nabla L_f|^2$, $\frac{\partial}{\partial t_1} L_f$ and $\triangle L_f$ under this assumption.

Lemma 6.9
The reduced distance $L_f(q,t_1)$ has the gradient properties

$$|\nabla L_f(q,t_1)|^2 = -4t_1 S \hat{+} \tfrac{4}{\sqrt{t_1}} K + \tfrac{2}{\sqrt{t_1}} L_f(q,t_1), \tag{6.18}$$

$$\tfrac{\partial}{\partial t_1} L_f(q,t_1) = 2\sqrt{t_1} S \hat{-} \tfrac{1}{t_1} K - \tfrac{1}{2t_1} L_f(q,t_1), \tag{6.19}$$

where

$$K := \int_0^{t_1} t^{3/2} \mathcal{H}(\mathcal{S}, \hat{-}X)\, dt$$

and $\mathcal{H}(\mathcal{S}, \hat{-}X)$ is the Harnack type expression from Definition 6.5, evaluated at time t. Remember that in the backwards case we interpret $\mathcal{H}(\mathcal{S}, X)$ as in (6.8).

Proof. A minimizing curve satisfies $G_f(X) = 0$, hence the first variation formula above yields

$$\delta_Y L_f(q,t_1) = 2\sqrt{t_1} \langle X(t_1), Y(t_1)\rangle = \langle \nabla L_f(q,t_1), Y(t_1)\rangle .$$

Thus, the gradient of L_f must be $\nabla L_f(q,t_1) = 2\sqrt{t_1} X(t_1)$. This yields

$$|\nabla L_f|^2 = 4t_1 |X|^2 = -4t_1 S + 4t_1\big(S + |X|^2\big). \tag{6.20}$$

Moreover, we compute

$$\begin{aligned} \tfrac{\partial}{\partial t_1} L_f(q,t_1) &= \tfrac{d}{dt_1} L_f(q,t_1) - \nabla_X L_f(q,t_1) = \sqrt{t_1}\big(S + |X|^2\big) - \langle \nabla L_f(q,t_1), X\rangle \\ &= \sqrt{t_1}\big(S + |X|^2\big) - 2\sqrt{t_1}|X|^2 = 2\sqrt{t_1} S - \sqrt{t_1}\big(S + |X|^2\big). \end{aligned} \tag{6.21}$$

Note that $\frac{\partial}{\partial t_1}$ denotes the partial derivative with respect to t_1 keeping the point q fixed, while $\frac{d}{dt_1}$ refers to differentiation along an $\mathcal{L}$-geodesic, i.e. simultaneously varying the time t_1 and the point q. Next, we determine $\big(S + |X|^2\big)$ in terms of L_f. With the Euler-Lagrange equation (6.10), we get

$$\begin{aligned} \tfrac{d}{dt}\big(S(\eta(t)) + |X(t)|^2\big) &= \tfrac{\partial}{\partial t} S + \nabla_X S + 2\langle \nabla_X X, X\rangle \hat{-} 2\mathcal{S}(X,X) \\ &= \tfrac{\partial}{\partial t} S + 2\langle \nabla S, X\rangle - \tfrac{1}{t}|X|^2 \hat{+} 2\mathcal{S}(X,X) \\ &= \hat{+}\mathcal{H}(\mathcal{S}, \hat{-}X) - \tfrac{1}{t}\big(S + |X|^2\big). \end{aligned}$$

From this we obtain

$$t^{3/2} \tfrac{d}{dt}\big(S + |X|^2\big) = \hat{+} t^{3/2} \mathcal{H}(\mathcal{S}, \hat{-}X) - \sqrt{t}\big(S + |X|^2\big) \tag{6.22}$$

and thus by integrating and using the notation $K = \int_0^{t_1} t^{3/2}\mathcal{H}(\mathcal{S}, \hat{-}X)dt$, we conclude

$$\begin{aligned} \hat{+}K - L_f(q,t_1) &= \int_0^{t_1} t^{3/2} \tfrac{d}{dt}\big(S + |X|^2\big) dt \\ &= t_1^{3/2}\big(S(\eta(t_1)) + |X(t_1)|^2\big) - \int_0^{t_1} \tfrac{3}{2}\sqrt{t}\big(S + |X|^2\big) dt \\ &= t_1^{3/2}\left(S + |X|^2\right) - \tfrac{3}{2} L_f(q,t_1). \end{aligned}$$

Hence, we have

$$t_1^{3/2}\big(S + |X|^2\big) = \hat{+}K + \tfrac{1}{2}L_f(q, t_1). \tag{6.23}$$

If we insert this into (6.20) and (6.21), we get (6.18) and (6.19), respectively. □

To estimate the Laplacian $\triangle L_f$, we compute the second variation of $\mathcal{L}_f(\eta)$, using the following claim.

Claim 1: Under the flow $\frac{\partial}{\partial t} g_{ij} = \hat{-}2S_{ij}$, we have

$$\begin{aligned}\tfrac{\partial}{\partial t}\langle \nabla_Y Y, X\rangle = \langle \nabla_X \nabla_Y Y, X\rangle + \langle \nabla_Y Y, \nabla_X X\rangle \mathbin{\hat{-}} 2\mathcal{S}(\nabla_Y Y, X)\\ \mathbin{\hat{-}} 2(\nabla_Y \mathcal{S})(Y, X) \mathbin{\hat{+}} (\nabla_X \mathcal{S})(Y, Y).\end{aligned} \tag{6.24}$$

Proof. We start with

$$\tfrac{\partial}{\partial t}\langle \nabla_Y Y, X\rangle = \langle \nabla_X \nabla_Y Y, X\rangle + \langle \nabla_Y Y, \nabla_X X\rangle \mathbin{\hat{-}} 2\mathcal{S}(\nabla_Y Y, X) + \langle \dot{\nabla}_Y Y, X\rangle, \tag{6.25}$$

where $\dot{\nabla} := \frac{\partial}{\partial t}\nabla$. From [45, page 21], we know that under the flow $\frac{\partial}{\partial t} g = h$, we have

$$\langle \dot{\nabla}_U V, W\rangle = \tfrac{1}{2}(\nabla_U h)(V, W) + \tfrac{1}{2}(\nabla_V h)(U, W) - \tfrac{1}{2}(\nabla_W h)(U, V).$$

Hence, with $U = V = Y$, $W = X$ and $h = \hat{-}2\mathcal{S}$, we get

$$\langle \dot{\nabla}_Y Y, X\rangle = \hat{-}2(\nabla_Y \mathcal{S})(Y, X) \mathbin{\hat{+}} (\nabla_X \mathcal{S})(Y, Y).$$

Inserting this into (6.25) proves the claim. □

Using Claim 1, we can now write $2\langle \nabla_Y \nabla_X Y, X\rangle$ as

$$\begin{aligned}2\langle \nabla_Y \nabla_X Y, X\rangle &= 2\langle \nabla_X \nabla_Y Y, X\rangle + 2\langle \mathrm{Rm}(Y, X)Y, X\rangle\\ &= 2\tfrac{\partial}{\partial t}\langle \nabla_Y Y, X\rangle - 2\langle \nabla_Y Y, \nabla_X X\rangle \mathbin{\hat{+}} 4\mathcal{S}(\nabla_Y Y, X)\\ &\quad \mathbin{\hat{+}} 4(\nabla_Y \mathcal{S})(Y, X) \mathbin{\hat{-}} 2(\nabla_X \mathcal{S})(Y, Y) + 2\langle \mathrm{Rm}(Y, X)Y, X\rangle,\end{aligned}$$

and a partial integration yields

$$\begin{aligned}\int_0^{t_1} 2\sqrt{t}\,\langle \nabla_Y \nabla_X Y, X\rangle\, dt = 2\sqrt{t}\,\langle \nabla_Y Y, X\rangle\,\Big|_0^{t_1} - \int_0^{t_1} \sqrt{t}\,\tfrac{1}{t}\,\langle \nabla_Y Y, X\rangle\, dt\\ - \int_0^{t_1} 2\sqrt{t}\,\big\langle \nabla_Y Y, \nabla_X X \mathbin{\hat{-}} 2\mathcal{S}(X, \cdot)\big\rangle\, dt\\ \hat{+} \int_0^{t_1} \sqrt{t}\,(4(\nabla_Y \mathcal{S})(Y, X) - 2(\nabla_X \mathcal{S})(Y, Y))\, dt\\ + \int_0^{t_1} 2\sqrt{t}\,\langle \mathrm{Rm}(Y, X)Y, X\rangle\, dt.\end{aligned} \tag{6.26}$$

If the geodesic equation (6.10) holds, we can write the first two integrals on the right hand side of (6.26) as

$$-2\int_0^{t_1} \sqrt{t}\,\big\langle \nabla_Y Y, \tfrac{1}{2t}X + \nabla_X X \mathbin{\hat{-}} 2\mathcal{S}(X, \cdot)\big\rangle\, dt = -\int_0^{t_1} \sqrt{t}\,\langle \nabla_Y Y, \nabla S\rangle\, dt,$$

and equation (6.26) becomes

$$\begin{aligned}\int_0^{t_1} 2\sqrt{t}\,\langle \nabla_Y \nabla_X Y, X\rangle\, dt &= 2\sqrt{t}\,\langle \nabla_Y Y, X\rangle\,\big|_0^{t_1} - \int_0^{t_1} \sqrt{t}\,\langle \nabla_Y Y, \nabla S\rangle\, dt \\ &\quad \hat{+} \int_0^{t_1} \sqrt{t}\,(4(\nabla_Y \mathcal{S})(Y,X) - 2(\nabla_X \mathcal{S})(Y,Y))\, dt \\ &\quad + \int_0^{t_1} 2\sqrt{t}\,\langle \mathrm{Rm}(Y,X)Y, X\rangle\, dt.\end{aligned} \tag{6.27}$$

We can now compute the second variation of $\mathcal{L}_f(\eta)$ for $\mathcal{L}_f$-geodesics η where $G_f(X) = 0$ is satisfied. Using the first variation

$$\delta_Y \mathcal{L}(\eta) = \int_{t_0}^{t_1} \sqrt{t}(\nabla_Y S + 2\,\langle \nabla_Y X, X\rangle)dt$$

from the last section, we compute

$$\begin{aligned}\delta_Y^2 \mathcal{L}(\eta) &= \int_0^{t_1} \sqrt{t}\big(\tfrac{\partial}{\partial s}\,\langle \nabla S, Y\rangle + 2\,\langle \nabla_Y \nabla_Y X, X\rangle + 2\,|\nabla_Y X|^2\big)dt \\ &= \int_0^{t_1} \sqrt{t}\big(\,\langle \nabla S, \nabla_Y Y\rangle + \nabla_Y \nabla_Y S + 2\,|\nabla_X Y|^2 + 2\,\langle \nabla_Y \nabla_X Y, X\rangle\,\big)dt \\ &= 2\sqrt{t}\,\langle \nabla_Y Y, X\rangle\,\big|_0^{t_1} + \int_0^{t_1} \sqrt{t}\,\big(\nabla_Y \nabla_Y S + 2\,|\nabla_X Y|^2\big)\, dt \\ &\quad \hat{-} \int_0^{t_1} \sqrt{t}(2(\nabla_X \mathcal{S})(Y,Y) - 4(\nabla_Y \mathcal{S})(Y,X))dt \\ &\quad + \int_0^{t_1} 2\sqrt{t}\,\langle \mathrm{Rm}(Y,X)Y, X\rangle\, dt,\end{aligned} \tag{6.28}$$

where we used (6.27) in the last step. Now, choose the test variation $Y(t)$ such that

$$\nabla_X Y = \hat{+}\mathcal{S}(Y,\cdot) + \tfrac{1}{2t} Y, \tag{6.29}$$

which implies $\frac{d}{dt}\,|Y|^2 = \hat{-}2\mathcal{S}(Y,Y) + 2\,\langle \nabla_X Y, Y\rangle = \frac{1}{t}\,|Y|^2$. Hence, if $|Y(t_1)| = 1$, we obtain $|Y(t)|^2 = t/t_1$, in particular $Y(0) = 0$. We have

$$\begin{aligned}\mathrm{Hess}_{L_f}(Y,Y) &= \nabla_Y \nabla_Y L_f = \delta_Y^2(L_f) - \langle \nabla_Y Y, \nabla L_f\rangle \\ &\leq \delta_Y^2 \mathcal{L}_f - 2\sqrt{t_1}\,\langle \nabla_Y Y, X\rangle\,(t_1),\end{aligned} \tag{6.30}$$

where the Y in $\mathrm{Hess}_{L_f}(Y,Y) = \nabla_Y \nabla_Y L_f$ denotes a vector $Y(t_1) \in T_q M$, while in $\delta_Y^2 \mathcal{L}_f$ it denotes the associated variation of the curve, i.e. the vector field $Y(t)$ along η which solves the above ODE (6.29). Note that (6.30) holds with equality if Y is an $\mathcal{L}_f$-Jacobi field. We obtain

$$\begin{aligned}\mathrm{Hess}_{L_f}(Y,Y) &\leq \int_0^{t_1} \sqrt{t}\big(\nabla_Y \nabla_Y S + 2\,|\nabla_X Y|^2 + 2\,\langle \mathrm{Rm}(Y,X)Y, X\rangle\,\big)dt \\ &\quad \hat{-} \int_0^{t_1} \sqrt{t}(2(\nabla_X \mathcal{S})(Y,Y) - 4(\nabla_Y \mathcal{S})(Y,X))dt.\end{aligned} \tag{6.31}$$

Lemma 6.10
For K defined as in Lemma 6.9, and under the assumption $\mathcal{D}(\mathcal{S}, Z) \geq 0$, $\forall Z \in \Gamma(TM)$, the distance function $L_f(q, t_1)$ satisfies

$$\triangle L_f(q, t_1) \leq \frac{m}{\sqrt{t_1}} \hat{+} 2\sqrt{t_1} S - \frac{1}{t_1} K. \tag{6.32}$$

Proof. Note that with (6.29) we find

$$\begin{aligned} |\nabla_X Y|^2 &= |\mathcal{S}(Y, \cdot)|^2 \hat{+} \tfrac{1}{t}\mathcal{S}(Y, Y) + \tfrac{1}{4t^2} |Y(t)|^2 \\ &= |\mathcal{S}(Y, \cdot)|^2 \hat{+} \tfrac{1}{t}\mathcal{S}(Y, Y) + \tfrac{1}{4t\,t_1}, \end{aligned} \tag{6.33}$$

as well as

$$\begin{aligned} \tfrac{d}{dt}\mathcal{S}(Y(t), Y(t)) &= (\tfrac{\partial}{\partial t}\mathcal{S})(Y, Y) + (\nabla_X \mathcal{S})(Y, Y) + 2\mathcal{S}(\nabla_X Y, Y) \\ &= (\tfrac{\partial}{\partial t}\mathcal{S})(Y, Y) + (\nabla_X \mathcal{S})(Y, Y) + \tfrac{1}{t}\mathcal{S}(Y, Y) \hat{+} 2 |\mathcal{S}(Y, \cdot)|^2 . \end{aligned} \tag{6.34}$$

Using (6.33), (6.34) and a partial integration, we get from (6.31)

$$\begin{aligned} \mathrm{Hess}_{L_f}(Y, Y) \leq& \int_0^{t_1} \sqrt{t}\big(\nabla_Y \nabla_Y S + 2\langle \mathrm{Rm}(Y, X)Y, X\rangle\big) dt \\ &\hat{-} \int_0^{t_1} \sqrt{t}\big(2(\nabla_X \mathcal{S})(Y, Y) - 4(\nabla_Y \mathcal{S})(Y, X)\big) dt \\ &+ \int_0^{t_1} \sqrt{t}\big(2 |\mathcal{S}(Y, \cdot)|^2 \hat{+} \tfrac{2}{t}\mathcal{S}(Y, Y) + \tfrac{1}{2t\,t_1}\big) dt \\ =& \frac{1}{\sqrt{t_1}} - \int_0^{t_1} \sqrt{t}\, \mathcal{H}(\mathcal{S}, \hat{-}X, Y) dt \hat{+} \int_0^{t_1} \sqrt{t}\big(2(\nabla_X \mathcal{S})(Y, Y) \hat{+} 4 |\mathcal{S}(Y, \cdot)|^2\big) dt \\ &\hat{+} \int_0^{t_1} \sqrt{t}\big(\tfrac{3}{t}\mathcal{S}(Y, Y) + 2(\tfrac{\partial}{\partial t}\mathcal{S})(Y, Y)\big) dt \\ =& \frac{1}{\sqrt{t_1}} - \int_0^{t_1} \sqrt{t}\, \mathcal{H}(\mathcal{S}, \hat{-}X, Y) dt \hat{+} \int_0^{t_1} \sqrt{t}\big(2\tfrac{d}{dt}\mathcal{S}(Y, Y) + \tfrac{1}{t}\mathcal{S}(Y, Y)\big) dt \\ =& \frac{1}{\sqrt{t_1}} \hat{+} 2\sqrt{t_1}\mathcal{S}(Y, Y) - \int_0^{t_1} \sqrt{t}\, \mathcal{H}(\mathcal{S}, \hat{-}X, Y) dt. \end{aligned}$$

Here, $\mathcal{H}(\mathcal{S}, \hat{-}X, Y)$ denotes the Harnack type expression from Definition 6.5 evaluated at time t. Remember that in the backwards case $\mathcal{H}(\mathcal{S}, X, Y)$ has to be interpreted as in (6.9). Now let $\{Y_i(t_1)\}$ be an orthonormal basis of $T_q M$, and define $Y_i(t)$ as above, solving the ODE (6.29). We compute

$$\begin{aligned} \tfrac{\partial}{\partial t} \langle Y_i, Y_j \rangle &= \hat{-}2\mathcal{S}(Y_i, Y_j) + \langle \nabla_X Y_i, Y_j \rangle + \langle Y_i, \nabla_X Y_j \rangle \\ &= \hat{-}2\mathcal{S}(Y_i, Y_j) + \big\langle \hat{+}\mathcal{S}(Y_i, \cdot) + \tfrac{1}{2t} Y_i, Y_j \big\rangle + \big\langle Y_i, \hat{+}\mathcal{S}(Y_j, \cdot) + \tfrac{1}{2t} Y_j \big\rangle \\ &= \tfrac{1}{t} \langle Y_i, Y_j \rangle . \end{aligned}$$

Thus the $\{Y_i(t)\}$ are orthogonal with $\langle Y_i(t), Y_j(t)\rangle = \frac{t}{t_1}\langle Y_i(t_1), Y_j(t_1)\rangle = \frac{t}{t_1}\delta_{ij}$. In particular, there exist orthonormal vector fields $e_i(t)$ along η with $Y_i(t) = \sqrt{t/t_1}\, e_i(t)$. Summing over $\{e_i\}$ yields

$$\begin{aligned}
\triangle L_f(q,t_1) &\leq \sum_i \Big(\frac{1}{\sqrt{t_1}} \hat{+} 2\sqrt{t_1}\mathcal{S}(Y_i,Y_i) - \int_0^{t_1} \sqrt{t}\,\mathcal{H}(\mathcal{S}, \hat{-}X, Y_i)dt\Big)\\
&= \frac{m}{\sqrt{t_1}} \hat{+} 2\sqrt{t_1}S - \frac{1}{t_1}\int_0^{t_1} t^{3/2}\sum_i \mathcal{H}(\mathcal{S}, \hat{-}X, e_i)dt\\
&= \frac{m}{\sqrt{t_1}} \hat{+} 2\sqrt{t_1}S - \frac{1}{t_1}\int_0^{t_1} t^{3/2}\big(\mathcal{H}(\mathcal{S}, \hat{-}X) + \mathcal{D}(\mathcal{S}, \hat{-}X)\big)dt\\
&\leq \frac{m}{\sqrt{t_1}} \hat{+} 2\sqrt{t_1}S - \frac{1}{t_1}K,
\end{aligned}$$

using Lemma 6.6 and the assumption $\mathcal{D}(\mathcal{S}, \hat{-}X) \geq 0$. □

The three formulas from Lemma 6.9 and Lemma 6.10 can now be combined to one evolution inequality for the reduced distance function $\ell_f(q,t_1) = \frac{1}{2\sqrt{t_1}}L_f(q,t_1)$. From (6.18), (6.19) and (6.32), we get

$$\begin{aligned}
|\nabla \ell_f|^2 &= \frac{1}{4t_1}|\nabla L_f|^2 = -S + \frac{1}{t_1}\ell_f \hat{+} \frac{1}{t_1^{3/2}}K,\\
\tfrac{\partial}{\partial t_1}\ell_f &= -\frac{1}{4t_1^{3/2}}L_f + \frac{1}{2\sqrt{t_1}}\tfrac{\partial}{\partial t_1}L_f = -\frac{1}{t_1}\ell_f + S \hat{-} \frac{1}{2t_1^{3/2}}K,\\
\triangle \ell_f &= \frac{1}{2\sqrt{t_1}}\triangle L_f \leq \frac{m}{2t_1} \hat{+} S - \frac{1}{2t_1^{3/2}}K,
\end{aligned}$$

and thus with a lot of cancellations

$$\triangle \ell_f \hat{+} \tfrac{\partial}{\partial t_1}\ell_f \hat{+} |\nabla \ell_f|^2 \hat{-} S - \tfrac{m}{2t} \leq 0. \tag{6.35}$$

This is equivalent to

$$(\tfrac{\partial}{\partial t} \hat{+} \triangle \hat{-} S)v_f(q,t) \leq 0, \tag{6.36}$$

where $v_f(q,t) := (4\pi t)^{-m/2}e^{\hat{+}\ell_f(q,t)}$ is the density function for the reduced volume $\tilde{V}_f(t)$.

Remember that so far we pretended that $L_f(q,t_1)$ is smooth. In the general case it is obvious that the inequality (6.35) holds in the classical sense at all points where L_f is smooth. But what happens at the other points? This question is answered by the following lemma.

Lemma 6.11
The inequality (6.35) holds on $M \times (0,T)$ in the barrier sense*, i.e. for all $(q_*, t_*) \in M \times (0,T)$ there exists a neighborhood U of q in M, some $\varepsilon > 0$ and a smooth upper barrier $\tilde{\ell}_f$ defined on $U \times (t_* - \varepsilon, t_* + \varepsilon)$ with $\tilde{\ell}_f \geq \ell_f$ and $\tilde{\ell}_f(q_*, t_*) = \ell_f(q_*, t_*)$ which satisfies (6.35). Moreover, (6.35) holds on $M \times (0,T)$ in the* distributional sense.

Proof. Given $(q_*, t_*) \in M \times (0, T)$, let $\eta : [0, t_*] \to M$ be a minimal $\mathfrak{L}_f$-geodesic from p to q_*, so that $\ell_f(q_*, t_*) = \frac{1}{2\sqrt{t_*}}\mathfrak{L}_f(\eta)$. Extend η to a smooth $\mathfrak{L}_f$-geodesic $\eta : [0, t_* + \varepsilon] \to M$ for some $\varepsilon > 0$. For a given orthonormal basis $\{Y_i(t_*)\}$ of $T_{q_*}M$, solve the ODE (6.29) on $[0, t_* + \varepsilon]$ and let $\eta_i(s, t)$ be a variation of $\eta(t)$ in the direction of Y_i, i.e. $\eta_i(0, t) = \eta(t)$ and $\frac{\partial}{\partial s}\eta_i(s, t)|_{s=0} = Y_i(t)$. Finally, for a small neighborhood U of q_* we choose a smooth family of curves $\sigma_{q,t_1} : [0, t_1] \to M$ from $\sigma_{q,t_1}(0) = p$ to $\sigma_{q,t_1}(t_1) = q \in U$, $t_1 \in (t_* - \varepsilon, t_* + \varepsilon)$, with the following property:

$$\sigma_{\eta_i(s,t),t} = \eta_i(s, \cdot)|_{[0,t]}, \quad \forall t \in (t_* - \varepsilon, t_* + \varepsilon) \text{ and } |s| < \varepsilon.$$

Define $\tilde{L}_f(q, t_1) := \mathfrak{L}_f(\sigma_{q,t_1})$ and $\tilde{\ell}_f(q, t_1) = \frac{1}{2\sqrt{t_1}}\tilde{L}_f(q, t_1)$. By construction, we have $\sigma_{q_*,t_*} = \eta|_{[0,t_*]}$ and hence $\tilde{L}_f(q, t_1)$ is a smooth upper barrier for $L_f(q, t_1)$ with $\tilde{L}_f(q_*, t_*) = L_f(q_*, t_*)$. Moreover, $\tilde{L}_f$ satisfies the formulas in Lemma 6.9 and Lemma 6.10. Thus $\tilde{\ell}_f(q, t_1)$ is a smooth upper barrier for $\ell_f(q, t_1)$ that satisfies (6.35).

To see that (6.35) holds in the distributional sense, we use the general fact that if a differential inequality of the type (6.32) holds in the barrier sense and we have a bound on $|\nabla L_f|$, then the inequality also holds in the distributional sense, see for example [10, Lemma 7.125]. Using the same lemma, (6.18) and (6.19) also hold in the distributional sense, since they hold in the barrier sense. Together, the claim from the lemma follows. □

Proof of Theorem 6.4. Since (6.35) and hence also (6.36) hold in the distributional sense, we simply compute, using $\frac{\partial}{\partial t}dV = \hat{-}SdV$,

$$\begin{aligned}\frac{d}{dt}\tilde{V}_f(t) &= \int_M v_f(q,t)\tfrac{\partial}{\partial t}dV + \int_M \tfrac{\partial}{\partial t}v_f(q,t)dV \\ &\leq \int v_f(q,t)\cdot(\hat{-}S)dV \,\hat{+} \int_M (S - \triangle)v_f(q,t)dV \\ &= \hat{-}\int_M \triangle v_f(q,t)dV = 0.\end{aligned} \tag{6.37}$$

Thus, the reduced volume $\tilde{V}_f(t)$ is non-increasing in t. □

No local collapsing

The following section follows Perelman's results for the Ricci flow from [48, Section 7], as well as the notes on his paper by Kleiner and Lott [36] and the book by Morgan and Tian [42]. We have seen in Chapter 3 that the metrics $g(t)$ along the $(RH)_\alpha$ flow are uniformly equivalent as long as the curvature on M stays uniformly bounded. But it could happen that at a singularity (i.e. when Rm blows up) the solution collapses geometrically in the following sense

Definition 6.12
Let $(g(t),\phi(t))_{t\in[0,T)}$ be a maximal solution of $(RH)_\alpha$, or more generally of any flow of the form $\frac{\partial}{\partial t}g_{ij} = -2S_{ij}$. We say that this solution is locally collapsing at time T, if there is a sequence of times $t_k \nearrow T$ and a sequence of balls $B_k := B_{g(t_k)}(x_k, r_k)$ at time t_k, such that the following holds. The ratio r_k^2/t_k is bounded, the curvature satisfies $|\mathrm{Rm}| \leq r_k^{-2}$ on the parabolic neighborhood $B_k \times [t_k - r_k^2, t_k]$ and $r_k^{-m}\operatorname{vol}(B_k) \to 0$ as $k \to \infty$.

Using the monotonicity of the reduced volume, we obtain the following result.

Theorem 6.13
Let $(g(t),\phi(t))_{t\in[0,T)}$ be a solution of $(RH)_\alpha$ with non-increasing $\alpha(t) \in [\underline{\alpha}, \bar{\alpha}]$, $0 < \underline{\alpha} \leq \bar{\alpha} < \infty$ and $T < \infty$. Then this solution is not locally collapsing at T.

Remark. For the proof, we only need the interior gradient estimates from Proposition A.5 (a local version of Theorem 3.10) and the monotonicity of the backwards reduced volume. Hence, every flow $\frac{\partial}{\partial t}g = -2\mathcal{S}$ that satisfies the assumption of Theorem 6.4 and some interior estimates for $\mathcal{S}$, ∇S in the spirit of Proposition A.5 will also satisfy the non-collapsing result. Note that for the $(RH)_\alpha$ flow, it is possible to obtain a slightly stronger result using the monotonicity of $\mu_\alpha(g,\phi,\tau)$ from Chapter 5 instead of the monotonicity of the backwards reduced volume. In the special case of List's flow (0.6), this can be found in List's dissertation [41, Section 7]. The proof in the case of $(RH)_\alpha$ is completely analogous. However, the result here is more general in the sense that it may be adopted to other flows $\frac{\partial}{\partial t}g = -2\mathcal{S}$ in the way explained above. This is the reason why we prefer this proof here.

Proof. The proof is by contradiction. Assume that there is some sequence of times $t_k \nearrow T$ and some sequence of balls $B_k := B_{g(t_k)}(x_k, r_k)$ at each time t_k, such that r_k^2 is bounded, the curvature is bounded by $|\mathrm{Rm}| \leq r_k^{-2}$ on the parabolic neighborhood $B_k \times [t_k - r_k^2, t_k]$ and $r_k^{-m}\operatorname{vol}(B_k) \to 0$ as $k \to \infty$. Define $\varepsilon_k := r_k^{-1}\operatorname{vol}(B_k)^{1/m}$, then $\varepsilon_k \to 0$ for $k \to \infty$. For each k, we set $\tau_k(t) = t_k - t$ and define the backwards reduced volume $\tilde{V}_k$ using curves going backward in real time from the base point (x_k, t_k), i.e. forward in time τ_k from $\tau_k = 0$. The goal is to estimate the reduced volumes $\tilde{V}_k(\varepsilon_k r_k^2)$, where $\tau_k = \varepsilon_k r_k^2$ corresponds to the real time $t = t_k - \varepsilon_k r_k^2$, which is very close to t_k and hence close to T. We first prove

Claim 1: $\lim_{k\to\infty} \tilde{V}_k(\varepsilon_k r_k^2) = 0$.

Proof. As we have seen above, an $\mathcal{L}_b$-geodesic $\eta(\tau)$ starting at $\eta(0) = x_k$ is uniquely defined through its initial vector $v = \lim_{\tau\to 0} 2\sqrt{\tau}X = \lim_{\lambda\to 0}\tilde{X}$. We first show that if $|v| \leq \frac{1}{8}\varepsilon_k^{-1/2}$ with respect to the metric at (x_k, t_k), then $\eta(\tau)$ does not escape from $B_k^{1/2} := B_{g(t_k)}(x_k, r_k/2)$ in time $\tau = \varepsilon_k r_k^2$. Write $\hat{t}_k = t_k - r_k^2$. Since $|\mathrm{Rm}| \leq r_k^{-2}$ on

$B_k \times [\hat{t}_k, t_k]$ by assumption, we obtain from Proposition A.5

$$|\nabla\phi|^2 \leq \frac{C}{t-\hat{t}_k}, \quad |\nabla^2\phi|^2 \leq \frac{C}{(t-\hat{t}_k)^2}, \quad |\nabla \mathrm{Rm}|^2 \leq \frac{C}{(t-\hat{t}_k)^3}, \quad \text{on } B_k^{1/2} \times (\hat{t}_k, t_k),$$

for some constant C independent of k. Without loss of generality, $\varepsilon_k \leq \frac{1}{2}$ so that $t - \hat{t}_k \geq \frac{1}{2}r_k^2$ whenever $t \in [t_k - \varepsilon_k r_k^2, t_k)$. This means that

$$|\nabla\phi|^2 \leq C r_k^{-2}, \quad |\nabla^2\phi| \leq C r_k^{-2} \quad |\nabla \mathrm{Rm}| \leq C r_k^{-3}, \quad \text{on } B_k^{1/2} \times [t_k - \varepsilon_k r_k^2, t_k).$$

Together with the assumption $|\mathrm{Rm}| \leq r_k^{-2}$, this yields

$$\begin{aligned} |\mathcal{S}| &\leq |\mathrm{Rc}| + |\nabla\phi|^2 \leq C r_k^{-2}, \\ |\nabla S| &\leq |\nabla R| + |\nabla\phi||\nabla^2\phi| \leq C r_k^{-3}, \end{aligned} \tag{6.38}$$

on $B_k^{1/2} \times [t_k - \varepsilon_k r_k^2, t_k)$. Plugging this into the estimate (6.12), we get

$$\tfrac{\partial}{\partial\lambda}|\tilde{X}| \leq \lambda C |\tilde{X}| r_k^{-2} + \lambda^2 C r_k^{-3} \leq C|\tilde{X}|\varepsilon_k^{1/2} r_k^{-1} + C\varepsilon_k r_k^{-1}, \tag{6.39}$$

for $\lambda = \sqrt{\tau} \leq \sqrt{\varepsilon_k r_k^2} = \varepsilon_k^{1/2} r_k$. Since $|\tilde{X}(0)| = |v| \leq \frac{1}{8}\varepsilon_k^{-1/2}$ we obtain the estimate $|\tilde{X}(\lambda)| \leq \frac{1}{4}\varepsilon_k^{-1/2}$ for all $\tau \in [0, \varepsilon_k r_k^2]$ if k is large enough, i.e. ε_k small enough. With an integration, we find

$$\int_0^{\varepsilon_k r_k^2} |X(\tau)| d\tau = \int_0^{\sqrt{\varepsilon_k}\, r_k} |\tilde{X}(\lambda)| d\lambda \leq \int_0^{\sqrt{\varepsilon_k}\, r_k} \tfrac{1}{4}\varepsilon_k^{-1/2} d\lambda \leq \tfrac{1}{4} r_k.$$

Since the metrics $g(\tau = 0)$ and $g(\tau = \varepsilon_k r_k^2)$ are close to each other, the length of the curve η measured with respect to $g(\tau = 0) = g(t_k)$ will be at most $r_k/2$ for large enough k. This means that indeed

$$(\eta(\tau), t_k - \tau) \in B_k^{1/2} \times [t_k - \varepsilon_k r_k^2, t_k), \qquad \forall 0 < \tau \leq \varepsilon_k r_k^2. \tag{6.40}$$

With the bounds from (6.38) and the lower bound in (6.13), we obtain

$$\mathcal{L}_b(\eta) \geq -C r_k^{-2}(\varepsilon_k r_k^2)^{3/2} = -C\varepsilon_k^{3/2} r_k, \quad \text{i.e.} \quad \ell_b(q, \varepsilon_k r_k^2) \geq -C\varepsilon_k$$

Thus, the contribution to the reduced volume $\tilde{V}_k(\varepsilon_k r_k^2)$ coming from $\mathcal{L}_b$-geodesics with initial vector $|v| \leq \frac{1}{8}\varepsilon_k^{-1/2}$ is bounded above for large k by

$$\int_{B_k^{1/2}} (4\pi\varepsilon_k r_k^2)^{-m/2} e^{C\varepsilon_k} dV \leq C\varepsilon_k^{-m/2} r_k^{-m} \operatorname{vol}(B_k^{1/2}) \leq C\varepsilon_k^{m/2} \to 0 \quad (k \to \infty).$$

Next, we compute the contribution of geodesics with large initial vector $|v| > \frac{1}{8}\varepsilon_k^{-1/2}$ to the reduced volume $\tilde{V}_k(\varepsilon_k r_k^2)$. To this end, we note that we can write the reduced volume with base point (x_k, t_k) as

$$\tilde{V}_k(\tau_1) = \int_M (4\pi\tau_1)^{-m/2} e^{-\ell(q,\tau_1)} dV(q) = \int_{\Omega(\tau_1,k)} (4\pi\tau_1)^{-m/2} e^{-\ell(\mathcal{L}_b\exp_{x_k}^{\tau_1}(v),\tau_1)} J(v,\tau_1) dv.$$

Here, $\mathcal{L}_b\exp_{x_k}^{\tau_1}$ is the $\mathcal{L}_b$-exponential map defined in a previous section, taking v to $\eta(\tau_1)$ with η being the $\mathcal{L}_b$-geodesic with initial vector v, $J(v,\tau_1) = \det d(\mathcal{L}_b\exp_{x_k}^{\tau_1})$ denotes the Jacobian of $\mathcal{L}_b\exp_{x_k}^{\tau_1}$ and $\Omega(\tau_1, k) \subset T_{x_k}M$ is a set which is mapped bijectively to M up to a set of measure zero under the map $\mathcal{L}_b\exp_{x_k}^{\tau_1}$. We claim that the integrand

$$f(v,\tau_1) := (4\pi\tau_1)^{-m/2} e^{-\ell(\mathcal{L}_b\exp_{x_k}^{\tau_1}(v),\tau_1)} J(v,\tau_1) \tag{6.41}$$

is non-increasing in τ_1 for fixed v and has the limit $\lim_{\tau_1\to 0} f(v,\tau_1) = \pi^{-m/2}e^{-|v|^2}$. To prove this, let $\eta(\tau)$ be an $\mathcal{L}_b$-geodesic with initial vector v. We note that by (6.21) and (6.23) and the first variation formula

$$\tfrac{d}{d\tau}\big|_{\tau=\tau_1} L_b(\eta(\tau),\tau) = \sqrt{\tau_1}(S + |X|^2) = -\tfrac{1}{\tau_1}K + \tfrac{1}{2\tau_1}L_b(\eta(\tau_1),\tau_1)$$

or equivalently

$$\tfrac{d}{d\tau}\big|_{\tau=\tau_1} \ell_b(\eta(\tau),\tau) = -\tfrac{1}{2\tau_1^{3/2}}K. \tag{6.42}$$

To estimate the Jacobian, let $\{Y_i\}$ be a basis for the Jacobi fields along η which vanish at $\tau = 0$ and satisfy $\langle Y_i(\tau_1), Y_2(\tau_1)\rangle = \delta_{ij}$. This yields

$$\tfrac{d}{d\tau}\big|_{\tau=\tau_1} \log J(v,\tau) = \tfrac{d}{d\tau}\big|_{\tau=\tau_1} \log\sqrt{\det(Y_i(\tau), Y_j(\tau))} = \tfrac{1}{2}\sum_{i=1}^m \tfrac{\partial}{\partial\tau}\big|_{\tau=\tau_1}|Y_i|^2.$$

Next, note that for the family of geodesics $\eta_s(\tau)$ that corresponds to the Jacobi field Y_i, we find

$$\mathrm{Hess}_{L_b}(Y_i(\tau_1), Y_i(\tau_1)) = \tfrac{d^2}{ds^2} L_b(\eta_s(\tau_1),\tau_1) = 2\sqrt{\tau_1}\langle \nabla_X Y(\tau_1), Y(\tau_1)\rangle, \tag{6.43}$$

and hence

$$\begin{aligned}\tfrac{\partial}{\partial\tau}\big|_{\tau=\tau_1}|Y_i|^2 &= 2\mathcal{S}(Y_i(\tau_1), Y_i(\tau_1)) + 2\langle \nabla_X Y_i, Y_i\rangle(\tau_1)\\ &= 2\mathcal{S}(Y_i(\tau_1), Y_i(\tau_1)) + \tfrac{1}{\sqrt{\tau_1}}\mathrm{Hess}_{L_b}(Y_i(\tau_1), Y_i(\tau_1)).\end{aligned} \tag{6.44}$$

Let $\tilde{Y}_i$ be solutions of the ODE (6.29) which agree with Y_i at time τ_1. Then, completely analogous to our computation of $\triangle L$ in the previous section, we have

$$\begin{aligned}\tfrac{1}{2}\sum_{i=1}^m \tfrac{\partial}{\partial\tau}\big|_{\tau=\tau_1}|Y_i|^2 &= \tfrac{1}{2\sqrt{\tau_1}}\sum_i \Big(2\sqrt{\tau_1}\mathcal{S}(Y_i(\tau_1), Y_i(\tau_1)) + \mathrm{Hess}_{L_b}(Y_i(\tau_1), Y_i(\tau_1))\Big)\\ &\le \tfrac{1}{2\sqrt{\tau_1}}\sum_i \Big(\tfrac{1}{\sqrt{\tau_1}} - \int_0^{\tau_1}\sqrt{\tau}\mathcal{H}(\mathcal{S}, X, \tilde{Y}_i)d\tau\Big)\\ &\le \tfrac{1}{2\sqrt{\tau_1}}\Big(\tfrac{m}{\sqrt{\tau_1}} - \tfrac{1}{\tau_1}K\Big) = \tfrac{m}{2\tau_1} - \tfrac{1}{2\tau_1^{3/2}}K.\end{aligned}$$

From this estimate and the estimate (6.42), we conclude

$$\begin{aligned}\tfrac{d}{d\tau}\big|_{\tau=\tau_1}\log\big((4\pi)^{m/2}f(v,\tau)\big) &= -\tfrac{m}{2\tau_1} - \tfrac{d}{d\tau}\big|_{\tau=\tau_1}\ell_b(\eta(\tau),\tau) + \tfrac{d}{d\tau}\big|_{\tau=\tau_1}\log J(v,\tau)\\ &\le -\tfrac{m}{2\tau_1} + \tfrac{1}{2\tau_1^{3/2}}K + \big(\tfrac{m}{2\tau_1} - \tfrac{1}{2\tau_1^{3/2}}K\big) = 0,\end{aligned}$$

which proves the claimed monotonicity.

Remark. Together with $\Omega(\tau') \subset \Omega(\tau)$ for $\tau \leq \tau'$ this yields an alternative proof of the monotonicity of the reduced volumes.

The limit statement for f follows directly from the fact, that for $\tau \to 0$ the reduced volume approaches the reduced volume on Euclidean space. Thus, the contribution to the reduced volume $\tilde{V}_k(\varepsilon_k r_k^2)$ coming from $\mathfrak{L}_b$-geodesics with initial vector $|v| > \frac{1}{8}\varepsilon_k^{-1/2}$ can be bounded by

$$\int_{|v|>\frac{1}{8}\varepsilon_k^{-1/2}} \pi^{-m/2} e^{-|v|^2} dv \leq C e^{-\frac{1}{64\varepsilon_k}} \to 0 \quad (k \to \infty), \tag{6.45}$$

which completes the proof of Claim 1. □

Claim 2: $\tilde{V}_k(t_k)$ is bounded below away from zero.

Proof. Let us remark that $\tau = t_k$ corresponds to real time $t = 0$. We assume that k is large enough, so that $t_k \geq T/2$. The idea behind the proof is to go from (x_k, t_k) to some point q_k at the real time $T/2$ (i.e. $\tau = t_k - T/2$) for which the reduced $\mathfrak{L}_b$-distance $\ell_b(q_k, t_k - T/2)$ is small. From the upper bound on L_b from Lemma 6.7 we see that for small τ it is possible to find a point $q_k(\tau)$ such that $\ell_b(q_k(\tau), \tau) \leq \frac{m}{2}$. On the other hand, combining the evolution equations for $\frac{\partial}{\partial\tau}\ell_b$ and $\triangle\ell_b$, we obtain

$$\frac{\partial}{\partial\tau}\Big|_{\tau=\tau_1} \ell_b + \triangle\ell_b \leq -\frac{1}{\tau_1}\ell_b + \frac{m}{2\tau_1} \tag{6.46}$$

(in the barrier sense) and hence for the minimum of $\ell_{min}(\tau) = \min_{q\in M} \ell_b(q,\tau)$

$$\frac{\partial}{\partial\tau}\Big|_{\tau=\tau_1} \ell_{min} \leq -\frac{1}{\tau_1}\ell_{min} + \frac{m}{2\tau_1} \tag{6.47}$$

(in the sense of difference quotients). The latter is obtained by applying the maximum principle to a smooth barrier. The inequality (6.47) shows that there is some point $q_k(\tau)$ with $\ell_b(q_k(\tau), \tau) \leq \frac{m}{2}$ for every τ. As mentioned above, we choose q_k at the real time $T/2$ with $\ell_b(q_k, t_k - T/2) \leq \frac{m}{2}$. Let $\eta : [0, t_k - T/2] \to M$ be an $\mathfrak{L}_b$-geodesic realizing this length. Moreover, let $\eta_p : [t_k - T/2, t_k] \to M$ be $g(t=0)$-geodesics (i.e. a $g(\tau = t_k)$-geodesic) from q_k at time $\tau = t_k - T/2$ to $p \in B^{q_k} := B_{g(\tau=t_k)}(q_k, 1) = B_{g(t=0)}(q_k, 1)$ at time $\tau = t_k$. Since $|\mathrm{Rm}|$ is uniformly bounded for $t \in [0, T/2]$ (i.e. $\tau \in [t_k - T/2, t_k]$), we get a uniform bound for S along this family of $g(\tau = t_k)$-geodesics. Since all the metrics $g(\tau)$ with $\tau \in [t_k - T/2, t_k]$ are uniformly equivalent, we get an uniform upper bound for the $\mathfrak{L}_b$-length of all η_p. From this, we see that the concatenations $(\eta \smile \eta_p) : [0, t_k] \to M$ connecting x_k to $p \in B^{q_k}$ have uniformly bounded $\mathfrak{L}_b$-length, independent of p and k. This gives a uniform bound $\ell_b(p, t_k) \leq C$, for all $p \in B^{q_k}$ and $k \in \mathbb{N}$ large enough. We can then estimate

$$\begin{aligned} \tilde{V}_k(t_k) &= \int_M (4\pi t_k)^{-m/2} e^{-\ell_b(q,t_k)} dV(q) \\ &\geq \int_{B^{q_k}} (4\pi t_k)^{-m/2} e^{-C} dV \\ &\geq C \inf_{q_k\in M} \mathrm{vol}(B^{q_k}), \end{aligned} \tag{6.48}$$

which is bounded below away from zero, independently of k. This proves Claim 2. □

Since the backwards reduced volumes $\tilde{V}_k$ are non-increasing in τ (i.e. non-decreasing in real time t) according to Theorem 6.4, we obtain $\tilde{V}_k(t_k) \leq \tilde{V}_k(\varepsilon_k r_k^2)$ for k large enough. But since $\tilde{V}_k(t_k)$ is bounded below away from zero by Claim 2 while $\tilde{V}_k(\varepsilon_k r_k^2)$ converges to zero with $k \to \infty$ by Claim 1, we obtain the desired contradiction that proves the theorem. □

Appendix

Background material from geometric analysis

i) Basic Riemannian geometry. Let TM and T^*M denote the tangent and co-tangent bundle of a manifold M, respectively. Then a (p,q)-tensor B is a smooth section of the bundle $(T^*M)^{\otimes p} \otimes (TM)^{\otimes q}$. In a system of local coordinates $(x^1, \ldots, x^n)$, induced by a chart $\phi : U \to \mathbb{R}^n$, $U \subseteq M$, the tensor B has the coordinate representation

$$B = B^{k_1 \ldots k_q}_{\ell_1 \ldots \ell_p} \, dx^{\ell_1} \otimes \ldots \otimes dx^{\ell_p} \otimes \tfrac{\partial}{\partial x^{k_1}} \otimes \ldots \otimes \tfrac{\partial}{\partial x^{k_q}},$$

with

$$B^{k_1 \ldots k_q}_{\ell_1 \ldots \ell_p} = B\big(\tfrac{\partial}{\partial x^{\ell_1}}, \ldots, \tfrac{\partial}{\partial x^{\ell_p}}, dx^{k_1}, \ldots, dx^{k_q}\big).$$

Slightly abusing this notation, we often write $B = B^{k_1 \ldots k_q}_{\ell_1 \ldots \ell_p}$ as an abbreviation. In particular, we write v^i for vectors and v_j for co-vectors, meaning $v^i \frac{\partial}{\partial x^i}$ and $v_j dx^j$, respectively. We also abbreviate the derivative $\frac{\partial}{\partial x^k}$ by ∂_k.

The Riemannian metric on M is denoted by $g_{ij} = g_{ji}$, its inverse by $g^{k\ell}$, so that $g_{ij} g^{jk} = \delta^k_i$. For the induced measure on M we write $dV = \sqrt{\det(g_{ij})}\, dx$. We write $\langle X, Y \rangle := g(X,Y) = g_{ij} X^i Y^j$ for the induced inner product of the metric g. We always use the extended Einstein summation convention, where $X^k Y_k$ means $\sum_{k=1}^n X^k Y_k$, and $X_k Y_k$ denotes $g^{k\ell} X_k Y_\ell = \sum_{k,\ell=1}^n g^{k\ell} X_k Y_\ell$.

The Levi-Cività connection $\nabla_i v^k = \partial_i v^k + \Gamma^k_{ij} v^j$ is determined by the Christoffel symbols

$$\Gamma^k_{ij} := \frac{1}{2} g^{k\ell} (\partial_i \, g_{j\ell} + \partial_j \, g_{i\ell} - \partial_\ell \, g_{ij}), \tag{A.1}$$

and the Riemannian curvature tensor

$$\mathrm{Rm}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = R^k_{ij\ell} X^i Y^j Z^\ell$$

has the local representation

$$R^k_{ij\ell} = \partial_i \, \Gamma^k_{j\ell} - \partial_j \, \Gamma^k_{i\ell} + \Gamma^m_{j\ell} \, \Gamma^k_{im} - \Gamma^m_{i\ell} \, \Gamma^k_{jm}. \tag{A.2}$$

If we lower the upper index to the third position, we get $R_{ijk\ell} = g_{kh} R^h_{ij\ell}$, a tensor which is anti-symmetric in (i,j) and (k,ℓ), and symmetric in the interchange of these pairs,

$R_{k\ell ij} = R_{ijk\ell}$. The Ricci tensor Rc is defined by $R_{ik} = g^{j\ell} R_{ijk\ell}$, the scalar curvature as its trace $R = g^{ik} R_{ik} = g^{ik} g^{j\ell} R_{ijk\ell}$. It is easy to see that

$$R_{ki} = R_{ik} = g^{h\ell} R_{hi\ell k} = g^{h\ell} g_{\ell j} R^j_{hik} = R^j_{jik}.$$

It is a standard fact that in Riemannian normal coordinates (induced by the exponential map) we have

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ipjq} x^p x^q + O(|x|^3), \tag{A.3}$$

see for example [45, Lemma 1.1]. In particular, this implies

$$g_{ij}(0) = \delta_{ij}, \quad \partial_k\, g_{ij}(0) = 0, \quad \Gamma^k_{ij}(0) = 0.$$

Another important symmetry of the curvature tensor is the first Bianchi identity

$$\mathrm{Rm}(X,Y)Z + \mathrm{Rm}(Y,Z)X + \mathrm{Rm}(Z,X)Y = 0.$$

The Levi-Cività connection $\nabla = \nabla^{TM}$ on the tangent bundle TM over (M,g) induces a connection ∇^{T^*M} on the co-tangent bundle T^*M via

$$\big(\nabla^{T^*M}_X \omega\big)(Y) = X(\omega(Y)) - \omega\big(\nabla^{TM}_X Y\big), \qquad X, Y \in \Gamma(TM),\ \omega \in \Gamma(T^*M).$$

With the product rule and compatibility with contractions we obtain natural connections ∇ on all bundles $(T^*M)^{\otimes p} \otimes (TM)^{\otimes q}$ which we call again Levi-Cività connections. In particular, we obtain covariant derivatives of the Riemannian metric (which always vanish, $\nabla_k g_{ij} = 0\ \forall i,j,k$) and of the different curvature tensors. The Riemannian curvature tensor satisfies the second Bianchi identity

$$\langle \nabla_X \mathrm{Rm}(Y,Z)V, W\rangle + \langle \nabla_Y \mathrm{Rm}(Z,X)V, W\rangle + \langle \nabla_Z \mathrm{Rm}(X,Y)V, W\rangle = 0,$$

which implies the formula

$$\nabla_h R_{ijk\ell} + \nabla_k R_{ij\ell h} + \nabla_\ell R_{ijhk} = 0 \tag{A.4}$$

in coordinates. Tracing this with g^{ih} yields

$$\nabla_i R_{ijk\ell} = g^{ih} \nabla_h R_{ijk\ell} = -g^{ih} \nabla_k R_{ij\ell h} - g^{ih} \nabla_\ell R_{ijhk} = \nabla_k R_{j\ell} - \nabla_\ell R_{jk}.$$

By tracing again with $g^{j\ell}$, we get

$$\nabla_i R_{ik} = g^{j\ell} \nabla_i R_{ijk\ell} = g^{j\ell} \nabla_k R_{j\ell} - g^{j\ell} \nabla_\ell R_{jk} = \nabla_k R - \nabla_j R_{jk}.$$

By changing the indices in the last term, this implies

$$\nabla_i R_{ik} = \tfrac{1}{2} \nabla_k R. \tag{A.5}$$

From the definition of the Riemannian curvature tensor, we see that for a vector field X in local coordinates

$$[\nabla_i, \nabla_j] X^k = \nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k = R^k_{ijm} X^m = g^{k\ell} R_{ij\ell m} X^m. \tag{A.6}$$

The commutator for a co-vector field θ can then be obtained by using the metric dual,

$$[\nabla_i, \nabla_j]\theta_\ell = \nabla_i\nabla_j\,\theta_\ell - \nabla_j\nabla_i\,\theta_\ell = g_{k\ell}R^k_{ijm}(g^{mq}\theta_q) = R_{ij\ell m}\theta_m.$$

Similarly, for a general (p,q)-tensor B, we find the commutator identity

$$[\nabla_i, \nabla_j]B^{k_1\dots k_q}_{\ell_1\dots\ell_p} = \sum_{r=1}^{q} R^{k_r}_{ijm}B^{k_1\dots k_{r-1}mk_{r+1}\dots k_q}_{\ell_1\dots\ell_p} + \sum_{s=1}^{p} R_{ij\ell_s m}B^{k_1\dots k_q}_{\ell_1\dots\ell_{s-1}m\ell_{s+1}\dots\ell_p}. \tag{A.7}$$

ii) Connections and commutator identities on more complicated bundles. Assume that we have given Levi-Cività connections for all (p,q) tensors over (M,g) and over (N,γ). In particular, they satisfy $\nabla g = 0$ and $\nabla\gamma = 0$. For a map $\phi : (M,g) \to (N,\gamma)$, there is a canonical notion of pull-back bundle ϕ^*TN over M with sections $\phi^*V = V \circ \phi$ for $V \in \Gamma(TN)$. The Levi-Cività connection ∇^{TN} on TN also induces a connection ∇^{ϕ^*TN} on this pull-back bundle via

$$\nabla^{\phi^*TN}_X \phi^*V = \phi^*\big(\nabla^{TN}_{\phi_*X}V\big), \qquad X \in \Gamma(TM),\ V \in \Gamma(TN).$$

Again, we obtain connections on all product bundles over M with factors TM, T^*M, ϕ^*TN and ϕ^*T^*N via the product rule and compatibility with contractions. Defining the metric $\tilde\gamma := \gamma \circ \phi$, then these connections satisfy $\nabla\tilde\gamma = 0 = \nabla g$.

Take coordinates x^k on M, $k = 1,\dots,m = \dim M$, and y^μ on N, $\mu = 1,\dots,n = \dim N$, and write ∂_k for $\frac{\partial}{\partial x^k}$ and ∂_μ for $\frac{\partial}{\partial y^\mu}$. The curvature tensor on ϕ^*TN is given by

$$\mathrm{Rm}(X,Y)V = \nabla_X\nabla_Y V - \nabla_Y\nabla_X V - \nabla_{[X,Y]}V = R^\kappa_{ij\lambda}X^iY^jV^\lambda$$

for $X = X^i\partial_i, Y = Y^j\partial_j \in \Gamma(TM)$ and $V = V^\lambda\phi^*(\partial_\lambda) \in \Gamma(\phi^*TN)$. To lower the index κ, we need the metric $\tilde\gamma = \gamma \circ \phi$,

$$R_{ij\mu\lambda} = \tilde\gamma_{\mu\kappa}R^\kappa_{ij\lambda}.$$

With this definition, we obtain the commutator rule

$$\nabla_i\nabla_j V_\kappa - \nabla_j\nabla_i V_\kappa = R_{ij\kappa\lambda}V_\lambda.$$

Let us compute $R_{ij\kappa\lambda}$ now. In the coordinates x^i, y^μ, this is

$$\begin{aligned} R_{ij\kappa\lambda}(x) &= \big\langle\, \mathrm{Rm}(\partial_i,\partial_j)\phi^*\big(\partial_\lambda\big), \phi^*\big(\partial_\kappa\big)\big\rangle_{\phi^*TN}(x) \\ &= \big\langle\, {}^N\mathrm{Rm}\big(\phi_*\partial_i, \phi_*\partial_j\big)\partial_\lambda, \partial_\kappa\big\rangle_{TN}(\phi(x)) \\ &= {}^N R_{\mu\nu\kappa\lambda}(\phi(x))\nabla_i\phi^\mu(x)\nabla_j\phi^\nu(x), \end{aligned}$$

where we used $\phi_*\partial_i = \nabla_i\phi^\mu\,\partial_\mu$. We can then extend (A.7) to mixed tensors, for example

$$[\nabla_i, \nabla_j]B^{k\kappa}_{\ell\lambda} = R^k_{ijp}B^{p\kappa}_{\ell\lambda} + R_{ij\ell p}B^{k\kappa}_{p\lambda} + R^\kappa_{ij\varrho}B^{k\varrho}_{\ell\lambda} + R_{ij\lambda\varrho}B^{k\kappa}_{\ell\varrho}. \tag{A.8}$$

The standard example that will be used quite often is the following. The derivative $\nabla\phi$ of $\phi : M \to N$ is a section of $T^*M \otimes \phi^*TN$. Thus, the intrinsic second order derivative is built with the connection on this bundle, i.e.

$$\nabla_i \nabla_j \phi^\lambda = \partial_i \partial_j \phi^\lambda - \Gamma_{ij}^k \partial_k \phi^\lambda + {}^N\Gamma_{\mu\nu}^\lambda \partial_i \phi^\mu \partial_j \phi^\nu \tag{A.9}$$

and similar for higher derivatives. Using (A.8), we obtain

$$\begin{aligned} \nabla_i \nabla_j \nabla_\ell \phi^\beta - \nabla_j \nabla_i \nabla_\ell \phi^\beta &= R_{ij\ell p} \nabla_p \phi^\beta + R_{ij\lambda}^\beta \nabla_\ell \phi^\lambda \\ &= R_{ij\ell p} \nabla_p \phi^\beta + {}^N R_{\mu\nu\lambda}^\beta \nabla_\ell \phi^\lambda \nabla_i \phi^\mu \nabla_j \phi^\nu. \end{aligned} \tag{A.10}$$

There is also a different way to obtain these formulas, which is especially useful when ϕ is evolving and we also want to include time derivatives. We learned this from [37]. Here, we interpret $\nabla^k\phi$ as a k-linear TN-valued map along $\phi \in C^\infty(M, N)$ rather than as a section in $(T^*M)^{\otimes k} \otimes \phi^*TN$. To obtain the desired formulas, let ω be any such k-linear TN-valued map along ϕ, i.e.

$$\omega(x) : (T_xM)^{\times k} \to T_{\phi(x)}N.$$

The covariant derivative $\nabla\omega$ is a $(k+1)$-linear TN-valued map along ϕ, defined by

$$(\nabla\omega)(X_0, \ldots, X_k) := X_0(\omega(X_1, \ldots, X_k)) - \sum_{s=1}^k \omega(X_1, \ldots, \nabla_{X_0} X_s, \ldots, X_k).$$

This is analogous to the definition of the covariant derivative of a 1-form above. The curvature tensor ${}^k\mathrm{Rm}$ for the k-linear map ω is defined by

$${}^k\mathrm{Rm}(X, Y)\omega = \nabla_X \nabla_Y \omega - \nabla_Y \nabla_X \omega - \nabla_{[X,Y]}\omega \tag{A.11}$$

and can be computed using

$$\begin{aligned} ({}^k\mathrm{Rm}(X, Y)\omega)(X_1, \ldots, X_k) = {}&{}^N\mathrm{Rm}(\nabla\phi(X), \nabla\phi(Y))\omega(X_1, \ldots, X_k) \\ &- \sum_{s=1}^k \omega(X_1, \ldots, \mathrm{Rm}(X, Y)X_s, \ldots, X_k). \end{aligned} \tag{A.12}$$

Of course, this agrees with the definition above, where we used the bundle interpretation. If ϕ is time-dependent, we simply interpret it as a map $\tilde{\phi} : M \times I \to N$ and interpret $\nabla^k\phi$ as k-linear TN-valued maps on $M \times I$ along $\tilde{\phi}$. The formalism stays exactly the same.

Note that $\frac{\partial}{\partial t}$ induces a covariant time derivative ∇_t (on all bundles over $M \times I$) that agrees with $\frac{\partial}{\partial t}$ for time-dependent functions. Choose coordinates x^i for M with

$$\nabla_t(\tfrac{\partial}{\partial t}) = \nabla_i(\partial_j) = \nabla_t(\partial_i) = \nabla_i(\tfrac{\partial}{\partial t}) = 0, \quad \forall i, j = 1, \ldots, m \tag{A.13}$$

at some base point (p, t) in $M \times I$. Then, from (A.11) and (A.12), we obtain for $\omega = \nabla\phi$

$$\begin{aligned} \nabla_t(\nabla_i \nabla_j \phi) &= \nabla_t((\nabla_i \omega)(\partial_j)) = (\nabla_t \nabla_i \omega)(\partial_j) = (\nabla_i \nabla_t \omega + {}^1\mathrm{Rm}(\tfrac{\partial}{\partial t}, \partial_i)\omega)(\partial_j) \\ &= \nabla_i((\nabla_t \omega)(\partial_j)) + {}^N\mathrm{Rm}(\tfrac{\partial}{\partial t}\phi, \nabla_i \phi)\omega(\partial_j) - \omega({}^{M\times I}\mathrm{Rm}(\tfrac{\partial}{\partial t}, \partial_i)\partial_j) \\ &= \nabla_i \nabla_j \tfrac{\partial}{\partial t}\phi + {}^N\mathrm{Rm}(\tfrac{\partial}{\partial t}\phi, \nabla_i \phi)\nabla_j \phi. \end{aligned} \tag{A.14}$$

Remark. If we also vary the metric g on M in time, we will get an additional term from the evolution of $\nabla_i\nabla_j$. From (A.9), we see that this additional term must be $-(\frac{\partial}{\partial t}\Gamma^k_{ij})\nabla_k\phi$. Note that $\frac{\partial}{\partial t}\Gamma$ is a tensor, while Γ itself is not.

iii) The maximum principle. The weak maximum principle for parabolic partial differential equations with a nonlinear reaction term states that a solution of the corresponding ordinary differential equation yields pointwise bounds for the solutions of the PDE. Since we work on an evolving manifold, we need a slightly generalised version. The following result is proved in [9, Theorem 4.4].

Proposition A.1
Let $u : M \times [0,T) \to \mathbb{R}$ be a smooth function satisfying

$$\tfrac{\partial}{\partial t}u \geq \triangle_{g(t)}u + \langle X(t), \nabla u\rangle_{g(t)} + F(u), \tag{A.15}$$

where $g(t)$ is a smooth 1-parameter family of metrics on M, $X(t)$ a smooth 1-parameter family of vector fields on M, and $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function. Suppose that $u(\cdot, 0)$ is bounded below by a constant $C_0 \in \mathbb{R}$ and let $\phi(t)$ be a solution to

$$\tfrac{\partial}{\partial t}\phi = F(\phi), \quad \phi(0) = C_0.$$

Then $u(x,t) \geq \phi(t)$ for all $x \in M$ and all $t \in [0,T)$ for which $\phi(t)$ exists.

Similarly, if (A.15) is replaced by

$$\tfrac{\partial}{\partial t}u \leq \triangle_{g(t)}u + \langle X(t), \nabla u\rangle_{g(t)} + F(u)$$

and $u(\cdot, 0)$ is bounded from above by C_0, then $u(x,t) \leq \phi(t)$ for all $x \in M$ and $t \in [0,T)$ for which the solution $\phi(t)$ of the corresponding ODE exists.

Collection of some additional proofs

i) The Ricci flow of warped products. This paragraph closely follows a result by Li [39]. Let (M^m, g_{ij}) be a Riemannian manifold and let $(\tilde{M}, \tilde{g})$ be the product $\tilde{M} = S^1 \times M^m$ of M^m and a circle S^1, equipped with the warped product metric $\tilde{g} = \psi^2(dx^0)^2 + g_{ij}\,dx^i \otimes dx^j$, where $0 < \psi \in C^\infty(M)$.

Lemma A.2
The Ricci flow $\frac{\partial}{\partial t}\tilde{g} = -2\,\mathrm{Rc}(\tilde{g})$ on $\tilde{M}$ is given by the system of equations

$$\begin{cases} \frac{\partial}{\partial t}g_{ij} = -2R_{ij} + \frac{2}{\psi}\nabla_i\nabla_j\psi, \\ \frac{\partial}{\partial t}\psi = \triangle_g\psi. \end{cases} \tag{A.16}$$

Proof. Since the components

$$\tilde{R}^k_{ij\ell} = \big(\partial_i\tilde{\Gamma}^k_{j\ell} + \tilde{\Gamma}^p_{j\ell}\tilde{\Gamma}^k_{ip}\big) - \big(\partial_j\tilde{\Gamma}^k_{i\ell} + \tilde{\Gamma}^p_{i\ell}\tilde{\Gamma}^k_{jp}\big)$$

of the Riemann curvature tensor of $(\tilde{M},\tilde{g})$ obviously agree with the components $R^k_{\ell ij}$ of the curvature tensor of (M,g) if all indices differ from zero, it is convenient to consider the other components separately. Let $i \neq 0$ and $j \neq 0$. Then we find for the Ricci tensor of $(\tilde{M},\tilde{g})$

$$\tilde{R}_{0j} = \tilde{R}_{i0} = 0, \qquad \tilde{R}_{00} = \tilde{R}^i_{i00}, \qquad \tilde{R}_{ij} = R_{ij} + \tilde{R}^0_{0ij},$$

and the Ricci flow takes the form (using $\tilde{g}_{00} = \psi^2$)

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2R_{ij} - 2\tilde{R}^0_{0ij}, \\ \frac{\partial}{\partial t}\psi = \frac{1}{2\psi}\frac{\partial}{\partial t}\tilde{g}_{00} = -\frac{1}{\psi}\tilde{R}^i_{i00}. \end{cases} \tag{A.17}$$

We compute, using $\tilde{g}_{00} = \psi^2$,

$$\begin{aligned} \tilde{R}^0_{0ij} &= \big(\partial_0\tilde{\Gamma}^0_{ij} + \tilde{\Gamma}^p_{ij}\tilde{\Gamma}^0_{0p}\big) - \big(\partial_i\tilde{\Gamma}^0_{0j} + \tilde{\Gamma}^p_{0j}\tilde{\Gamma}^0_{ip}\big) \\ &= \big(0 + \tfrac{1}{2}\Gamma^p_{ij}\tilde{g}^{00}\partial_p\tilde{g}_{00}\big) - \big(\tfrac{1}{2}\partial_i\big(\tilde{g}^{00}\partial_j\tilde{g}_{00}\big) + \tfrac{1}{4}\big(\tilde{g}^{00}\partial_j\tilde{g}_{00}\big)\big(\tilde{g}^{00}\partial_i\tilde{g}_{00}\big)\big) \\ &= \tfrac{1}{2\psi^2}\Gamma^p_{ij}\partial_p\psi^2 - \tfrac{1}{2}\partial_i\big(\tfrac{1}{\psi^2}\partial_j\psi^2\big) - \tfrac{1}{4\psi^4}\partial_i\psi^2\partial_j\psi^2 \\ &= \tfrac{1}{\psi}\Gamma^p_{ij}\partial_p\psi - \big(\tfrac{1}{\psi}\partial_i\partial_j\psi - \tfrac{1}{\psi^2}\partial_i\psi\partial_j\psi\big) - \tfrac{1}{\psi^2}\partial_i\psi\partial_j\psi \\ &= -\tfrac{1}{\psi}\nabla_i\nabla_j\psi, \end{aligned}$$

as well as

$$\begin{aligned} \tilde{R}^i_{i00} &= \big(\partial_i\tilde{\Gamma}^i_{00} + \tilde{\Gamma}^p_{00}\tilde{\Gamma}^i_{ip}\big) - \big(\partial_0\tilde{\Gamma}^i_{i0} + \tilde{\Gamma}^p_{i0}\tilde{\Gamma}^i_{0p}\big) \\ &= \big(-\tfrac{1}{2}\partial_i\big(g^{i\ell}\partial_\ell\tilde{g}_{00}\big) - \tfrac{1}{4}\big(g^{p\ell}\partial_\ell\tilde{g}_{00}\big)\big(g^{ik}\partial_p g_{ik}\big)\big) - \big(0 - \tfrac{1}{4}\big(\tilde{g}^{00}\partial_i\tilde{g}_{00}\big)\big(g^{i\ell}\partial_\ell\tilde{g}_{00}\big)\big) \\ &= -\tfrac{1}{2}\,\big(g^{i\ell}\partial_i\partial_\ell\tilde{g}_{00} - \big(g^{ik}g^{p\ell}\partial_i g_{pk} - \tfrac{1}{2}g^{p\ell}g^{ik}\partial_p g_{ik}\big)\partial_\ell\tilde{g}_{00}\big) + \tfrac{1}{4}\tilde{g}^{00}\big(g^{i\ell}\partial_i\tilde{g}_{00}\partial_\ell\tilde{g}_{00}\big) \\ &= -\tfrac{1}{2}\big(g^{i\ell}\partial_i\partial_\ell\tilde{g}_{00} - g^{ik}\Gamma^\ell_{ik}\partial_\ell\tilde{g}_{00}\big) + \tfrac{1}{4}\tilde{g}^{00}|\nabla\tilde{g}_{00}|^2 \\ &= -\tfrac{1}{2}\triangle_g\psi^2 + \tfrac{1}{4\psi^2}|\nabla\psi^2|^2 \\ &= -\psi\triangle_g\psi. \end{aligned}$$

□

Lemma A.3
If we define $\phi := \frac{1}{\sqrt{2}}\log\psi$, such that $\psi = e^{\sqrt{2}\phi}$, the Ricci flow of the warped product metric $\tilde{g}$ on $\tilde{M}$ is equivalent to

$$\begin{cases} \frac{\partial}{\partial t} g = -2\mathrm{Rc} + 4\nabla\phi\otimes\nabla\phi, \\ \frac{\partial}{\partial t}\phi = \triangle_g\phi, \end{cases} \tag{A.18}$$

modulo the action of a family of diffeomorphisms. This is a special case of $(RH)_\alpha$ with $N = \mathbb{R}$ and $\alpha \equiv 2$.

Proof. Using Lemma A.2, we obtain

$$\begin{aligned}\tfrac{\partial}{\partial t} g_{ij} &= -2R_{ij} + \tfrac{2}{\psi}\nabla_i\nabla_j\psi \\ &= -2R_{ij} + 2e^{-\sqrt{2}\phi}\nabla_i\nabla_j\big(e^{\sqrt{2}\phi}\big) \\ &= -2R_{ij} + 4\nabla_i\phi\nabla_j\phi + 2\sqrt{2}\nabla_i\nabla_j\phi,\end{aligned}$$

as well as

$$\tfrac{\partial}{\partial t}\phi = \tfrac{1}{\sqrt{2}}e^{-\sqrt{2}\phi}\tfrac{\partial}{\partial t}\psi = \tfrac{1}{\sqrt{2}}e^{-\sqrt{2}\phi}\triangle_g\big(e^{\sqrt{2}\phi}\big) = \triangle_g\phi + \sqrt{2}|\nabla\phi|^2.$$

Let $X = -\sqrt{2}\nabla\phi$. Then we find for the Lie derivatives $(\mathfrak{L}_X g)_{ij} = -2\sqrt{2}\nabla_i\nabla_j\phi$ and $\mathfrak{L}_X\phi = -\sqrt{2}|\nabla\phi|^2$. Thus the Ricci flow of the warped product metric may be written as

$$\begin{cases} \tfrac{\partial}{\partial t} g = -2\mathrm{Rc} + 4\nabla\phi\otimes\nabla\phi - \big(\mathfrak{L}_X g\big), \\ \tfrac{\partial}{\partial t}\phi = \triangle_g\phi - \big(\mathfrak{L}_X\phi\big). \end{cases} \tag{A.19}$$

Therefore, if Ψ_t denotes the family of diffeomorphisms generated by X, then the pull-backs $\Psi_t^* g$, $\Psi_t^*\phi$ satisfy (A.18). □

ii) Conformal covariance of the operator L_m^μ. In (1.10), we introduced the operator

$$L_g(\psi) = L_m^\mu(\psi) = \triangle_g\psi - \langle\nabla\psi,\nabla f\rangle_g + \frac{1}{m-2}(\triangle_g f + R_g)\psi$$

where f is the density function with $d\mu = e^{-f}dV_g$. If we change the metric conformally, $h = e^{2\omega}g$ while keeping the measure μ fixed, the operator becomes

$$L_h(\psi) = \triangle_h\psi - \langle\nabla\psi,\nabla\hat{f}\rangle_h + \frac{1}{m-2}(\triangle_h\hat{f} + R_h)\psi,$$

where $\hat{f}$ is the density function with $d\mu = e^{-\hat{f}}dV_h$. Since $e^{-\hat{f}}dV_h = d\mu = e^{-f}dV_g = e^{-f}e^{-m\omega}dV_h$, the density functions are related by $\hat{f} = f + m\omega$. The goal of this paragraph is the following proposition.

Proposition A.4
L_g is conformally covariant with bi-degree $(-1,1)$, i.e. $e^{-\omega}L_g(e^{-\omega}\psi) - L_h(\psi) = 0$.

Proof. First, we note that

$$\triangle_h\psi = e^{-m\omega}\tfrac{1}{\sqrt{|g|}}\partial_i\big(e^{(m-2)\omega}g^{ij}\sqrt{|g|}\partial_j\psi\big) = e^{-2\omega}(\triangle_g\psi + (m-2)\langle\nabla\psi,\nabla\omega\rangle_g) \tag{A.20}$$

and by the product and chain rule

$$e^{-\omega}\triangle_g(e^{-\omega}\psi) = e^{-2\omega}((-\triangle_g\omega + |\nabla\omega|^2)\psi - 2\langle\nabla\omega,\nabla\psi\rangle_g + \triangle_g\psi).$$

Thus, the difference of the leading order terms of the two operators is

$$e^{-\omega}\triangle_g(e^{-\omega}\psi) - \triangle_h\psi = e^{-2\omega}((-\triangle_g\omega + |\nabla\omega|^2)\psi - m\langle\nabla\psi,\nabla\omega\rangle_g). \tag{A.21}$$

Next, we deal with the first order terms. An easy computation shows that

$$\begin{aligned}\langle\nabla\psi,\nabla\hat{f}\rangle_h - &e^{-\omega}\langle\nabla(e^{-\omega}\psi),\nabla f\rangle_g\\ &= e^{-2\omega}\big(\langle\nabla\psi,\nabla f\rangle_g + m\langle\nabla\psi,\nabla\omega\rangle_g - \langle\nabla\psi,\nabla f\rangle_g + \langle\nabla\omega,\nabla f\rangle_g\psi\big)\\ &= e^{-2\omega}(m\langle\nabla\psi,\nabla\omega\rangle_g + \langle\nabla f,\nabla\omega\rangle_g\psi).\end{aligned} \tag{A.22}$$

Then, we observe that with (A.20), we get

$$\begin{aligned}\triangle_h\hat{f} &= e^{-2\omega}(\triangle_g\hat{f} + (m-2)\langle\nabla\hat{f},\nabla\omega\rangle_g)\\ &= e^{-2\omega}(\triangle_g f + m\triangle_g\omega + (m-2)\langle\nabla f,\nabla\omega\rangle_g + m(m-2)|\nabla\omega|_g^2),\end{aligned}$$

and hence the last difference becomes

$$\begin{aligned}e^{-\omega}&\tfrac{1}{m-2}(\triangle_g f + R_g)e^{-\omega}\psi - \tfrac{1}{m-2}(\triangle_h\hat{f} + R_h)\psi\\ &= e^{-2\omega}\big(-\tfrac{m}{m-2}\triangle_g\omega - \langle\nabla f,\nabla\omega\rangle_g - m|\nabla\omega|_g^2 + \tfrac{1}{m-2}(R_g - e^{2\omega}R_h)\big)\psi.\end{aligned} \tag{A.23}$$

Adding (A.21), (A.22) and (A.23), we arrive at

$$e^{-\omega}L_g(e^{-\omega}\psi) - L_h(\psi) = e^{-2\omega}\big(-\tfrac{2(m-1)}{m-2}\triangle_g\omega - (m-1)|\nabla\omega|^2 + \tfrac{1}{m-2}(R_g - e^{2\omega}R_h)\big)\psi.$$

Hence, in order to prove the proposition, we need to show that

$$e^{2\omega}R_h = R_g - 2(m-1)\triangle_g\omega - (m-1)(m-2)|\nabla\omega|^2.$$

But this is simply the Gauss equation

$$R_h = u^{-\frac{m+2}{m-2}}\big(R_g u - \tfrac{4(m-1)}{m-2}\triangle_g u\big) \tag{A.24}$$

for $h = u^{4/(m-2)}g = e^{2\omega}g$, rewritten in terms of ω. □

iii) Scaling invariance of the functional $\bar{\lambda}_\alpha(g,\phi)$. We prove that the functional

$$\bar{\lambda}_\alpha(g,\phi)(t) := V(t)^{2/m}\lambda_\alpha(g,\phi)(t) = V(t)^{2/m}\inf_f\left\{\mathcal{F}_\alpha(g,\phi,f)\;\middle|\;\int_M e^{-f}dV_g = 1\right\}$$

is scaling invariant with respect to the scaling $\tilde{g} = cg$. Here, $V(t) := \int_M dV_{g(t)}$ is the volume of M at time t and $\mathcal{F}_\alpha(g,\phi,f)$ is the energy functional from (1.1).

To this end, define for every f with $\int_M e^{-f}dV_g = 1$ a function $\tilde{f} = f + \frac{m}{2}\log c$, and note that

$$e^{-\tilde{f}}dV_{\tilde{g}} = e^{-f}e^{-\frac{m}{2}\log c}c^{m/2}dV_g = e^{-f}dV_g. \tag{A.25}$$

This means that for every admissible test function f in the definition of $\lambda_\alpha(g,\phi)$, $\tilde{f}$ is an admissible test function for $\lambda_\alpha(\tilde{g},\phi)$ and vice versa. With the scaling behavior $S_{\tilde{g}} = \frac{1}{c}S_g$, $|\nabla f|_{\tilde{g}}^2 = \frac{1}{c}|\nabla f|_g^2$, $dV_{\tilde{g}} = c^{m/2}dV_g$ and

$$V_{\tilde{g}}^{2/m} = \left(\int_M dV_{\tilde{g}}\right)^{2/m} = \left(\int_M c^{m/2}dV_g\right)^{2/m} = c\,V_g^{2/m},$$

we obtain

$$\begin{aligned}\bar{\lambda}_\alpha(\tilde{g},\phi) &= V_{\tilde{g}}^{2/m}\inf_{\tilde{f}}\Big\{\int_M \big(S_{\tilde{g}}+|\nabla\tilde{f}|^2_{\tilde{g}}\big)e^{-\tilde{f}}dV_{\tilde{g}}\ \Big|\ \int_M e^{-\tilde{f}}dV_{\tilde{g}}=1\Big\}\\ &= c\,V_g^{2/m}\inf_{f}\Big\{\int_M \big(S_{\tilde{g}}+|\nabla f|^2_{\tilde{g}}\big)e^{-f}dV_g\ \Big|\ \int_M e^{-f}dV_g=1\Big\}\\ &= c\,V_g^{2/m}\inf_{f}\Big\{\int_M \tfrac{1}{c}\big(S_g+|\nabla f|^2_g\big)e^{-f}dV_g\ \Big|\ \int_M e^{-f}dV_g=1\Big\}\\ &= \bar{\lambda}_\alpha(g,\phi).\end{aligned}$$

iv) A local version of the gradient estimates from Theorem 3.10. Here, we prove a corollary to Theorem 3.10, in which we only assume a curvature bound in some set $B\times[0,T')$ for $T'<T<\infty$ and allow T to be a singular time. The setting is made in such a way to perfectly fit for the proof of the non-collapsing result in Chapter 6.

Proposition A.5
Let $(g(t),\phi(t))_{t\in[0,T)}$ solve $(RH)_\alpha$ with non-increasing $\alpha(t)\in[\underline{\alpha},\bar{\alpha}]$, $0<\underline{\alpha}\leq\bar{\alpha}<\infty$ and $T'<T<\infty$. Let $B:=B_{g(T')}(x,r)$ be a ball around x with radius r, measured with respect to the metric at time T'. Assume that $|\mathrm{Rm}|\leq R_0$ on the set $B\times[0,T')$. Then there exist constants $K=K(\underline{\alpha},\bar{\alpha},R_0,T,m,N)<\infty$ and $C_k=C_k(k,\bar{\alpha},m,N)$ for $k\in\mathbb{N}$, $C_0=1$, such that the following estimates hold for $k\geq 0$

$$|\nabla\phi|^2\leq\tfrac{K}{t}\quad\textit{and}\quad|\mathrm{Rm}|\leq\tfrac{K}{t},\quad\forall(x,t)\in B^{1/2}\times(0,T'),\tag{A.26}$$

$$|\nabla^k\mathfrak{T}|^2=|\nabla^k\mathrm{Rm}|^2+|\nabla^{k+2}\phi|^2\leq C_k\big(\tfrac{K}{t}\big)^{k+2},\quad\forall(x,t)\in B^{1/2}\times(0,T'),\tag{A.27}$$

where $B^{1/2}:=B_{g(T')}(x,r/2)$ is the ball of half the radius and the same center as B.

Proof. The statement (A.26) follows exactly as in Theorem 3.10. Then, we can simply continue to follow the proof of this theorem, except if we need to apply the maximum principle. For these cases we need a cut-off function to ensure that the maxima are attained in the interior of our set B. Indeed, let us fix a smooth cut-off function $\eta:B\to[0,1]$ with the properties $\eta\equiv 1$ on $B^{1/2}$, $\eta\equiv 0$ in a neighborhood of the boundary ∂B and $|\nabla\eta|,|\triangle\eta|\leq L$ for some constant $L<\infty$. From (3.27), we get

$$\big(\tfrac{\partial}{\partial t}-\triangle\big)(\eta f)\leq\eta\big(\tfrac{C}{t}f+\tfrac{C}{t}-t^3|\nabla^2\phi|^4\big)-2\langle\nabla\eta,\nabla f\rangle+Lf.$$

Since (ηf) vanishes near the boundary of B, a maximum can only be attained in the interior, and we can again apply the maximum principle, Proposition A.1, to obtain some bound $(\eta f)\leq D$. Since $\eta\equiv 1$ on $B^{1/2}$, we obtain $f\leq D$ on $B^{1/2}\times[0,T')$ and hence we get (3.28) on this set.

Then, in each inductive step, we follow the proof of Theorem 3.10 until we arrive at (3.31). Here we use again our cut-off function and apply the maximum principle to (ηh) instead of h. Repeating this argument, the corollary follows. $\square$

Geometric flows with monotone reduced volumes

For the Ricci flow coupled with harmonic map heat flow, we have seen in Chapter 6 that

$$\mathcal{D}(S_{ij}, X) = 2\alpha \left|\tau_g\phi - \nabla_X\phi\right|^2 - 2\dot{\alpha}e(\phi) \tag{A.28}$$

for all X on M. Hence, we can apply Theorem 6.4 if $\alpha(t)$ is positive and non-increasing. In this short section, we list a few other examples of flows $\frac{\partial}{\partial t}g_{ij} = -2S_{ij}$, or special cases of $(RH)_\alpha$ for which the quantity $\mathcal{D}(S_{ij}, X)$ from Definition 6.3 is nonnegative for all vector fields X and where Theorem 6.4 can hence be applied to obtain a monotonicity result.

i) Bernhard List's flow. Since List's flow (0.6) is a special case of our flow $(RH)_\alpha$ with $\alpha \equiv 2$ and $N = \mathbb{R}$ (thus $\tau_g\phi = \triangle_g\phi$), we immediately obtain

$$\mathcal{D}(S_{ij}, X) = 4\left|\triangle\psi - \nabla_X\psi\right|^2 \geq 0 \tag{A.29}$$

for all vector fields X on M. Thus, the backwards and forwards reduced volume monotonicity results hold for List's flow without any further assumptions. This result was first developed in a joint work of the author of the present thesis together with Valentina Vulcanov. This was part of Vulcanov's Master's thesis [64].

ii) The Ricci flow. This is again a special case of $(RH)_\alpha$ with $\alpha \equiv 0$. Thus, $\mathcal{D}(\mathrm{Rc}, X)$ vanishes identically on M, a fact which can also easily be seen from the evolution equation $\frac{\partial}{\partial t}R = \triangle R + 2\left|R_{ij}\right|^2$ for the scalar curvature under Ricci flow together with the twice traced second Bianchi identity $\nabla_i R_{ij} = \frac{1}{2}\nabla_j R$. Hence the conclusion of the theorem follows again without any further assumptions. Note that $\mathcal{H}(\mathrm{Rc}, X, Y)$ and $\mathcal{H}(\mathrm{Rc}, X)$ from Definition 6.5 denote Hamilton's matrix and trace Harnack quantities for the Ricci flow from [25]. The backwards reduced volume corresponds to the one defined by Perelman in [48], the forwards reduced volume and the proof of its monotonicity were developed by Feldman, Ilmanen and Ni in [21].

iii) The static case. Let (M, g) be a Riemannian manifold and set $S_{ij} = 0$ so that g is fixed. Then the quantity $\mathcal{D}$ reduces to $\mathcal{D}(0, X) = 2R_{ij}X_iX_j = 2\,\mathrm{Rc}(X, X)$, i.e. the conclusion holds true if M has nonnegative Ricci curvature. This means for example that the backwards reduced volume

$$V_b(\tau) = \int_M (4\pi\tau)^{-n/2} e^{-\ell_b(q,\tau)} dV$$

is non-increasing in τ if $\mathrm{Rc} \geq 0$, where

$$\ell_b(q, \tau_1) := \inf_{\eta \in \Gamma} \left\{ \frac{1}{2\sqrt{\tau_1}} \int_0^{\tau_1} \sqrt{\tau}\left|\tfrac{\partial}{\partial\tau}\eta\right|^2 d\tau \right\}.$$

Note that the assumption $\mathrm{Rc} \geq 0$ is necessary for the monotonicity, a result which we already proved in [45, page 72].

iv) The mean curvature flow. Let $M^n(t) \subset \mathbb{R}^{n+1}$ denote a family of hypersurfaces evolving by mean curvature flow. Then the induced metrics evolve by $\frac{\partial}{\partial t} g_{ij} = -2HA_{ij}$, where A_{ij} denote the components of the second fundamental form A of M and $H = g^{ij}A_{ij}$ denotes the mean curvature of M. Letting $S_{ij} = HA_{ij}$ with trace $S = H^2$, the expression $\mathcal{H}(\mathcal{S}, X)$ from Definition 6.5 becomes

$$\begin{aligned}\mathcal{H}(\mathcal{S}, X) &= \tfrac{\partial}{\partial t} H^2 + \tfrac{1}{t} H^2 - 2\big\langle \nabla H^2, X\big\rangle + 2HA(X,X)\\ &= 2H\big(\tfrac{\partial}{\partial t} H + \tfrac{1}{2t} H - 2\left\langle \nabla H, X\right\rangle + A(X,X)\big),\end{aligned}$$

that is $2H$ times Hamilton's differential Harnack expression for the mean curvature flow defined in [28]. Moreover, the quantity $\mathcal{D}(\mathcal{S}, X)$ again has a sign for all vector fields X, but unfortunately the wrong one for our purpose. Indeed, one finds $\mathcal{D}(\mathcal{S}, X) = -2\left|\nabla H - A(X,\cdot)\right|^2 \leq 0$, $\forall X \in \Gamma(TM)$, and Theorem 6.4 can't be applied. But fortunately the sign changes if we consider mean curvature flow in Minkowski space, as suggested by Mu-Tao Wang. More general, let $M^n(t) \subset L^{n+1}$ be a family of spacelike hypersurfaces in an ambient Lorentzian manifold, evolving by Lorentzian mean curvature flow. Then the induced metric solves $\frac{\partial}{\partial t} g_{ij} = 2HA_{ij}$, i.e. we have $S_{ij} = -HA_{ij}$ and $S = -\left|H\right|^2$. Marking the curvature with respect to the ambient manifold with a bar, we have the Gauss equation

$$R_{ij} = \bar{R}_{ij} - HA_{ij} + A_{i\ell}A_{\ell j} + \bar{R}_{i0j0},$$

the Codazzi equation

$$\nabla_i A_{jk} - \nabla_k A_{ij} = \bar{R}_{0jki},$$

as well as the evolution equation for the mean curvature

$$\tfrac{\partial}{\partial t} H = \triangle H - H(|A|^2 + \overline{\mathrm{Rc}}(\nu,\nu)),$$

cf. Holder [31, Section 2.1 and 4.1]. Here, ν denotes the future-oriented timelike normal vector, represented by 0 in the index-notation. Combining the three equations above, we find

$$\mathcal{D}(\mathcal{S}, X) = 2\left|\nabla H - A(X,\cdot)\right|^2 + 2\,\overline{\mathrm{Rc}}(H\nu - X, H\nu - X) + 2\left\langle \overline{\mathrm{Rm}}(X,\nu)\nu, X\right\rangle. \quad \text{(A.30)}$$

In particular, if L^{n+1} has nonnegative sectional curvatures, we get $\mathcal{D}(\mathcal{S}, X) \geq 0$ and Theorem 6.4 yields a monotonicity result in this case.

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Acknowledgements

I want to express my gratitude to my supervisor Michael Struwe for his support and the stimulating discussions during the last years. Both as an undergraduate and as a graduate student, I benefited very much from his skills and his way to look at mathematical problems. Moreover, I thank my co-examiners Tom Ilmanen and Peter Topping as well as my colleague Robert Haslhofer for valuable remarks and suggestions during the last months.

I thank the Geometric Analysis Groups in Zürich, Berlin and Potsdam for interesting discussions. In particular, I thank Klaus Ecker and Gerhard Huisken for their invitation to spend a four months research visit at the Free University in Berlin and the Albert Einstein Institute in Potsdam. I thank Valentina Vulcanov for our collaboration concerning a special case of the reduced volume monotonicity discussed in Section 6 of this work. Let me also mention my former and actual office workmates: I thank Fabian, Melanie and Michael (at ETH Zürich), Amos (at FU Berlin) as well as Mariel and Marilyn (at AEI Potsdam) for a mostly pleasant working atmosphere and some interesting discussions about mathematics and about life. Furthermore, I thank the Swiss National Science Foundation that partially supported my research carried out in the present thesis.

I want to thank my friends at ETH with whom I usually ate lunch, spent my coffee breaks, went to parties or played Tichu, in particular Alex, Ben, Christian, Eveline, Julia, Luca, Manuel, Martin, Michele, Robert and most importantly Thomas, who was not only the most loyal visitor for coffee in my office, but also solved all my computer problems and answered all my LaTeX questions.

Last but not least, I owe many thanks to everybody outside the university who gave me additional energy and motivation through the last years, in particular my parents Robert and Marianne, my sister Annett and my friends André and Marc. Moreover, I thank André, Kevin and all the rock bands out there for playing the music that I love. Without blues and rock 'n' roll, Earth would be a boring planet! I dedicate this work to the person who became most important for me during the last years; thank you Eveline for all your love and your constant support!

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Oct 2000 – Mar 2005	Diploma Studies in Mathematics, ETH Zürich, Switzerland Diploma with distinction and ETH Medal for Diploma Thesis *Differential Harnack inequalities for parabolic equations*
Aug 1993 – Feb 2000	Gymnasium with focus on natural sciences, Kantonsschule Zürcher Oberland Filiale Glattal, Switzerland Matura "Typus C"
Apr 1987 – Jul 1993	Primary School, Uster, Switzerland

Professional Experience

Apr 2005 – Mar 2009	Teaching Assistant Departement of Mathematics, ETH Zürich, Switzerland
Oct 2006 – Feb 2007	Four Month Research Visit *Ricci flow equations* FU Berlin and MPI for Gravitational Physics in Golm
Oct 2002 – Mar 2005	Tutor and Junior Assistant Departement of Mathematics, ETH Zürich, Switzerland