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Time - Optimal Control of Mechanical Systems

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Abstract

We treat the problem of time-optimal control of dynamical systems with the help of differential geometry.

For the solution of time-optimal control problems, a system of first-order differential equations, the so-called adjoint system, has to be integrated. In most cases this cannot be done, neither analytically nor numerically. To reduce the complexity of the adjoint system we suggest to reduce its dimension by using First Integrals. In order to find those we formulate our problem in the language of differential geometry and apply the tools of the theory of dynamical systems.

We state a rule for obtaining First Integrals for several classes of systems and show how the adjoint system can be reduced.

Our procedure and the results we gained with it, constitute a step towards optimal control of nonlinear systems.

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Zusammenfassung

Das Problem der Zeitoptimalen Regelung dynamischer Systeme wird behandelt; dazu wird die Differentialgeometrie verwendet.

Bei der Behandlung des Problems der Zeitoptimalen Regelung muss ein System von Differentialgleichungen erster Ordnung integriert werden, das sogenannte adjungierte System. Normalerweise ist dies nicht möglich, weder numerisch noch analytisch. Um die Komplexität der adjungierten Gleichungen zu vermindern, reduzieren wir die Dimension des adjungierten Systems durch ein Erstes Integral.

Um die Ersten Integrale zu finden, formulieren wir unser Problem in der Sprache der Differentialgeometrie, und wir verwenden die Werkzeuge der Theorie der dynamischen Systeme.

Wir finden eine Regel für verschiedene Klassen von Systemen, wie Erste Integrale zu erhalten sind, und wir zeigen, wie man das adjungierte System reduziert.

Mit unseren Methoden und den Resultaten haben wir einen Schritt in Richtung einer neuen Vorgehensweise zur Optimierung nichtlinearer Systeme unternommen.

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Symbols

$\boldsymbol{\alpha}$	Direction of control (Section 7.3).
$\mathbf{C}$	Liouville vector field (Section 11.1.3, Proposition 11.9).
$\mathcal{X}(K)$	Space of smooth vector fields on K.
$(\cdot)^{\mathbf{C}}$	Complete lift (Section 4.2.1).
$\lrcorner$	Contraction operator.
$\nabla^{\mathbf{\Gamma}}$	Covariant derivative on $(TM, \mathbf{S})$ (Section 4.2.3).
E	Bundle $(E, \pi, T^*(TM))$ (Chapter 3).
E_L	Energy function (Section 5.1.2).
F_{H^o}	First Integral, $F_{H^o} : K \to \mathbb{R}$.
g_{ij}	Riemannian metric.
Γ^k_{ij}	Christoffel symbols.
$\mathbf{\Gamma}$	Connection on $(TM, \mathbf{S})$ (Definition 4.5).
$\mathbf{h}$	Projection (Equation (4.7)).
H	$H : E \to \mathbb{R}$ (Section 5.3).
H'	$H' : E \to \mathbb{R}$ (Section 5.3).
H^o	$H^o : K \to \mathbb{R}$ Hamiltonian (Section 5.3).
$(\cdot)^H$	Horizontal lift (Proposition 11.11).
K	Manifold $K \subset E$ (Chapter 3).
L	Lagrange function $TM \to \mathbb{R}$.
L_{perf}	Performance index (Chapter 1).
$\mathcal{L}_{(\cdot)}(\cdot)$	Lie-derivative.
M	Smooth manifold of dimension n (Chapter 3).
$\mathbf{\Phi}$	Symplectic Transformation (Chapter 6).
$\boldsymbol{\omega}$	Symplectic 2-form.
π_{TM}	Projection (Chapter 3).
$\mathbf{S}$	Almost tangent structure (Section 4.1).
S	$S : T^*(TM) \to \mathbb{R}$ Hamilton function (Section 5.3).
s	$s : E \to \mathbb{R}$ (Section 7.3).
sgrad	Operator sgrad : $C^\infty(K, \mathbb{R}) \to \mathcal{X}(K)$ (Equation (6.1)).
τ_M	Projection (Chapter 3).
τ_{TM}	Projection (Chapter 3).
$\mathbf{u}$	Section of bundle E, control (Chapter 3).
$\mathbf{u}^o$	Section of bundle E, optimal control (Chapter 3).
Υ^t	Flow of the symmetry vector field (Chapter 6).
$\mathbf{v}$	Projection (Equation (4.7)).
$(\cdot)^V$	Vertical lift (Section 11.1.3).
$\boldsymbol{\xi}$	Semispray (Definition 4.1).
$\boldsymbol{\xi}_L$	Semispray of a Lagrangian system with Lagrange function L.
$\boldsymbol{\xi}^*$	Deviation of a semispray $\boldsymbol{\xi}$ (Definition 4.2).
$[\cdot, \cdot]$	Lie bracket.
$\{\cdot, \cdot\}$	Poisson bracket.

Contents

1 Introduction

We consider the problem of time-optimal control of holonomic mechanical systems. Let the system equations be given by

$$\dot{x}^i = f^i(\mathbf{x}, \mathbf{u}) \qquad i = 1, \ldots, n \tag{1.1}$$

where, locally, $\mathbf{x} \in \mathbb{R}^n$ are the states and, locally, $\mathbf{u} \in \mathbb{R}^m$ the controls of the system. We aim to determine a control $\mathbf{u}$ such that the system can be transferred from an initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ to some final state $\mathbf{x}(t_f) = \mathbf{x}_f$ in the shortest possible time. There are a number of ways to describe the final state. It could be given, for instance, by a point in $\mathbb{R}^n$ or by a function $z(\mathbf{x}_f) = 0$, to mention just two possibilities. However, it does not make sense to perform time-optimal control without constraints for the control $\mathbf{u}$ (for example $|\mathbf{u}| < u_{max}$ with the Riemannian metric δ^i_j on $\mathbb{R}^m$ as measure).

To embed time-optimal control in the framework of the calculus of variation, we introduce a performance index

$$J = \int_{t_0}^{t_f} L_{perf}\, dt = \int_{t_0}^{t_f} 1\, dt\ , \tag{1.2}$$

which has to be minimized with (1.1) as constraint. The value of (1.2) corresponds to the elapsed time.

Applying the methods of calculus of variation we end up with a boundary value problem, the so-called **adjoint** system

$$\begin{aligned} \dot{x}^i &= \frac{\partial H^o}{\partial \lambda_i}(x^1, \ldots, x^n, \boldsymbol{\lambda}) \\ \dot{\lambda}_i &= -\frac{\partial H^o}{\partial x^i}(x^1, \ldots, x^n, \boldsymbol{\lambda}) \end{aligned} \qquad \begin{aligned} &H^o = \min_{\mathbf{u}}(1 + \lambda_i f^i) \\ &i = 1, \ldots, n \end{aligned} \tag{1.3}$$

with boundary conditions

$$\mathbf{x}(t_0) = \mathbf{x}_0 \qquad \text{and} \qquad \boldsymbol{\lambda}(t_f) = \cdots\ .$$

The conditions for $\boldsymbol{\lambda}(t_f)$ depend on the type of conditions for the final state [6]. The variables $\boldsymbol{\lambda}(t)$ are Lagrange multipliers. Furthermore, $\mathbf{u}$ is determined by

$$\frac{\partial H}{\partial \mathbf{u}} = 0\ . \tag{1.4}$$

Problems may occur if H is not differentiable and convex in $\mathbf{u}$. Pontryagin [20] proved that $\mathbf{u}$ has to be determined such that we have

$$H^o = \min_{\mathbf{u}}(H) \tag{1.5}$$

with control $\mathbf{u}$ admissible (Pontryagin's Minimum principle). If H is differentiable and convex this condition is equivalent to (1.4). We have to solve the boundary value problem (1.3) with Hamiltonian H^o together with (1.5), $\mathbf{x}(t_0) = \mathbf{x}_0$, and conditions for $\mathbf{x}(t_f)$ and $\boldsymbol{\lambda}(t_f)$.

In most cases it is not possible to find an analytic solution of (1.3), and a numerical treatment of the problem normally fails due to its high dimension and the amount of time thus needed to arrive at a solution.

One possibility, however, to get somewhat closer to a solution is to reduce the dimension of the adjoint system by using a First Integral (symmetry) of it. With the help of a First Integral we are able to find a symplectic coordinate transformation $\Phi : \mathbf{x} \mapsto \bar{\mathbf{x}}, \boldsymbol{\lambda} \mapsto \bar{\boldsymbol{\lambda}}$ such that the transformed system is of the form

$$
\begin{aligned}
\dot{\bar{x}}^i &= \frac{\partial \bar{H}^o}{\partial \bar{\lambda}_i}(\bar{x}^2, \ldots, \bar{x}^n, \bar{\boldsymbol{\lambda}}) \\
\dot{\bar{\lambda}}_1 &= 0 \\
\dot{\bar{\lambda}}_j &= -\frac{\partial \bar{H}^o}{\partial \bar{x}^j}(\bar{x}^2, \ldots, \bar{x}^n, \bar{\boldsymbol{\lambda}}) \,.
\end{aligned}
\qquad
\begin{aligned}
&\bar{H}^o = \min_{\bar{\mathbf{u}}}(1 + \bar{\lambda}_i \bar{f}^i) \\
&i = 1, \ldots, n \\
&j = 2, \ldots, n
\end{aligned}
$$

The question now is how to find a First Integral of (1.3). One way consists in finding a First Integral of the uncontrolled system ($\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u} \equiv 0)$) and then investigating what happens with it when the system becomes controlled.

Since we use differential geometry to find First Integrals of (1.3) we must formulate our problem in the terminology of differential geometry. We have to relate the elements of our system to mathematical objects such as manifolds, sections, and so on. We will construct a **setting**, that is a collection of manifolds, bundles, etc. onto which our system can be placed. On this setting the uncontrolled system is represented in Lagrangian and Hamiltonian form and the controlled and the optimally controlled system (adjoint system) appear on it only in Hamiltonian form. This enables us to apply the apparatus of differential geometry.

We relate the second-order differential equation (Euler-Lagrange equation) of the uncontrolled system to the setting. The structure of the latter can be recovered by this equation, i.e., we are able to determine a connection on the setting. This connection defines a covariant derivative with whose help we are able to find a symmetry of the uncontrolled system in Lagrangian form. This symmetry reappears in the Hamiltonian representation of the uncontrolled system. It even turns out to exist in the optimally controlled system.

Following these steps we obtain a rule to determine First Integrals. We supply a simple example where the step of reduction is carried out.

The contents in brief

Chapter 2 gives an overview of the tools of differential geometry from the point of view of dynamical systems. Since the aim is to provide engineers with an intuitive understanding for the mathematical tools we apply throughout this work, we refrain from overloading the text of this chapter with definitions. Instead, we provide references to relevant literature where the definitions may be looked up.

In Chapter 3 we introduce the setting, i.e., all the manifolds, bundles, and their relations. At this stage we make no reference to physics or dynamical systems.

In Chapter 4 we introduce the notion of semispray, a vector field which describes a second-order differential equation (or our system in Lagrangian form). It also determines most of the structure (connection) of our setting.

In Chapter 5 we add some physics to our setting. We explain how the adjoint system is represented on the setting. We also show how the uncontrolled system is represented in Hamiltonian and in Lagrangian form.

Chapter 6 gives an introduction to First Integrals (symmetries) of Hamiltonian systems. In addition, the reduction process is presented.

In Chapter 7 we show how to find a particular First Integral (symmetry) of the uncontrolled system in Lagrangian form. The First Integral found is then related to affine transformations. We describe how it evolves when the uncontrolled system is expressed in Hamiltonian form, and what it looks like in the adjoint system.

In Chapter 8 we deal with some classes of systems that can be reduced by First Integrals.

In Chapter 9.1 we give an example and perform the reduction procedure.

In Chapter 10 we summarize what we have achieved and list some of the remaining problems.

Chapter 11 contains some calculations needed for Chapters 4 and 7. The second part is concerned with the *Maple* routines (symbolic computation) of Chapter 9.1.

2 Tools, Basics

The aim of Chapter 2 is to provide engineers with an intuitive understanding for the mathematical tools we apply throughout this work. Readers should then be able to understand what we mean when we use expressions such as "manifold" or "vector field." This introduction should also enable readers to understand the main ideas of this work and help them decide whether or not to familiarize themselves with this domain. However, supplying the exact mathematical definitions would exceed the scope of this work. Throughout the text, references are given to the literature where the definitions can be looked up.

We wish to emphasize that all the topics considered are explained from the point of view of system dynamics, especially the geometric interpretation. The properties from a complete mathematical view are not treated, even though for a deep understanding, this would of course be absolutely necessary. We are aware of the fact that, in some cases, mathematical correctness suffers for the benefit of vividness of the explanations.

2.1 Nonlinear Spaces, Manifolds

2.1.1 Charts and Atlases

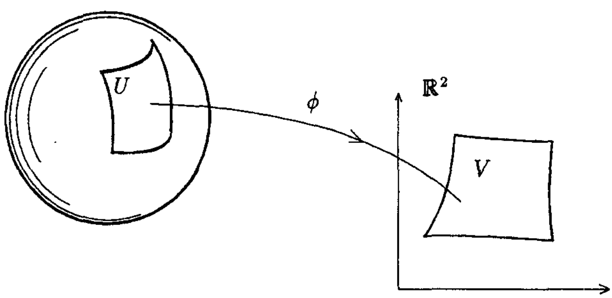

Figure 1: Chart

The concept of ***manifolds*** is very convenient to the thinking of engineers. Manifolds are very important in system theory. However, when one studies differential geometry it is not easy to see the relation between these mathematical objects and physical systems.

An n-dimensional ***manifold*** is an item that is locally related to $\mathbb{R}^n$.

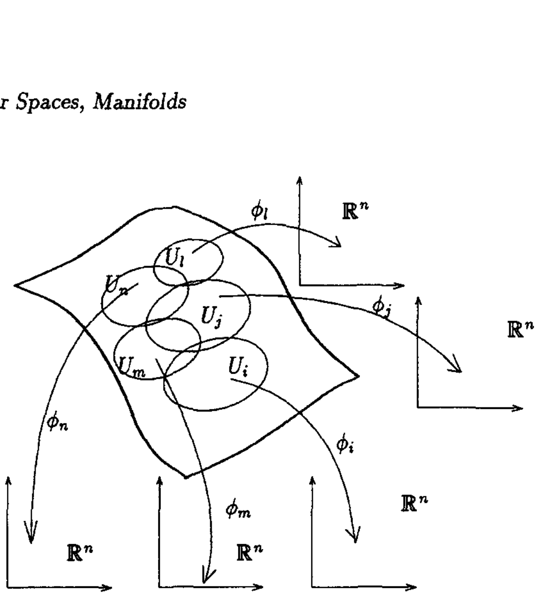

Figure 2: Atlas

Example: 2.1 *Sphere in* $\mathbb{R}^3$(Figure 1), ([3] p. 78).

There is a map ϕ that maps the region U of the sphere to $V \subset \mathbb{R}^2$. Such a map is called a ***chart*** ([4] p. 54). It is easy to imagine that through such a chart a coordinate system is induced on the manifold. A collection of charts which covers the whole manifold M is called an ***atlas*** (Figure 2). What makes the concept of manifolds so important is shown in Figure 3. One thinks on the manifold, but the calculation is done in the representation. The representation is realized by the charts and the atlases. What we have calculated in the representation can be pushed back to the manifold, to the thinking.

The way vector spaces are related to manifolds is of great importance for us. At a certain point p of a manifold M the ***tangential space*** is a vector space with the same dimensions as the manifold M (Figure 4). It is denoted T_pM ([4] p. 74). Its ***dual*** is called T_p^*M. The union of all vector spaces is called TM, the ***tangent bundle***. The dimension of TM, $\dim TM$ is equal to $2 \cdot \dim M$.

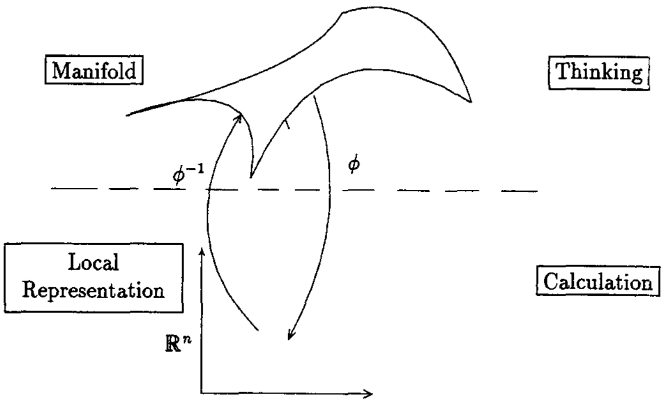

Figure 3: Concept of manifolds and representations

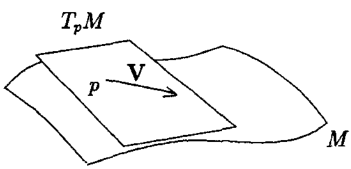

Figure 4: T_pM with vector field $\mathbf{V}$

2.2 Vector Fields

A vector field $\mathbf{V}$ on a manifold M in a point p of M is an element of the tangential space T_pM (Figure 4). The components of $\mathbf{V}$ are functions of the coordinates of p. In local coordinates a vector field is expressed as follows ([5] p. 110)

$$\mathbf{V} = V^1\frac{\partial}{\partial x^1} + V^2\frac{\partial}{\partial x^2} +, \cdots, + V^n\frac{\partial}{\partial x^n} = \sum_{i=1}^{n} V^i\frac{\partial}{\partial x^i},$$

where $(\partial/\partial x^i)$ form the base. The functions $V^i : M \to \mathbb{R}$ are the coordinate functions.

2.2.1 Systems and Vector Fields

We consider some trajectories of a system (Figure 5a): The set of all trajectories

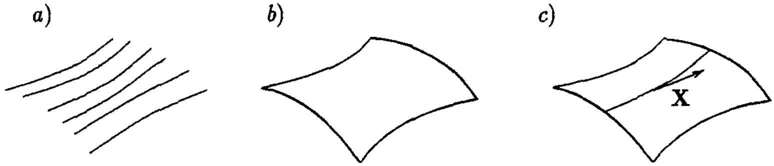

Figure 5: Configuration Manifold

forms the ***configuration manifold*** M (Figure 5b, [3] p. 133). We now take the configuration manifold and one trajectory on this manifold. On this trajectory we consider the vector field which points in the direction of the trajectory. It is the ***system vector field*** $\mathbf{X}$ (Figure 5c). For example, consider a system on a configuration manifold M that is described by the first-order equation $\dot{x}^i = f^i(\mathbf{x})$. The system vector field $\mathbf{X}$ in a point p is then described by

$$\mathbf{X} = f^i(\mathbf{x})\frac{\partial}{\partial x^i}$$

where $(\mathbf{x} = (x^1, \dots, x^n))$ is a coordinate system on M and hence $(\partial/\partial x^i)$ is the base of T_pM ([5], p. 110).

Example: 2.2 *What does the configuration manifold of a double pendulum look like?*
It is the Torus T^2, when we think of the pendulum being constructed such that $0 \leq \alpha, \beta \leq 2\pi$(Figure 6).

2.2.2 Flow Generated by a Vector Field

We consider the map which maps the system along a trajectory from point p_i to point p_{i+1} on the configuration manifold (Figure 7). This map describes the evolution of the system from time t_i to time t_{i+1}. We have

$$F_{t,t_{i+1}} : p_i \mapsto p_{i+1} ,$$

where $t_{i+1} - t_i$ is the elapsed time,

$$F_{t,t} = \text{identity} ,$$

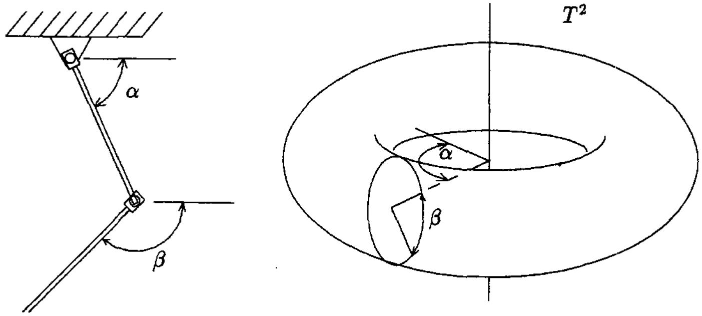

Figure 6: Double Pendulum

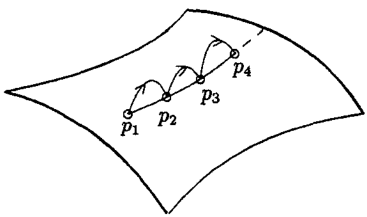

Figure 7: Flow on the configuration manifold

and

$$F_{t_1,t_2} \circ F_{t_0,t_1} = F_{t_0,t_2} \, .$$

We call F the ***flow*** of the system ([1] p. 249). This flow is ***generated*** by the ***system vector field*** $\mathbf{X}$. We see that

$$\mathbf{X}|_{p_i} = \frac{d}{dt} F_{0,t}\bigg|_{t_i} \, ,$$

where $\mathbf{X}$ is the system vector field.

2.2.3 Lie Derivative

Literature to this chapter can be found in ([1] p. 272, 269, and 271, respectively, and in [9] p. 207). We consider a vector field $\mathbf{W}$ that is given on a manifold M. Furthermore, there exist various other objects on this manifold M, such as functions, $f : M \to \mathbb{R}$, and tensors. We now wish to determine the derivative of these objects in the direction of the vector field $\mathbf{W}$. This ***directional derivative*** is the ***Lie derivative***. If we have a function f

$$f : M \to \mathbb{R}$$

then the directional derivative is given by

$$\mathcal{L}_{\mathbf{W}} f = W^i \frac{\partial f}{\partial x^i} .$$

The Lie derivative of a vector field $\mathbf{Y}$ in the direction of $\mathbf{W}$ is

$$\mathcal{L}_{\mathbf{W}} \mathbf{Y} = [\mathbf{W}, \mathbf{Y}] ,$$

where $[\cdot, \cdot]$ is the ***Lie bracket*** ([1] p. 271, p. 278, [3] p. 211, and [9] p. 209). The Lie derivative of a general tensor can be looked up in ([9] p. 209). Formulas for the Lie derivative are determined with the help of the flow generated by the vector fields.

2.3 Transformations

First we will consider the relations of **mappings** of a manifold M onto itself to **coordinate transformations** on M.

We consider a map

$$F : M \to M , \qquad F : p \mapsto q$$

where M is a manifold and p, q are points in M. In local coordinates this map is given by

$$\begin{aligned} z^1 &= z^1(y^1, \ldots, y^n) \\ &\vdots \\ z^n &= z^n(y^1, \cdots, y^n) , \end{aligned}$$

where $\mathbf{y}$ are the coordinates of p and $\mathbf{z}$ are the coordinates of q. Now, the mapping F can also be understood as a **change of the coordinate system** from the coordinates $\mathbf{y}$ to the coordinates $\mathbf{z}$. The question that arises is how mathematical objects, such as vector fields, are transformed by this change of coordinates. As an example we will calculate the transformation of a vector field $\mathbf{V}$. Let $\mathbf{V} \in TM$ be a vector field given in the coordinate system $\mathbf{y}$.

$$\mathbf{V}_y = V_y^i \frac{\partial}{\partial y^i} .$$

We would like to know the components $\mathbf{V}_z$ of $\mathbf{V}$ in the coordinates $\mathbf{z}$. The Lie derivative of a function $f : M \to \mathbb{R}$ with respect to $\mathbf{V}$ will be transformation invariant, which is expressed by

$$\mathcal{L}_{\mathbf{V}_z} f_z = \mathcal{L}_{\mathbf{V}_y} f_y \ ,$$

where f_y, f_z are f expressed in the coordinate systems $\mathbf{y}$ and $\mathbf{z}$, respectively. Instead of f we take the coordinate function z^k from which follows

$$\mathcal{L}_{\mathbf{V}_z} z^k = \mathcal{L}_{\mathbf{V}_y} z^k_y = \mathcal{L}_{\mathbf{V}_y} z^k(\mathbf{y})$$

which is, in local coordinates,

$$V_z^i \frac{\partial z^k}{\partial z^i} = V_y^i \frac{\partial z^k(\mathbf{y})}{\partial y^i}$$

yielding

$$V_z^k = V_y^i \frac{\partial z^k}{\partial y^i} \quad \text{or} \quad V_z^k = \frac{\partial F^k}{\partial y^i} V_y^i \ .$$

$\partial F / \partial \mathbf{y}$ is called the ***push forward*** of F and is denoted by F_*. It pushes an object forward. (The transformation rules for other objects are well described for example in ([9] p. 152)).

We distinguish different kinds of transformations. The type of transformation applied depends on the character of the map $F : M \to M$ which defines the transformation.

2.3.1 Isometry Transformation

We consider a manifold M with a flow F generated by a vector field $\mathbf{Y} \in TM$. F is a mapping

$$F : M \to M \ .$$

Let M be a Riemannian manifold with metric $\mathbf{g}$. If the flow F lets the metric be invariant, i.e.,

$$F_* \mathbf{g} = \mathbf{g} \tag{2.1}$$

then F is called an **isometry**. Equation (2.1) can be expressed as an infinitesimal relation by

$$\mathcal{L}_{\mathbf{Y}} \mathbf{g} = 0$$

which is the Lie derivative of $\mathbf{g}$ with respect to $\mathbf{Y}$.

2.3.2 Affine Transformation

Let q be a point in a manifold M with coordinates $(q^1, \ldots, q^n)$. A map $F : M \to M$, is called ***affine*** iff it is represented by

$$F^a(q) = a^a_i q^i + c^a ,$$

a^i_j being called the ***linear part***. We see that $F_* = \mathbf{a}$, hence it is a matrix ([9] p. 28). Every isometry is an affine transformation. An affine transformation is an isometry if $\mathbf{a}^T\mathbf{a} = \delta^i_j$ where $i, j = 1, \ldots, n$.

2.4 Lie Groups

Lie group theory is very important to the theory of dynamical systems ([7] p. 152). A Lie group G is a group that is also a manifold. (From linear algebra we know that a group is a set X together with a group operator $X \times X \to X$ by $(x, y) \mapsto xy$ such that

1. $$(xy)z = x(yz), \quad x, y, z \in X$$

2. there is an element $\mathrm{Id}_X \in X$ called the identity such that
$$z\mathrm{Id}_X = \mathrm{Id}_X z = z \quad \forall z \in X$$

3. for each $x \in X$ there is an element of X called the inverse of x, written x^{-1}, such that $x^{-1}x = xx^{-1} = \mathrm{Id}_X$)

Going back to (Figure 7), we see the maps which transform the system from one state into another. The idea is now to "form a manifold" out of those maps, and to look for relations between this new manifold and the flow. We call this new manifold ***Lie group*** G. This Lie group G **acts** onto the configuration manifold. This ***action*** is called ***the action of the Lie group*** onto manifold M. The action is a mapping σ with

$$\begin{aligned}\sigma &: G \times M \to M \\ \sigma_x &: G \to M \\ \sigma_g &: M \to M ,\end{aligned}$$

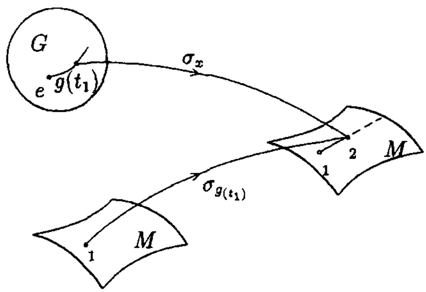

Figure 8: Action of Lie group G onto M

where $g \in G$ and $x \in M$. The evolution of the system is described by a ***one-parameter subgroup*** $g(t) \in G$ with $\sigma_x : g(t) \mapsto M$ describing the trajectory of our system (Figure 8). It is obvious that a large part of the systems characteristics is determined by the structure of the Lie group. Many of the properties of the Lie group can be gleaned from the tangential space of G at the identity element $e \in G$, T_eG, ([7] pp. 153, 156) which in turn is related to the so-called ***Lie algebra*** $\mathfrak{g}$ of G ([7] p. 156).

The Lie algebra $\mathfrak{g}$ is of great importance to the theory of dynamical systems. The importance of Lie group theory to dynamical systems cannot be overemphasized.

2.5 Bundles, Fibre Bundles, Projections

We will give a short introduction to bundles ([7] p. 124). They play an important role in the construction of the structure onto which our optimized systems will move. The simplest example of a bundle is the so-called ***Cartesian bundle*** $\mathbb{R}^2 \times \mathbb{R}$. A point in the Cartesian bundle is described by

$$(x_b^1, x_b^2, x_{fi})$$

where x_{fi} is the coordinate in the fibre (Figure 9). The first two coordinates are the coordinates in $\mathbb{R}^2$. We can imagine a projection π_1 that maps the point p down to the $\mathbb{R}^2$ space. This projection deletes the third coordinate from which follows

$$\begin{aligned} \pi_1 &: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \\ \pi_1 &: (x_b^1, x_b^2, x_{fi}) \mapsto (x_b^1, x_b^2) \,. \end{aligned}$$

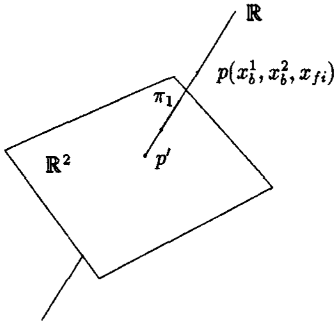

Figure 9: Cartesian bundle

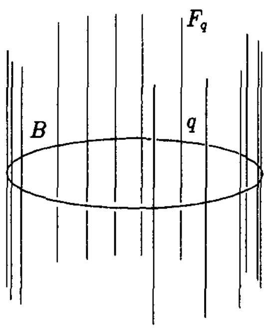

Figure 10: Cylinder as a vector bundle

Point p is said to be projected to the ***base*** manifold $\mathbb{R}^2$. $\mathbb{R}$ is called the ***fibre***.

A bundle is described by the triple (P, B, π) where P is the bundle, π the ***projection*** and B the base manifold.

In the Cartesian example the bundle would be described by

$$(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2, \pi_1)$$

where π_1 is the so-called first projection. The fibre F_q of a bundle is given by

$$\pi^{-1}(q) ,$$

where q is a point in the base manifold. In the Cartesian example

$$\pi_1^{-1}(x_b^1, x_b^2) = (x_b^1, x_b^2, \textit{every point in } \mathbb{R})$$

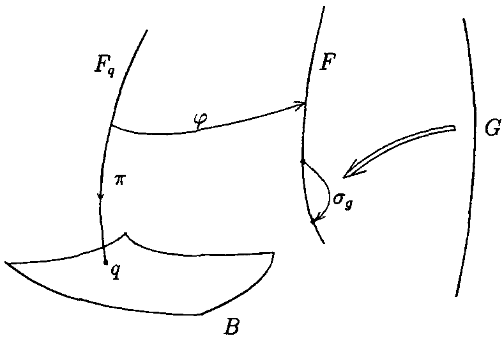

Figure 11: Fibre bundle

and $F_{q\,Cartesian} = \mathbb{R}$.

Different types of bundles are distinguished by the character of the fibre. For example, if the fibre is a ***vector space***, we speak of a ***vector bundle*** ([7] p. 125, [1] p. 174). We understand that the tangential space TM is a vector bundle with the fibre T_qM, the tangential space, fixed at some point q.

Example: 2.3 *The cylinder with $B = S^1$ and fibres $= \mathbb{R}$ is a vector bundle* (Figures 10, 12).

We speak of a ***typical fibre*** F if there is a map φ between the fibre F_q and a manifold F (homeomorphic map). This F is a sort of model for all fibres of this bundle.

A ***fibre bundle*** (P, B, π, G) ([7] p. 125) is a bundle (P, B, π) together with a typical fibre F and a group G of maps of F onto itself (Figure 11). We speak of a ***principal fibre bundle*** if F and G are identical. G can be understood as a Lie group which acts onto fibre F.

If we take a strip of paper, twist it once and glue it, we get a **Möbius band.** Without twisting it, we would have got a cylinder (Figure 10). The Möbius band is a vector bundle, the base manifold being S^1 and the fibre equal to $\mathbb{R}$. Nevertheless it is completely different from a cylinder. How is the twisting of the Möbius band described? Consider Figure 12, where we have maps φ_1 and φ_2 which map the fibres over the neighborhoods U_1 and U_2 to the typical fibre, where U_1 and U_2 intersect.

The relationship between φ_1 and φ_2 describes the twisting of the Möbius band or, more generally, the structure of the fibre bundle. The pairs (U_i, φ_i) are ***local trivializations*** of the bundle ([7] p. 129).

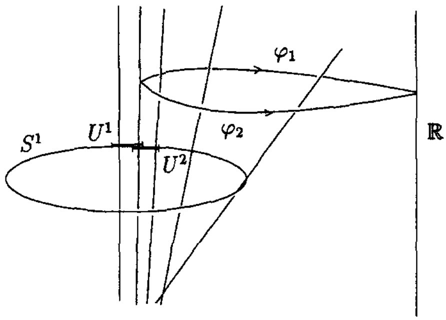

Figure 12: Möbius Band

2.6 Metric, Riemannian Metric

Let $p_1,p_2 \in M$, where M is a manifold. We want to measure the distance between two points p_1 and p_2 which are given by their coordinates (chart). We know how to do this in **Euclidean** space: There, the distance is the length of a "straight" line between the two points. The distance is equal to the length of the vector $\mathbf{V}$ between p_1 and p_2

$$\mid \mathbf{V} \mid^2 = \mathbf{V}^T\mathbf{g}\mathbf{V} = [V^1,\dots,V^n]\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} V^1 \\ \vdots \\ V^n \end{bmatrix} = V^iV^j\delta_{ij} \, .$$

The character of the metric to be Euclidean is expressed by the fact that $\mathbf{g}$ is represented by a diagonal matrix with constant entries. In other spaces $\mathbf{g}$ will neither be diagonal nor constant.

We are particularly interested in ***Riemannian*** metric ([9] p. 17). The condition on the elements g_{ij} of $(0,2)$ tensor $\mathbf{g}$ to be a Riemannian metric are: $\mathbf{g}$ is positive definite and the g_{ij} expressed in a coordinate system z transforms by a coordinate transformation form z to z' as

$$g'_{ij} = \frac{\partial z^k}{\partial z'^i} g_{kl} \frac{\partial z^l}{\partial z'^j} \, .$$

We speak of Riemannian manifold M if a Riemannian metric exists on M. If we have given a curve $x^i(t)$ on a manifold with Riemannian metric, then the arc length is given by

$$l = \int_a^b \sqrt{g_{ij}(\mathbf{x}(t))\frac{dx^i}{dt}\frac{dx^j}{dt}} \; dt \, ,$$

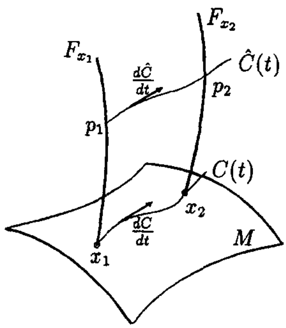

Figure 13: Parallel transport

g being a (0,2) tensor ([1], p. 393).
The shortest curve (in the sense of the metric) is called ***geodesic***. In a Euclidean space, a geodesic is a straight line.

2.7 Connections

A ***connection*** on a principal fibre bundle P ([7] p. 358) leads to a correspondence between any two fibres along a curve $C(t)$ in M, where M is the base of the bundle P. A point p_1 of the fibre over a point x_1 of the curve $C(t)$ is parallely translated along this curve by means of this correspondence to the point p_2 (Figure 13). The curve $\hat{C}(t)$ described by the ***parallel transport*** of P is a ***horizontal lift*** of the curve $C(t)$. What is important is that by such a connection a splitting of the tangential space of the bundle is achieved:

$$T_{p_1}P = H \oplus V$$

where H is the space which is formed out of the horizontal vector fields. The tangent vector fields to $\hat{C}(t)$, $d\hat{C}(t)/dt$, are horizontal vector fields. $T_{p_1}P = H \oplus V$ means that $T_{p_1}P$ can be separated into two vector spaces H and V, i.e., means that H and V have no common elements. They are complementary subspaces. V is the vertical space and it is related to the Lie algebra $\mathfrak{g}$ of G ([7] p. 359).

By the theory of the so-called ***associated vector bundle*** ([7] p. 367), connections are defined in the tangential bundle TM of M. These connections are the ***Christoffel symbols***, Γ^k_{ij} ([7], p. 301, p. 366). If the manifold M is a Riemannian manifold with Riemannian metric g_{ij}, then the equations for the geodesic of this metric are given by the differential equation

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0 ,$$

or, as solution of the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 ,$$

with

$$L = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j .$$

The Γ^m_{ij} are determined out of $\mathbf{g}$ by

$$\Gamma^m_{ij} = \frac{1}{2} g^{km}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) \tag{2.2}$$

([9], p. 289).

2.8 First Integrals and Reduction

2.8.1 First Integrals and Symmetries

On a manifold N let there be given a Hamiltonian system (Section 5.1.1) with Hamiltonian $H : N \to \mathbb{R}$; its system vector field $\mathbf{X}_H = \mathrm{sgrad} H$ (6.1), is on N. A ***First Integral*** f of this system is a mapping

$$f : N \to \mathbb{R}$$

which is constant along the trajectory of the system. This is expressed by the Lie derivative of f with respect to $\mathbf{X}_H$,

$$\mathcal{L}_{\mathbf{X}_H} f = 0 . \tag{2.3}$$

A ***symmetry vector field*** $\mathbf{Y}_f = \mathrm{sgrad} f$ belongs to every First Integral f of a Hamiltonian system ([11] p. 77), with

$$\mathcal{L}_{\mathbf{Y}_f} f = 0,\ \mathcal{L}_{\mathbf{Y}_f} H = 0 \text{ and } [\mathbf{Y}_f, \mathbf{X}_H] = 0 .$$

These two items, the First Integral and the symmetry vector field, can be applied to **reduce the dimension** of the Hamiltonian system by two ([17] p. 89).

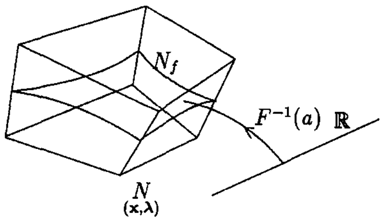

Figure 14: Hypersurface N_f

2.8.2 Reduction

First step:
Equation (2.3) expresses the fact that the system will move in such a way that f will be constant along a trajectory. We consider the hypersurface $N_f = f^{-1}(a) \quad a \in \mathbb{R}$ (Figure 14). Once the system moves on the hypersurface N_f, it will always remain on N_f, $\dim N_f = \dim N - 1$. We have thus reduced the system's dimension by one.

Second step:
On N_f we consider the integral curves of $\mathbf{Y}_f$ (Figure 15). We perform a co-

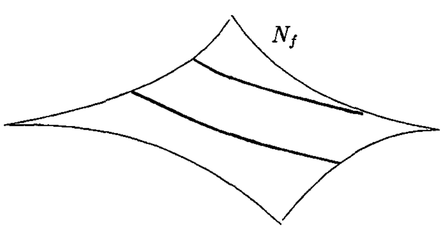

Figure 15: Hypersurface N_f with integral curves

ordinate transformation in such a way that the integral curves of $\mathbf{Y}_f$ become a coordinate in the new coordinate system, let us say z. In the transformed system it must hold

$$\mathcal{L}_{\bar{\mathbf{Y}}_f} \bar{H} = 0 \,, \tag{2.4}$$

where $\bar{H}$ is the transformed Hamiltonian and $\bar{\mathbf{Y}}_f$ the transformed vector field, which now points exactly in the direction of the coordinate z and is given by

$$\bar{\mathbf{Y}}_f = 1\frac{\partial}{\partial z} \, .$$

In order to satisfy (2.4), $\bar{H}$ must no longer depend on the coordinate z. One more coordinate is thus eliminated.

3 Setting

Our goal is to apply tools from differential geometry and the theory of dynamical systems to get information about the interior structure of our optimally controlled mechanical system.
In this section, we introduce the manifolds and bundles which are necessary for our investigations. They all are derived from a smooth manifold M. In Chapters 4 and 5 we will see that this manifold M is given by our system, namely it is the systems configuration manifold. Let M be a smooth manifold of dimension n. An atlas gives a coordinate system $(\mathbf{x} = (x^1, \dots, x^n))$ on M. Consider the tangent bundle (TM, τ_M, M) with projection τ_M. The coordinate system $(\mathbf{x})$ induces a coordinate system $(\mathbf{x} = (x^1, \dots, x^n), \mathbf{v} = (v^1, \dots, v^n))$ on TM.

From the tangent bundle we get the vector bundle $(T(TM), \tau_{TM}, TM)$ with τ_{TM} as its projection. As for TM out of M, we get a coordinate system on $T(TM)$

$$(\mathbf{x} = (x^1, \dots, x^n), \mathbf{v} = (v^1, \dots, v^n), \delta\mathbf{x} = (\delta x^1, \dots, \delta x^n), \delta\mathbf{v} = (\delta v^1, \dots, \delta v^n))$$

out of TM. In these coordinates the projection map $\tau_{TM} : T(TM) \to TM$ is given by $\tau_{TM} : (\mathbf{x}, \mathbf{v}, \delta\mathbf{x}, \delta\mathbf{v}) \mapsto (\mathbf{x}, \mathbf{v})$.

Out of TM we can also form the dual bundle of $T(TM)$, i.e., $(T^*(TM), \pi_{TM}, TM)$. On this bundle we have the coordinates

$$(\mathbf{x} = (x^1, \dots, x^n), \mathbf{v} = (v^1, \dots, v^n), \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n), \boldsymbol{\mu} = (\mu_1, \dots, \mu_n)) \, .$$

$$\begin{tikzcd}
\underset{(\mathbf{x},\mathbf{v},\boldsymbol{\lambda},\boldsymbol{\mu})}{T^*(TM)} \arrow[dr, "\pi_{TM}"'] & & \underset{(\mathbf{x},\mathbf{v},\delta\mathbf{x},\delta\mathbf{v})}{T(TM)} \arrow[dl, "\tau_{TM}"] \\
& \underset{(\mathbf{x},\mathbf{v})}{TM} \arrow[dl, "\tau_M"] & \\
\underset{(\mathbf{x})}{M} & &
\end{tikzcd}$$

Figure 16: Vector Bundles of the Setting

Up to now our configuration has been consisting entirely of vector bundles (Figure 16).

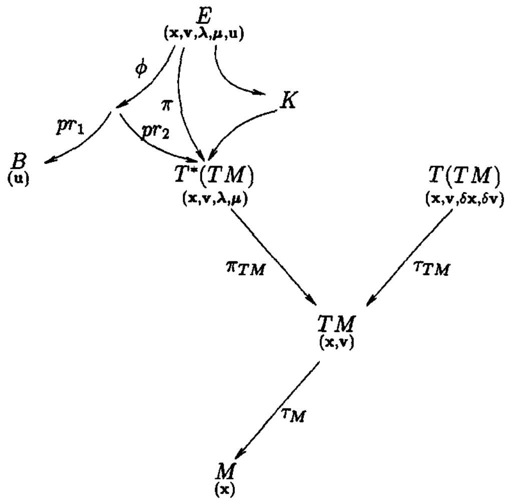

Figure 17: Setting

Now, let us assume that we have a fibre bundle $(E, \pi, T^*(TM))$, with typical fibre B. Let $\{U_i, \phi_i\}$ be a family of local trivializations of E, i.e., we have diffeomorphisms

$$\phi_i : \pi^{-1}(U_i) \to U_i \times B$$

for an open covering $\{U_j \mid j \in J\}$ of $T^*(TM)$. On this fibre bundle we have fibre coordinates

$$(\mathbf{x} = (x^1, \dots, x^n), \mathbf{v} = (v^1, \dots, v^n), \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n), \boldsymbol{\mu} = (\mu_1, \dots, \mu_n),$$

$$\mathbf{u} = (u^1, \dots, u^m)) ,$$

where $(\mathbf{u} = (u^1, \dots, u^m))$ is a coordinate system of B (over a preferred U_i). Furthermore, we consider a submanifold K of E, which is defined by a section $\mathbf{u}^o : T^*(TM) \to E$ of the bundle E, i.e., $K = \mathbf{u}^o(T^*(TM))$. An illustration of all the manifolds bundles and their relations is given in Figure 17. This arrangement of manifolds and bundles will be called ***setting*** in the further discussion.

4 Setting and Semisprays

We have to optimize a mechanical system that may be described by a second-order differential equation. In Chapter 1 our system was assumed to be described by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$. Of course it can equally be described by a second-order differential equation (Euler-Lagrange differential equation). We are going to relate this differential equation to our setting. For this purpose we assume the manifold M (in our setting) to be the configuration manifold of the system. The second-order differential equation is represented by a vector field on TM, a so-called semispray. Now, since M is considered to be the configuration manifold of the system, the structure of $TTM := T(TM)$ can be recovered by the semispray. In fact, we will see that a semispray determines a connection on TM, from which we can deduce the structure of TTM. It is therefore interesting for us to consider the relations of this connection and the semispray.

Definition: 4.1 ([2], p. 213), ([16], p. 193) *A* ***semispray*** *on M (or a second-order differential equation) is a vector field $\boldsymbol{\xi}$ on TM, such that $T\tau_M \circ \boldsymbol{\xi}$ is the identity on TM, whereas $T\tau_M$ is the differential of τ_M.*

We have the following commutative diagram

$$\begin{tikzcd}
 & TTM \arrow[dl, "T\tau_M"'] & \\
TM \arrow[rr, "Id"] & & TM \arrow[ul, "\boldsymbol{\xi}"'] \arrow[dl, "\boldsymbol{\xi}"] \\
 & TTM \arrow[ul, "\tau_{TM}"] &
\end{tikzcd} .$$

From now on, we will always use Einstein convention, i.e., if in an expression an index occurs twice, once as a lower index and once as an upper index, then summation over that index is assumed implicitly. Let $\boldsymbol{\xi}$ be a semispray on M. Locally $\boldsymbol{\xi}$ is given by

$$\boldsymbol{\xi} = \bar{\xi}^i \frac{\partial}{\partial x^i} + \xi^i \frac{\partial}{\partial v^i} , \tag{4.1}$$

where $(\mathbf{x} = (x^1, \dots, x^n), \mathbf{v} = (v^1, \dots, v^n))$ is a coordinate system of TM (which comes from a coordinate system $(\mathbf{x} = (x^1, \dots, x^n))$ of M). By calculation it can be shown that $T\tau_M$ and τ_{TM} are given by

$$\begin{aligned}
\tau_{TM} &: T(TM) \to TM, \quad (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \mapsto (\mathbf{a}, \mathbf{b}) \\
T\tau_M &: T(TM) \to TM, \quad (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \mapsto (\mathbf{a}, \mathbf{c}) ,
\end{aligned}$$

that the two mappings commute if and only if

$$\bar{\xi}^i = v^i$$

and that, therefore, we obtain

$$\boldsymbol{\xi} = v^i \frac{\partial}{\partial x^i} + \xi^i \frac{\partial}{\partial v^i} \ .$$

Intuitively, it is clear that such a semispray represents a second-order differential equation. As is well known, first-order differential equations are represented by vector fields on M.

Definition: 4.2 ([16], p. 194) *Let $\boldsymbol{\xi}$ be a semispray on M and $\mathbf{C}$ the Liouville vector field that is represented on our coordinate system on M by $\mathbf{C} = v^i(\partial/\partial v^i)$. The vector field $\boldsymbol{\xi}^* = [\mathbf{C}, \boldsymbol{\xi}] - \boldsymbol{\xi}$ is called the* ***deviation*** *of $\boldsymbol{\xi}$.*

Definition: 4.3 ([16], p. 195)*A semispray $\boldsymbol{\xi}$ is said to be a* ***spray*** *if $\boldsymbol{\xi}^* = 0$ and $\boldsymbol{\xi}$ is C^1 on the zero section. Moreover, if $\boldsymbol{\xi}$ is C^2 on the zero section, then $\boldsymbol{\xi}$ is called a* ***quadratic spray.***

Let $\boldsymbol{\xi}$ be a semispray. Since $\boldsymbol{\xi}^* = 0$ if and only if $[\mathbf{C}, \boldsymbol{\xi}] = \boldsymbol{\xi}$, we deduce that $\boldsymbol{\xi}$ is a spray if the functions ξ^i are homogeneous of degree 2 and C^1 on the zero section. Therefore, we see that a spray is the geometrical interpretation of a system of second-order differential equations homogeneous of degree 2 with respect to the first derivatives.

4.1 The Canonical Almost Tangent Structure on TM

In this section we introduce a geometric structure which is essential in the Lagrangian formulation of classical mechanics. We will see that on the tangent bundle (TM, τ_M, M) of any smooth manifold M (i.e., on our configuration manifold), we have a canonical almost tangent structure, hence the name).

Definition: 4.4 ([16], p. 111) *An* ***almost tangent structure*** $\mathbf{S}$ *on TM is a tensor field $\mathbf{S}$ of type $(1,1)$ on TM with constant rank n ($n = \dim M$) that satisfies $\mathbf{S}^2 = 0$. The pair $(TM, \mathbf{S})$ is called an* ***almost tangent manifold.***

Let $(\mathbf{x} = (x^1, \dots, x^n),\ \mathbf{v} = (v^1, \dots, v^n))$ be a local coordinate system of TM. Then, we can define a $(1,1)$ tensor field $\mathbf{S}$ on TM by

$$\mathbf{S} = \frac{\partial}{\partial v^i} \otimes dx^i \ .$$

Equivalently, $\mathbf{S}$ is given as a linear map $\mathbf{S} : TTM \to TTM$ by

$$\mathbf{S}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial v^i} \text{ and } \mathbf{S}\left(\frac{\partial}{\partial v^i}\right) = 0 \ .$$

This implies that $\operatorname{rank} \mathbf{S} = n$ and $\mathbf{S}^2 = 0$. Thus $\mathbf{S}$ is an almost tangent structure on TM, which is called the **canonical almost tangent structure** on TM. Moreover, we have

$$\ker \mathbf{S} = \operatorname{im} \mathbf{S} = V \ ,$$

where $V = \ker T\tau_M$.

4.2 Distributions and Connections in TM

In this section we are concerned with two important concepts from differential geometry. We start with the construction of a **distribution** on the tangent bundle (TTM, τ_{TM}, TM) of TM. This distribution will be determined by the semispray, hence by the second-order differential equation of the system we have to optimize. For $p \in TM$ this distribution will be of the form

$$T_p(TM) = V_p \oplus H_p$$

where $V_p = \ker T_p\tau_M$. Elements of V_p are called **vertical** vectors and elements of H_p **horizontal** vectors. A vector $\boldsymbol{\eta} \in T_p(TM)$ may therefore be decomposed into its two components

$$\boldsymbol{\eta} = \boldsymbol{\eta}_H + \boldsymbol{\eta}_V \ , \qquad \boldsymbol{\eta}_H \in H_p,\ \boldsymbol{\eta}_V \in V_p \ .$$

Later, we will introduce connections on the almost tangent manifold $(TM, \mathbf{S})$, where $\mathbf{S}$ is the canonical almost tangent structure. Out of a semispray $\boldsymbol{\xi}$ on TM, we will get a canonical connection on $(TM, \mathbf{S})$. Finally, we study how these two concepts, distribution and connection, are related to each other.

4.2.1 Vertical and Complete Lifts of Vector Fields

First we have to say something about vertical and complete lifts of vector fields. For $p = (\mathbf{x}, \mathbf{v}) \in T_xM$ we can define a linear map

$$(\cdot)^V : T_xM \to V_p \ , \quad \mathbf{U} \mapsto \mathbf{U}^V$$

($V_p = \ker T_p\tau_M$) as follows: for $\mathbf{U} \in T_xM$, $\mathbf{U}^V$ is the tangent vector of the curve $t \mapsto p + t \cdot \mathbf{U}$ in T_xM at time $t = 0$. The vertical vector $\mathbf{U}^V$ is called a ***vertical lift*** of $\mathbf{U}$. Furthermore, if $\mathbf{X}$ is a vector field on M, then we can define its vertical lift as the vector field $\mathbf{X}^V$ on TM by

$$\mathbf{X}^V(p) = (\mathbf{X}(\tau_M(p)))^V , \quad (p \in TM) .$$

If on the coordinate system $(U, \mathbf{x} = (x^1, \ldots, x^n))$ of M the vector field $\mathbf{X}$ is given by $\mathbf{X} = X^i(\partial/\partial x^i)$, then on $(TU, (\mathbf{x} = (x^1, \ldots, x^n)), \mathbf{v} = (v^1, \ldots, v^n))$ the vertical lift $\mathbf{X}^V$ is given by

$$\mathbf{X}^V = X^i \frac{\partial}{\partial v^i} .$$

Now for a vector field $\mathbf{X}$ on M, we can also define its ***complete lift*** $\mathbf{X}^C$ on TM by using its local flow, i.e., if ϕ_t is the local flow generating $\mathbf{X}$, then $\mathbf{X}^C$ is defined to be the vector field on TM, generated by the local flow $T\phi_t$. If $\mathbf{X} = X^i(\partial/\partial x^i)$ in our local coordinate system $(U, \mathbf{x})$, then

$$\mathbf{X}^C = X^i \frac{\partial}{\partial x^i} + v^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial v^i} . \tag{4.2}$$

Trivially, we have

$$\left(\frac{\partial}{\partial x^i}\right)^C = \frac{\partial}{\partial x^i} .$$

Note that from the local description of a vertical lift and of a complete lift follows that the vertical and complete lifts of vector fields on M span TTM at any point; this should be clear intuitively.

4.2.2 Distribution on TTM

Now we are going to construct a distribution, using the semispray $\boldsymbol{\xi}$ on TM. For this we first show that, for any vertical vector field $\mathbf{V}$, the Lie derivative of the almost tangent structure $\mathbf{S}$ is equal to $\mathbf{V}$, i.e., $\mathcal{L}_{\boldsymbol{\xi}}\mathbf{S}(\mathbf{V}) = \mathbf{V}$. It is sufficient to do this in the case where $\mathbf{V}$ is a vertical lift of some vector field on M, since the vertical lifts span the vertical subspace $V_p = \ker T_p\tau_M$ for all $p \in TM$. Let $\mathbf{U}^V$ be the vertical lift of some vector field $\mathbf{U}$ on M. Then, with (11.12) we get

$$\begin{aligned}
(\mathcal{L}_{\boldsymbol{\xi}}\mathbf{S})(\mathbf{U}^V) &= [\boldsymbol{\xi}, \mathbf{S}(\mathbf{U}^V)] - \mathbf{S}([\boldsymbol{\xi}, \mathbf{U}^V]) \\
&= \left[\boldsymbol{\xi}, \frac{\partial}{\partial v^j} \otimes dx^j \left(U^i \frac{\partial}{\partial v^i}\right)\right] - \mathbf{S}([\boldsymbol{\xi}, \mathbf{U}^V]) = [\boldsymbol{\xi}, 0] - \mathbf{S}([\boldsymbol{\xi}, \mathbf{U}^V]) \\
&= \mathbf{S}[\mathbf{U}^V, \boldsymbol{\xi}] .
\end{aligned}$$

On the other hand, since the Lie derivative of $\mathbf{S}$ by a vertical lift of a vector field on M is 0 (see (11.1)), together with (11.2) and (11.15), yields

$$\mathbf{S}([\mathbf{U}^V, \boldsymbol{\xi}]) = [\mathbf{U}^V, \mathbf{S}(\boldsymbol{\xi})] = [\mathbf{U}^V, \mathbf{C}] = \mathbf{U}^V ,$$

where $\mathbf{C}$ is the Liouville vector field. Therefore, we have proved that

$$\mathcal{L}_{\xi}\mathbf{S}(\mathbf{U}^V) = \mathbf{U}^V . \tag{4.3}$$

Consider next $(\mathcal{L}_{\xi}\mathbf{S})(\mathbf{U}^C)$ for any complete lift $\mathbf{U}^C$ of a vector field $\mathbf{U}$ on M. Using a similar argument to the one we used for $\mathbf{U}^V$, with (11.12) and (11.4) we obtain

$$\begin{aligned} \mathcal{L}_{\xi}\mathbf{S}(\mathbf{U}^C) &= [\boldsymbol{\xi}, \mathbf{S}(\mathbf{U}^C)] - \mathbf{S}([\boldsymbol{\xi}, \mathbf{U}^C]) \\ &= [\boldsymbol{\xi}, \mathbf{U}^V] + \mathbf{S}([\mathbf{U}^C, \boldsymbol{\xi}]) . \end{aligned}$$

On the other hand, taking into account (11.8), (11.2), (11.16), and (11.12) yields

$$\begin{aligned} 0 &= (\mathcal{L}_{\mathbf{U}^C}\mathbf{S})(\boldsymbol{\xi}) = [\mathbf{U}^C, \mathbf{S}(\boldsymbol{\xi})] - \mathbf{S}([\mathbf{U}^C, \boldsymbol{\xi}]) \\ &= [\mathbf{U}^C, \mathbf{C}] - \mathbf{S}([\mathbf{U}^C, \boldsymbol{\xi}]) \\ &= -\mathbf{S}([\mathbf{U}^C, \boldsymbol{\xi}]) . \end{aligned}$$

Thus,

$$(\mathcal{L}_{\xi}\mathbf{S})(\mathbf{U}^C) = [\boldsymbol{\xi}, \mathbf{U}^V] . \tag{4.4}$$

We have seen that $\mathbf{S}([\mathbf{U}^V, \boldsymbol{\xi}]) = \mathbf{U}^V$, which now together with (11.4), yields $\mathbf{S}(\mathbf{U}^C + [\boldsymbol{\xi}, \mathbf{U}^V]) = 0$. Therefore, $\mathbf{U}^C + [\boldsymbol{\xi}, \mathbf{U}^V]$ or due to (11.12) the vector field $\mathbf{U}^C + (\mathcal{L}_{\xi}\mathbf{S})(\mathbf{U}^C)$ is vertical. Using what we have already proved in (4.3),

$$(\mathcal{L}_{\xi}\mathbf{S})\left(\mathbf{U}^C + (\mathcal{L}_{\xi}\mathbf{S})(\mathbf{U}^C)\right) = \mathbf{U}^C + (\mathcal{L}_{\xi}\mathbf{S})(\mathbf{U}^C) , \tag{4.5}$$

follows

$$(\mathcal{L}_{\xi}\mathbf{S})^2(\mathbf{U}^C) = \mathbf{U}^C . \tag{4.6}$$

Since, as we noted before, the vertical lift and complete lift of TM span TTM at any point, we obtain from (4.3) and (4.6) that

$$(\mathcal{L}_{\xi}\mathbf{S})^2 = \mathrm{Id}_{TTM} .$$

Out of this property of $\mathbf{S}$ we are going to **construct projections** from TM onto TTM.

Consider now the two **tensor fields** $\mathbf{h}$ and $\mathbf{v}$ on TM given by

$$\boxed{\mathbf{h} = (1/2)(\mathrm{Id}_{TTM} - \mathcal{L}_{\xi}\mathbf{S})} , \qquad \boxed{\mathbf{v} = (1/2)(\mathrm{Id}_{TTM} + \mathcal{L}_{\xi}\mathbf{S})} . \tag{4.7}$$

From the fact that $(\mathcal{L}_\xi \mathbf{S})^2 = \mathrm{Id}_{TTM}$ follows that $\mathbf{h}$ and $\mathbf{v}$ have the properties

$$\mathbf{h}^2 = \mathbf{h}\ , \qquad \mathbf{v}^2 = \mathbf{v}\ ,$$
$$\mathbf{h} \circ \mathbf{v} = \mathbf{v} \circ \mathbf{h} = 0\ , \qquad \mathbf{h} + \mathbf{v} = \mathrm{Id}_{TTM}\ .$$

These tensor fields are projection operators corresponding to the direct sum decomposition of tangent space, i.e., $T_pTM = \mathrm{im}\,\mathbf{h}_p \oplus \mathrm{im}\,\mathbf{v}_p$ at each point p of TM. The kernel of $\mathbf{h}$ coincides with the image of $\mathbf{v}$, hence the kernels of $\mathbf{h}$ and $\mathbf{v}$ are complementary subspaces. Let

$$\ker \mathbf{h} = V\ , \qquad \ker \mathbf{v} = H\ , \tag{4.8}$$

then $TTM = V \oplus H$ which is the desired distribution. This construction works for any semispray or second-order differential equation. Note that, from the above construction, the vertical subspace V_p for $p \in TM$ is in fact $\ker T_p\tau_M$!

4.2.3 Connections on $(TM, \mathbf{S})$

We define a connection in the almost tangent manifold $(TM, \mathbf{S})$, where $\mathbf{S}$ is the canonical almost tangent structure. This definition was proposed by [12].

Definition: 4.5 *A* ***connection*** $\mathbf{\Gamma}$ *in* $(TM, \mathbf{S})$ *is a smooth tensor field of type* $(1,1)$ *on* TM*, such that*

$$\mathbf{S\Gamma} = \mathbf{S}\ , \qquad \mathbf{\Gamma S} = -\mathbf{S}\ , \tag{4.9}$$

where $\mathbf{S}$ *is the canonical almost tangent structure on* TM*.*

It can be shown that $\mathbf{\Gamma}$ is locally given by

$$\mathbf{\Gamma}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} - 2\Gamma_i^j \frac{\partial}{\partial v^j}\ ,$$

$$\mathbf{\Gamma}\left(\frac{\partial}{\partial v^i}\right) = -\frac{\partial}{\partial v^i}\ . \tag{4.10}$$

However, some further definitions have to be introduced.

Definition: 4.6 *A connection* $\mathbf{\Gamma}$ *in* $(TM, \mathbf{S})$ *is said to be* ***homogeneous*** *if* $\mathbf{\Gamma}$ *is homogeneous of degree 1* ([16], p. 202).

Obviously, $\mathbf{\Gamma}$ is homogeneous if the functions Γ_i^j are homogeneous of degree 1.

Definition: 4.7 *A homogeneous connection* $\mathbf{\Gamma}$ *is said to be a* ***linear connection*** *if* $\mathbf{\Gamma}$ *is* C^1 *on the zero section* ([16], p. 202).

If $\mathbf{\Gamma}$ is a linear connection on $(TM, \mathbf{S})$, then the functions Γ_i^j are linear on v^k, i.e., we may write

$$\Gamma_i^j(\mathrm{x}, \mathrm{v}) = v^k \Gamma_{ik}^j \,.$$

Definition: 4.8 *Let* $\mathbf{\Gamma}$ *be a connection in* $(TM, \mathbf{S})$ *and* $\boldsymbol{\xi}'$ *an arbitrary semispray on* TM. *Then we define the horizontal projection of* $\mathbf{\Gamma}$ *by* $\mathbf{h} = (1/2)(\mathrm{Id}_{TTM} + \mathbf{\Gamma})$ *and the vector field* $\boldsymbol{\xi}$ *by*

$$\boldsymbol{\xi} = \mathbf{h}\boldsymbol{\xi}' \,.$$

This vector field $\boldsymbol{\xi}$ *on* TM *is called the* ***associated semispray*** *to* $\mathbf{\Gamma}$.

If $\boldsymbol{\xi}''$ is any other semispray, then $\boldsymbol{\xi}' - \boldsymbol{\xi}''$ is vertical. Hence, the vector field $\boldsymbol{\xi}$ does not depend on the choice of $\boldsymbol{\xi}'$. By some calculations it can be shown that

$$\boldsymbol{\xi} = v^i \frac{\partial}{\partial x^i} - v^j \Gamma_j^i \frac{\partial}{\partial v^i} \,. \tag{4.11}$$

If $\mathbf{\Gamma}$ is a linear connection, then $\boldsymbol{\xi}$ becomes

$$\boldsymbol{\xi} = v^i \frac{\partial}{\partial x^i} - v^j v^k \Gamma_{jk}^i \frac{\partial}{\partial v^i} \,.$$

Proposition: 4.1 *Let* $\mathbf{S}$ *be the canonical almost tangent structure on* TM. *Then for any semispray* $\boldsymbol{\xi}$ *on* TM,

$$\mathbf{\Gamma} = -\mathcal{L}_{\xi}\mathbf{S} \,, \tag{4.12}$$

is a connection in $(TM, \mathbf{S})$.

Proof: 4.1 We have to prove that $\mathbf{\Gamma}$ satisfies property (4.9) from Definition 4.5. For this we first have to construct the so called Nijenhuis tensor (see [16], p. 116) $\mathbf{N}_S$ out of $\mathbf{S}$. This is a tensor field of type $(1, 2)$ which in general is given by

$$\mathbf{N_S}(\mathbf{Z}, \mathbf{X}) = [\mathbf{SZ}, \mathbf{SX}] - \mathbf{S}[\mathbf{SZ}, \mathbf{X}] - \mathbf{S}[\mathbf{Z}, \mathbf{SX}] \,, \tag{4.13}$$

where $\mathbf{X}$ and $\mathbf{Z}$ are vector fields on TM. Further calculation yields

$$0 = \mathbf{N}_S(\mathbf{Z}, \mathbf{X}) = [\mathbf{SZ}, \mathbf{SX}] - \mathbf{S}[\mathbf{SZ}, \mathbf{X}] - \mathbf{S}[\mathbf{Z}, \mathbf{SX}] \,, \tag{4.14}$$

and therefore,

$$[\mathbf{SZ}, \mathbf{SX}] = \mathbf{S}[\mathbf{SZ}, \mathbf{X}] + \mathbf{S}[\mathbf{Z}, \mathbf{SX}] \,. \tag{4.15}$$

Now we set $\mathbf{Z} = \boldsymbol{\xi}$ and out of (4.15) together with (11.2) we get

$$[\mathbf{C}, \mathbf{SX}] = \mathbf{S}[\mathbf{C}, \mathbf{X}] + \mathbf{S}[\boldsymbol{\xi}, \mathbf{SX}] \tag{4.16}$$

($\mathbf{C}$ being the Liouville vector field). On the other hand, according to (11.7) we have

$$-(\mathbf{SX}) = (\mathcal{L}_{\mathbf{C}}\mathbf{S})(\mathbf{X}) = [\mathbf{C}, \mathbf{SX}] - \mathbf{S}[\mathbf{C}, \mathbf{X}]\,, \tag{4.17}$$

hence

$$\mathbf{S}[\boldsymbol{\xi}, \mathbf{SX}] = -(\mathbf{SX})\,. \tag{4.18}$$

Using $\mathbf{S}^2 = 0$, this implies that

$$\mathbf{S}\boldsymbol{\Gamma}(\mathbf{X}) = -\mathbf{S}(\mathcal{L}_{\boldsymbol{\xi}}\mathbf{S})(\mathbf{X}) = -\mathbf{S}[\boldsymbol{\xi}, \mathbf{SX}] = \mathbf{SX} \tag{4.19}$$

and

$$\boldsymbol{\Gamma}\mathbf{S}(\mathbf{X}) = \mathbf{S}(\mathcal{L}_{\boldsymbol{\xi}}\mathbf{S}) = \mathbf{S}[\boldsymbol{\xi}, \mathbf{SX}] = -\mathbf{SX}\,. \tag{4.20}$$

This shows us that property (4.9) of Definition 4.5 is satisfied, so $\boldsymbol{\Gamma}$ is a connection in $(TM, \mathbf{S})$. □

As should be clear, the connection $\boldsymbol{\Gamma} = -\mathcal{L}_{\boldsymbol{\xi}}\mathbf{S}$ is determined by the semispray $\boldsymbol{\xi}$. Moreover, we have

Proposition: 4.2 *The **associated** semispray to the connection $\boldsymbol{\Gamma} = -\mathcal{L}_{\boldsymbol{\xi}}\mathbf{S}$ on $(TM, \mathbf{S})$ is given by $\boldsymbol{\xi} + (1/2)\boldsymbol{\xi}^*$, where $\boldsymbol{\xi}^* = [\mathbf{C}, \boldsymbol{\xi}] - \boldsymbol{\xi}$ is the deviation of $\boldsymbol{\xi}$.*

Proof: 4.2 Let us denote by $\boldsymbol{\xi}'$ the associated semispray. By its definition we have

$$\begin{aligned}
\boldsymbol{\xi}' &= \mathbf{h}\boldsymbol{\xi} = \frac{1}{2}(\mathrm{Id}_{TTM} + \boldsymbol{\Gamma})\boldsymbol{\xi} = \frac{1}{2}\boldsymbol{\xi} + \frac{1}{2}\boldsymbol{\Gamma}\boldsymbol{\xi} \\
&= \frac{1}{2}\boldsymbol{\xi} + \frac{1}{2}(-[\boldsymbol{\xi}, \mathbf{S}\boldsymbol{\xi}]) = \frac{1}{2}\boldsymbol{\xi} + \frac{1}{2}[\mathbf{C}, \boldsymbol{\xi}] \qquad \text{(by (11.2))} \\
&= \frac{1}{2}\boldsymbol{\xi} + \frac{1}{2}(\boldsymbol{\xi}^* + \boldsymbol{\xi}) \\
&= \boldsymbol{\xi} + \frac{1}{2}\boldsymbol{\xi}^*\,. \quad \square
\end{aligned}$$

Let us consider once more the connection $\mathbf{\Gamma} = -\mathcal{L}_{\boldsymbol{\xi}}\mathbf{S}$ on $(TM, \mathbf{S})$ for a semispray $\boldsymbol{\xi}$. For this connection, we can define a horizontal and a vertical projector, namely

$$\mathbf{h} = \frac{1}{2}(\mathrm{Id}_{TTM} + \mathbf{\Gamma})$$

and

$$\mathbf{v} = \frac{1}{2}(\mathrm{Id}_{TTM} - \mathbf{\Gamma}) .$$

These are in fact the horizontal and vertical tensor fields that we considered in (4.7). Therefore, we see that out of our connection $\mathbf{\Gamma} = -\mathcal{L}_{\boldsymbol{\xi}}\mathbf{S}$ on $(TM, \mathbf{S})$ in a canonical way we can get a distribution of TTM, which is given by (4.8). In fact $TTM = V \oplus H$, where $V = \ker \mathbf{h} = \ker T_p\tau_M$ and $H = \ker \mathbf{v}$.

Remark: 4.1 The denomination of $\mathbf{\Gamma}$ as a connection is not universal. In fact, in differential geometry, the term connection on a vector bundle (TM, τ_M, M) usually denotes a horizontal or vertical distribution or a covariant derivative (cf. Def. 4.9) on M. This is the reason why we always speak of a connection $\mathbf{\Gamma}$ on $(TM, \mathbf{S})$ rather than on TM. In our case, where $\mathbf{\Gamma} = -\mathcal{L}_{\boldsymbol{\xi}}\mathbf{S}$ on $(TM, \mathbf{S})$, we have seen how we can in a canonical way get a connection in the other sense.

Let us now introduce the notion of covariant derivative on a manifold M.

Definition: 4.9 [16] *A* ***covariant derivative*** *on a (smooth) manifold M is a function ∇ which associates a vector field $\nabla_{\mathbf{X}}\mathbf{Y}$ to two vector fields $\mathbf{X}$ and $\mathbf{Y}$ on M and which satisfies*

$$\nabla_{f\mathbf{X}_1+\mathbf{X}_2}\mathbf{Y} = f\nabla_{\mathbf{X}_1}\mathbf{Y} + \nabla_{\mathbf{X}_2}\mathbf{Y} ,$$

for $f \in C^\infty(M)$. If, moreover

$$\nabla_{\mathbf{X}}(f\mathbf{Y}_1 + \mathbf{Y}_2) = f\nabla_{\mathbf{X}}\mathbf{Y}_1 + \mathbf{X}(f)\mathbf{Y}_1 + \nabla_{\mathbf{X}}\mathbf{Y}_2$$

is satisfied, then ∇ is called a ***linear covariant derivative****.*

Remark: 4.2 In differential geometry, linear covariant derivatives usually are simply called covariant derivatives.

Let $\mathbf{x} = (x^1, \ldots, x^n)$ be a local coordinate system on M. In this local coordinate system we get

$$\nabla_{\mathbf{X}}\mathbf{Y} = \left(\Gamma^j_i(\mathbf{x}, \mathbf{v})X^i + X^i\frac{\partial Y^j}{\partial x^i}\right)\frac{\partial}{\partial x^j} .$$

If moreover ∇ is linear, then $\Gamma_i^k(\mathbf{x},\mathbf{v}) = \Gamma_{ij}^k v^j$, where Γ_{ij}^k are the usual Christoffel symbols.

Our goal is now to construct, out of a given distribution of TTM in horizontal and vertical space, a covariant derivative.

Recall that, for any $p \in TM$, we have a canonical isomorphism

$$J_p : T_{\tau_M(p)}M \to T_p(T_{\tau_M(p)}M)$$

$$\mathbf{W} \mapsto \left.\frac{d}{dt}\right|_{t=0} (p + t\mathbf{W}) .$$

Now let $TTM = H \oplus V$ be our distribution and let $\mathbf{h}$ and $\mathbf{v}$ be the horizontal and vertical projections, respectively. Then we can define a covariant derivative for vector fields $\mathbf{X}$ and $\mathbf{Y}$ on M and $p \in TM$ as

$$\nabla_{\mathbf{X}}^{\mathbf{\Gamma}}\mathbf{Y} := J_p^{-1}(\mathbf{v}(\mathbf{Y}_*(\mathbf{X}_{\tau_M(p)}))) \in T_{\tau_M(p)}M . \tag{4.21}$$

All this now shows us how to get a covariant derivative $\nabla^{\mathbf{\Gamma}}$ out of our connection $\mathbf{\Gamma} = -\mathcal{L}_\xi\mathbf{S}$ on $(TM,\mathbf{S})$, namely by first passing from $\mathbf{\Gamma}$ to its associated distribution of $TTM = H \oplus V$ and then going from this distribution to the covariant derivative $\nabla^{\mathbf{\Gamma}}$. Locally, (4.21) yields

$$\nabla_{\mathbf{X}}^{\mathbf{\Gamma}}\mathbf{Y} = \left(X^i\frac{\partial Y^j}{\partial x^i} + X^i\Gamma_i^j(\mathbf{x},\mathbf{v})\right)\frac{\partial}{\partial x^j} .$$

Remark: 4.3 If the connection $\mathbf{\Gamma}$ is linear, then the associated covariant derivative is a linear covariant derivative. In this case, $\Gamma_i^j(\mathbf{x},\mathbf{v}) = v^k\Gamma_{ik}^j$. Moreover, the above process is bijective. In fact, out of a linear covariant derivative we get a distribution of TTM in the following way: For $p \in TM$ and a section $\mathbf{X} : M \to TM$ of the tangent bundle, we can define

$$H_p = \{\mathbf{X}_*(\mathbf{Y}_m) \mid m = \pi(p) , \quad \nabla_{\mathbf{Y}_m}\mathbf{X} = 0\}$$

and

$$V_p = \ker T_p\tau_M .$$

Definition: 4.10 *Let* $\mathbf{\Gamma} = -\mathcal{L}_\xi\mathbf{S}$ *be our connection on* $(TM,\mathbf{S})$ *with associated covariant derivative* $\nabla^{\mathbf{\Gamma}}$. *A curve* $\sigma : I \to M$ *is said to be a* ***path of*** $\mathbf{\Gamma}$, *if the vector field* $\dot\sigma(t)$ *is parallel along* $\sigma(t)$, *i.e.,*

$$\nabla_{\dot\sigma}^{\mathbf{\Gamma}}\dot\sigma = 0 .$$

If we write $\sigma(t) = (x^1(t), \dots, x^n(t))$ in a local coordinate system $(\mathbf{x} = (x^1, \dots, x^n))$ of M, i.e., $x^i(t) = x^i \circ \sigma(t)$, then $\nabla^{\mathbf{\Gamma}}_{\dot\sigma}\dot\sigma = 0$ yields the second-order differential equation

$$\frac{d^2x^i(t)}{dt^2} + \Gamma^i_j\left(\mathbf{x}, \frac{d\mathbf{x}}{dt}\right)\frac{dx^j(t)}{dt} = 0 \quad (1 \le i \le n)\,. \tag{4.22}$$

Remark: 4.4 If $\mathbf{\Gamma}$ is linear, then the second-order differential equation of a geodesic can be recognized.

If $\boldsymbol{\xi} = v^i(\partial/\partial x^i) + \xi^i(\partial/\partial v^i)$, then

$$\begin{aligned}
\mathbf{\Gamma}\left(\frac{\partial}{\partial x^i}\right) &= -\mathcal{L}_{\boldsymbol{\xi}}\mathbf{S}\left(\frac{\partial}{\partial x^i}\right) = -\boldsymbol{\xi}(\mathbf{S}\left(\frac{\partial}{\partial x^i}\right)) + \mathbf{S}\left[\boldsymbol{\xi}, \frac{\partial}{\partial x^i}\right] \\
&= -\left[\boldsymbol{\xi}, \frac{\partial}{\partial v^i}\right] + 0 \\
&= -\left(-\frac{\partial v^l}{\partial v^i}\frac{\partial}{\partial x^l} - \frac{\partial \xi^l}{\partial v^i}\frac{\partial}{\partial v^l}\right) = \frac{\partial}{\partial x^i} + \frac{\partial \xi^j}{\partial v^i}\frac{\partial}{\partial v^j}\,.
\end{aligned}$$

The calculation of $\Gamma(\partial/\partial v^i)$ will be analogous. Then $\mathbf{\Gamma}$ is given by

$$\mathbf{\Gamma}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} + \frac{\partial \xi^j}{\partial v^i}\frac{\partial}{\partial v^j} \text{ and } \mathbf{\Gamma}\left(\frac{\partial}{\partial v^i}\right) = -\frac{\partial}{\partial v^i}\,. \tag{4.23}$$

For $\mathrm{h} = (1/2)(\mathrm{Id}_{TTM} + \mathbf{\Gamma})$ we get

$$\mathrm{h}\left(\frac{\partial}{\partial x^i}\right) = \frac{1}{2}(\mathrm{Id}_{TTM} - \mathcal{L}_{\boldsymbol{\xi}}\mathbf{S})\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} - \Gamma^j_i\frac{\partial}{\partial v^j}$$

from which follows that $\Gamma^j_i = -(1/2)(\partial\xi^j/\partial v^i)$. Based on (4.22), (4.1), and (4.23), we thus obtain the following

Proposition: 4.3 *The connection $\mathbf{\Gamma} = -\mathcal{L}_{\boldsymbol{\xi}}\mathbf{S}$ and its associated semispray $\bar{\boldsymbol{\xi}}$ have the same paths.*

Proof: 4.3 From equation (4.11), we know that we can write $\bar{\boldsymbol{\xi}}$ locally as

$$\bar{\boldsymbol{\xi}} = v^i\frac{\partial}{\partial x^i} - v^j\Gamma^i_j\frac{\partial}{\partial v^i}\,.$$

Since $\bar{\boldsymbol{\xi}}$ is the associated semispray to $\mathbf{\Gamma}$, we have $\Gamma^j_i = -(1/2)(\partial\xi^j/\partial v^i)$, hence

$$\bar{\boldsymbol{\xi}} = v^i\frac{\partial}{\partial x^i} + v^j\frac{1}{2}\frac{\partial \xi^i}{\partial v^j}\,.$$

For the path of $\bar{\boldsymbol{\xi}}$, we get $\ddot{x} = (1/2)v^j(\partial\xi^i/\partial v^j)$. Now, with equation (4.22) for the paths of $\mathbf{\Gamma}$ it follows that the paths of $\bar{\boldsymbol{\xi}}$ and $\mathbf{\Gamma}$ are the same. □

Proposition: 4.4 *If $\boldsymbol{\xi}$ is a spray, then $\boldsymbol{\Gamma} = -\mathcal{L}_{\xi}\mathbf{S}$ is a **homogeneous** connection whose **associated** semispray is precisely $\boldsymbol{\xi}$. Moreover, if $\boldsymbol{\xi}$ is a quadratic spray, then $\boldsymbol{\Gamma} = -\mathcal{L}_{\xi}\mathbf{S}$ is a linear connection.*

Proof: 4.4 By definition, for $\boldsymbol{\xi}$ to be a spray means $\boldsymbol{\xi}^* = 0$, hence by Proposition 4.2 it follows that

$$\boldsymbol{\xi} + \frac{1}{2}\boldsymbol{\xi}^* = \boldsymbol{\xi}\,.$$

By definition, a connection $\boldsymbol{\Gamma}$ is homogeneous if the functions Γ_i^j are homogeneous of degree 1. The condition for the homogeneity of Γ_i^j is

$$v^l \frac{\partial \Gamma_i^j}{\partial v^l} = \Gamma_i^j\,.$$

Now, with $\Gamma_i^j = -(1/2)(\partial \xi^j/\partial v^i)$, we get

$$-\frac{1}{2}v^l \frac{\partial^2 \xi^j}{\partial v^l \partial v^i} = v^l \frac{\partial \Gamma_i^j}{\partial v^l} = v^l \frac{\partial\left(-\frac{1}{2}\frac{\partial \xi^j}{\partial v^i}\right)}{\partial v^l} = -\frac{1}{2}\frac{\partial}{\partial v^i}\left[v^l \frac{\partial \xi^j}{\partial v^l} - \frac{\partial \xi^j}{\partial v^i}\right] = -\frac{1}{2}\frac{\partial \xi^j}{\partial v^i}\,,$$

since $\boldsymbol{\xi}$ is homogeneous of degree 1. Now, if moreover $\boldsymbol{\xi}$ is a quadratic spray, then $\boldsymbol{\xi}$ is C^2 on the zero section. Since $\Gamma_i^j = -(1/2)\partial\xi^j/\partial v^i$, we see that Γ_i^j is at least C^1 on the zero section, hence $\boldsymbol{\Gamma} = -\mathcal{L}_{\xi}\mathbf{S}$ is a linear connection. □

To finish this section, we present a result concerning the torsion $\mathbf{T}$ ([9], p. 298) of the connection $\boldsymbol{\Gamma}$.

Proposition: 4.5 *If $\boldsymbol{\xi}$ is a quadratic spray, then*

$$\boldsymbol{\Gamma} = -\mathcal{L}_{\xi}\mathbf{S}$$

is a symmetric ([9], p. 297) *connection, i.e., its torsion* $\mathbf{T}$ *vanishes.*

Proof: 4.5 Let us define the horizontal lift $\mathbf{U}^H$ of any vector field $\mathbf{U}$ on M as the horizontal projection of any vector field on TM that projects onto $\mathbf{U}$, i.e.,

$$\mathbf{U}^H = \mathbf{h}(\mathbf{U}^C) = \frac{1}{2}\left(\mathbf{U}^C - \mathcal{L}_{\xi}\mathbf{S}(\mathbf{U}^C)\right) = \frac{1}{2}\left(\mathbf{U}^C - [\boldsymbol{\xi}, \mathbf{U}^V]\right)\,. \tag{4.24}$$

The torsion of a linear connection can then be expressed as ([8])

$$(\mathbf{T}(\mathbf{U},\mathbf{V}))^V = [\mathbf{U}^H, \mathbf{V}^V] - [\mathbf{V}^H, \mathbf{U}^V] - [\mathbf{U},\mathbf{V}]^V\,. \tag{4.25}$$

Including (4.24) in (4.25) yields

$$\begin{aligned}
[\mathbf{U}^H,\mathbf{V}^V]-[\mathbf{V}^H,\mathbf{U}^V] &= \frac{1}{2}([\mathbf{U}^C,\mathbf{V}^V]-[[\boldsymbol{\xi},\mathbf{U}^V],\mathbf{V}^V]-[\mathbf{V}^C,\mathbf{U}^V]+[[\boldsymbol{\xi},\mathbf{V}^V],\mathbf{U}^V]) \\
&= \frac{1}{2}([\boldsymbol{\xi},[\mathbf{U}^V,\mathbf{V}^V]]+[\mathbf{U},\mathbf{V}]^V-[\mathbf{V},\mathbf{U}]^V)\,.
\end{aligned}$$

Based on ([8], p. 331) we have that $[\mathbf{V}^C,\mathbf{W}^V]=[\mathbf{V},\mathbf{W}]^V$, that $[\mathbf{V}^V,\mathbf{W}^V]=0$, and the Jacobi identity

$$[[\boldsymbol{\xi},\mathbf{U}^V],\mathbf{V}^V]+[[\mathbf{V}^V,\boldsymbol{\xi}],\mathbf{U}^V]+[[\mathbf{U}^V,\mathbf{V}^V],\boldsymbol{\xi}]=0\,,$$

from which follows that

$$[\mathbf{U}^H,\mathbf{V}^V]-[\mathbf{V}^H,\mathbf{U}^V]=[\mathbf{U},\mathbf{V}]^V$$

which then shows us that the torsion is zero. $\square$

5 Systems on the Setting

In Chapter 4 we considered the manifold M of our setting to be the configuration manifold of the system and we demonstrated how to represent a second-order differential equation on M. In this chapter, we will see how the equations of the adjoint Hamiltonian system to a second-order differential equation will be related to our setting. Moreover, we are interested in the representation of the uncontrolled mechanical system in Hamiltonian and Lagrangian formalism on our setting. We will then be able to recover how the different representations of our mechanical system (Hamiltonian and Lagrangian) are related to each other.

To compare the different representations we will write the systems in vector form.

5.1 Description of Hamiltonian and Lagrangian Systems

5.1.1 Hamiltonian systems

Let (P, ω) be a symplectic manifold ([3], p. 201). A vector field

$$\mathbf{X} : P \to TP$$

is said to be Hamiltonian ([3], p. 203) if the form

$$\mathbf{X} \lrcorner\, \omega$$

is exact where $\lrcorner$ is the contraction operator ([2], p. 341). A function $H : P \to \mathbb{R}$, such that

$$\mathbf{X} \lrcorner\, \omega = -dH$$

is called a Hamiltonian for $\mathbf{X}$. Since ω is closed,

$$\mathcal{L}_{\mathbf{X}}\omega = 0\ .$$

Example: 5.1 *We take the cotangent bundle T^*N as symplectic manifold and $\omega = d\boldsymbol{\lambda} \wedge d\mathbf{x}$ as symplectic two-form, where $(\mathbf{x}, \boldsymbol{\lambda})$ are the coordinates in T^*N and $\wedge$ is the wedge product*([3], p. 166). *With*

$$\mathbf{X} = \mathbf{a}\frac{\partial}{\partial \mathbf{x}} + \mathbf{b}\frac{\partial}{\partial \boldsymbol{\lambda}}$$

follows

$$\begin{aligned}
\mathbf{X} \lrcorner\, \omega &= \left(\mathbf{a}\frac{\partial}{\partial \mathbf{x}} + \mathbf{b}\frac{\partial}{\partial \boldsymbol{\lambda}}\right) \lrcorner\, d\boldsymbol{\lambda} \wedge d\mathbf{x} \\
&= -\mathbf{a}\, d\boldsymbol{\lambda} + \mathbf{b}\, d\mathbf{x} \\
&= -dH = -\frac{\partial H}{\partial \mathbf{x}} d\mathbf{x} - \frac{\partial H}{\partial \boldsymbol{\lambda}} d\boldsymbol{\lambda}\ .
\end{aligned}$$

Comparing the coefficients, we get

$$\mathbf{X} = \frac{\partial H}{\partial \boldsymbol{\lambda}}\frac{\partial}{\partial \mathbf{x}} - \frac{\partial H}{\partial \mathbf{x}}\frac{\partial}{\partial \boldsymbol{\lambda}} .$$

With $\mathbf{X} = (\partial H/\partial \boldsymbol{\lambda})(\partial/\partial \mathbf{x}) - (\partial H/\partial \mathbf{x})(\partial/\partial \boldsymbol{\lambda})$ *and* $\omega = d\boldsymbol{\lambda} \wedge d\mathbf{x}$ *we obtain*

$$\begin{aligned} \mathcal{L}_{\mathbf{X}}\omega &= d(\mathbf{X} \lrcorner\, \omega) + \mathbf{X} \lrcorner\, d\omega \\ &= d(-dH) + 0 \\ &= 0 . \end{aligned}$$

$\mathbf{X}$ *is called the* ***system vector** field and* T^*N *the* ***phase space**. We can write the Hamiltonian system in canonical form which yields*

$$\begin{aligned} \dot{x}^i &= \frac{\partial H}{\partial \lambda_i} \\ \dot{\lambda}_i &= -\frac{\partial H}{\partial x^i} \end{aligned}$$

where $\dot{\mathbf{x}}$ *and* $\dot{\boldsymbol{\lambda}}$ *are the components of the system vector field.*

5.1.2 Lagrangian Systems

Consider an n-dimensional manifold D with coordinate system ($\mathbf{q} = (q^1, \dots, q^n)$). On the tangent bundle (TD, τ_D, D) there is a coordinate system ($\mathbf{q} = (q^1, \dots, q^n)$, $\mathbf{v} = (v_1, \dots, v_n)$). To describe Lagrangian systems on a manifold D we will introduce a Lagrangian vector field $\boldsymbol{\xi}_L$ which is on TD and, since the Lagrange equation is a second-order differential equation, this vector field is a **semispray** (Chapter 4). The Lagrangian system is determined by the Lagrange function $L : TD \to \mathbb{R}$. To express the Euler-Lagrange equation in terms of vector fields we have to introduce a two-form ω_L on TD ([16], p. 301). By vertical derivation (cf. [16], p. 184) this two-form is given by

$$\omega_L = \frac{\partial^2 L}{\partial q^j \partial v^i} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v^i \partial v^j} dq^i \wedge dv^j .$$

The **Energy function**, $E_L : TD \to \mathbb{R}$, is defined by

$$E_L = \mathcal{L}_{\mathbf{C}} L - L$$

($\mathbf{C}$ being the Liouville vector field). By calculation it can be shown that the Euler-Lagrange equation may be written in terms of vector fields as

$$\boldsymbol{\xi}_L \lrcorner\, \omega_L = dE_L . \tag{5.1}$$

Thus, in coordinates, equation (5.1) becomes

$$\left(v^l\frac{\partial}{\partial q^l}+\xi_L^l\frac{\partial}{\partial v^l}\right)\lrcorner\left(\frac{\partial^2 L}{\partial q^j\partial v^i}dq^i\wedge dq^j+\frac{\partial^2 L}{\partial v^i v^j}dq^i\wedge dv^j\right)=d(E_L)$$

$$=d\left(v^l\frac{\partial L}{\partial v^l}-L\right)=v^l\frac{\partial^2 L}{\partial v^l\partial q^k}dq^k+v^l\frac{\partial^2 L}{\partial v^l\partial v^k}dv^k-\frac{\partial L}{\partial q^i}dq^i\,.$$

Comparing the coefficients yields

$$\xi_L^l\frac{\partial^2 L}{\partial v^i\partial v^l}+v^l\frac{\partial^2 L}{\partial q^l\partial v^i}-\frac{\partial L}{\partial q^i}=0\,.$$

If $\mathbf{q}(t)$ is a path of $\boldsymbol{\xi}_L$, we get

$$\ddot{q}^l\frac{\partial^2 L}{\partial \dot{q}^i\partial \dot{q}^l}+\dot{q}^l\frac{\partial^2 L}{\partial q^l\partial \dot{q}^i}-\frac{\partial L}{\partial q^i}=0\,,$$

which is of course the Euler-Lagrange equation. The vector field $\boldsymbol{\xi}_L$ is thus the vector field of the system and TD is the velocity phase space. If the system is not conservative, equation (5.1) becomes

$$\boldsymbol{\xi}_L\lrcorner\,\boldsymbol{\omega}_L=dE_L+\boldsymbol{\rho}\,,$$

where $\boldsymbol{\rho}$ is a semibasic form ([16], p. 190) on TD, called the ***force field***. We set $\boldsymbol{\rho}=\mathbf{f}(\mathrm{q},\mathrm{v})d\mathrm{q}$.

Definition: 5.1 ([16], p. 306) *A Lagrangian $L:TD\to\mathbb{R}$ is said to be **homogeneous** if the function L is homogeneous of degree 2, i.e., $\mathcal{L}_{\mathbf{C}}L=2L$, where $\mathbf{C}$ is the Liouville vector field.*

In ([16], p. 312) it is shown that for a homogeneous Lagrangian L the Euler-Lagrange vector field is a spray and there is one and only one homogeneous connection $\boldsymbol{\Gamma}$ whose geodesics form the solution of the Euler-Lagrange equation.

Example: 5.2 *If we have $L:TD\to\mathbb{R}$ with $L=(1/2)g_{ij}\dot{q}^i\dot{q}^j$ with g_{ij} as Riemannian metric on D, then L is a homogeneous Lagrangian which may represent the kinetic energy of a mechanical system. It can be shown* ([9], p. 317) *that the Euler-Lagrange equation of this system can be expressed by*

$$\ddot{q}^i=-\Gamma^i_{kj}\dot{q}^k\dot{q}^j \tag{5.2}$$

where Γ^i_{kj} are the Christoffel symbols compatible with the metric g_{ij}. Notice, that from Proposition 4.3 *and Equation* (4.22), it *follows that $q(t)$ is nothing else than a geodesic of the metric. Especially conservative holonomic mechanical systems are of this form.*

After this introduction into the vectorial representations of Hamiltonian and Lagrangian systems, it is our next goal to localize them on our setting.

5.2 Lagrange Systems on the Setting

The system we consider is a Lagrangian system on the manifold M (with coordinate system $(\mathbf{x} = (x^1, \dots, x^n))$ with Lagrange function $L : TM \to \mathbb{R}$. Then, in our setting ω_L is given as

$$\omega_L = \frac{\partial^2 L}{\partial x^j \partial v^i} dx^i \wedge dx^j + \frac{\partial^2 L}{\partial v^i \partial v^j} dx^i \wedge dv^j$$

and the semispray as

$$\boldsymbol{\xi}_L = v^i \frac{\partial}{\partial x^i} + \xi_L^i \frac{\partial}{\partial v^i} . \tag{5.3}$$

This system represents the uncontrolled mechanical system without forces.

5.3 Hamiltonian Systems on the Setting

System on $T^*(TM)$

Out of the spray which describes the Lagrangian system on M with Lagrangian $L = \frac{1}{2} g_{ij} v^i v^j$ we can form a Hamiltonian system on $T^*(TM)$ with Hamiltonian function $S : T^*(TM) \to \mathbb{R}$. We will show how this can be done. The symplectic two-form of the system corresponding to the local coordinates on $T^*(TM)$ is

$$\omega_S = d\boldsymbol{\lambda} \wedge d\mathbf{x} + d\boldsymbol{\mu} \wedge d\mathbf{v} .$$

The Hamiltonian vector field lives on $T^*(TM)$ with the components

$$\mathbf{X}_S = \frac{\partial S}{\partial \lambda_i} \frac{\partial}{\partial x^i} + \frac{\partial S}{\partial \mu_i} \frac{\partial}{\partial v^i} - \frac{\partial S}{\partial x^i} \frac{\partial}{\partial \lambda_i} - \frac{\partial S}{\partial v^i} \frac{\partial}{\partial \mu_i} .$$

This Hamiltonian system represents the uncontrolled system or, in other words, the controlled system with the control equal to zero. This corresponds to the zero section of our bundle E, as we will see later. As already mentioned, the Hamiltonian system is computed out of the semispray $\boldsymbol{\xi}_L = v^i(\partial/\partial x^i) + \xi_L^i(\partial/\partial v^i)$ of our system. The computation is as follows: Under the consideration of the forces f^i we have

$$\ddot{x}^i = \xi_L^i + f^i .$$

This second-order equation can be transformed into two first-order equations

$$\begin{aligned} \dot{x}^i &= v^i \\ \dot{v}^i &= \xi_L^i + f^i . \end{aligned}$$

Now, by adjoining the dual variables $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ to $(\mathbf{x}, \mathbf{v})$, we can construct the Hamiltonian function

$$S = v^i \lambda_i + \xi_L^i \mu_i + f^i \mu_i \, .$$

The Hamiltonian system in canonical representation then becomes

$$\begin{aligned}
\dot{x}^i &= \frac{\partial S}{\partial \lambda_i} = v^i \\
\dot{v}^i &= \frac{\partial S}{\partial \mu_i} = \xi_L^i + f^i \\
\dot{\lambda}_i &= -\frac{\partial S}{\partial x^i} = -\frac{\partial \xi_L^l}{\partial x^i} \mu_l \\
\dot{\mu}_i &= -\frac{\partial S}{\partial v^i} = -\frac{\partial \xi_L^l}{\partial v^i} \mu_l - \lambda_i \, .
\end{aligned}$$

We obtain the system vector field $\mathbf{X}_S$ as

$$\mathbf{X}_S = \operatorname{sgrad} S = v^i \frac{\partial}{\partial x^i} + \left(\xi_L^i + f^i\right) \frac{\partial}{\partial v^i} - \frac{\partial \xi_L^l}{\partial x^i} \mu_l \frac{\partial}{\partial \lambda_i} - \left(\frac{\partial \xi_L^l}{\partial v^i} \mu_l + \lambda_i\right) \frac{\partial}{\partial \mu_i}$$

which yields

$$\mathbf{X}_S \lrcorner\, \omega_S = -dS \, .$$

At first glance, this construction does not seem to be very reasonable, but this Hamiltonian system is a **preliminary stage** of the Hamiltonian system we have to build, namely, following Pontryagin's rule, the Hamiltonian adjoint system. With this construction at hand, we can relate the Lagrangian description of the system to Pontryagin's minimum principle. This should be clear, if we consider the equations for the Hamiltonian adjoint system (1.3). Considering Figure 17, we introduce the system on E, E being a fibre bundle (cf. Chapter 3).

System on E

Let us introduce the control $\mathbf{u}$. We assume the control $\mathbf{u}$ to be a section of the bundle E. Furthermore, we suppose the control system to be of the form

$$\ddot{x}^i = \xi_L^i + f^i + u^i \, .$$

When the system on $T^*(TM)$ gets controlled it could in some sense be understood as a Hamiltonian system on $T^*(TM)$ that is parameterized by the control $\mathbf{u}$. But of course, we cannot call it Hamiltonian system on E, since E is not a symplectic manifold. The system can be written as a system of first-order differential equations with a Hamiltonian $H : E \to \mathbb{R}$, $H = v^i \lambda_i + (\xi_L^i + f^i + u^i)\mu_i$. Moreover, it looks like a Hamiltonian system in canonical form, but there is no possibility to write it with the help of a symplectic two-form, because such a form does not exist on E. In the next step we derive the optimally controlled Hamiltonian system on K (cf. Figure 17).

System on K

Out of the system on E we will then construct the Hamiltonian adjoint system. We first introduce the function $H' : E \to \mathbb{R}$ by setting

$$H' = L_{perf} + H \ .$$

Since we perform time-optimal control it must be that $L_{perf} = 1$. According to Pontryagin we have to minimize H'. We set

$$H^o = \min_{\mathbf{u}} H' \ .$$

The section u of the bundle E that minimizes H' is called the optimal $\mathbf{u}$ and is denoted by $\mathbf{u}^o$. Since $\mathbf{u}^o$ is a section of bundle E the submanifold K (Chapter 3) of E can be defined. We recognize that K is a symplectic manifold, and

$$H^o : K \to \mathbb{R} \ .$$

The Hamiltonian system on K is thus given by the vector field

$$\mathbf{X}_{H^o} = \mathrm{sgrad} H^o \ .$$

6 Reduction of Hamiltonian System on K

In this chapter we introduce the notion of a First Integral on K and the symmetry vector field that is related to this First Integral. We show that reduction of dimension of the optimally controlled system on K by a First Integral can be performed. (See Chapters 3 and 5 for a closer description of K).

6.1 First Integrals and Symmetries

6.1.1 First Integrals

Let us define the notion of a First Integral on K.

Definition: 6.1 *A* ***First Integral*** *is a map $F_{H^\circ} : K \to \mathbb{R}$, such that $\mathcal{L}_{\mathbf{X}_{H^\circ}} F_{H^\circ} = 0$, where $\mathbf{X}_{H^\circ}$ is the Hamiltonian system vector field.*

Of course,

$$\frac{d}{dt} F_{H^\circ} = \mathcal{L}_{\mathbf{X}_{H^\circ}} F_{H^\circ} = 0 \; .$$

6.1.2 Symmetry Vector Fields

According to the famous theorem of Emmy Noether [19], there is a so-called symmetry vector field $\mathbf{Y}_{H^\circ}$ associated to each First Integral F_{H° of the Hamiltonian system on K. This vector field $\mathbf{Y}_{H^\circ}$ is defined by $\mathbf{Y}_{H^\circ} = \operatorname{sgrad} F_{H^\circ}$, where $\operatorname{sgrad} : C^\infty(K, \mathbb{R}) \to \mathcal{X}(K)$ and $\mathcal{X}(K)$ is the space of smooth vector fields on K. We define sgrad such that

$$\omega(\mathbf{V}, \operatorname{sgrad} F_{H^\circ}) = \mathcal{L}_{\mathbf{V}} F_{H^\circ} \tag{6.1}$$

where $\omega = d\boldsymbol{\lambda} \wedge d\mathbf{x} + d\boldsymbol{\mu} \wedge d\mathbf{v}$ is the symplectic two-form on K and $\mathbf{V} \in \mathcal{X}(K)$. With equation (6.1) the symmetry vector field becomes

$$\begin{aligned} \mathbf{Y}_{H^\circ} &= \operatorname{sgrad} F_{H^\circ} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \operatorname{grad} F_{H^\circ} \\ &= \frac{\partial F_{H^\circ}}{\partial \boldsymbol{\lambda}} \frac{\partial}{\partial \mathbf{x}} + \frac{\partial F_{H^\circ}}{\partial \boldsymbol{\mu}} \frac{\partial}{\partial \mathbf{v}} - \frac{\partial F_{H^\circ}}{\partial \mathbf{x}} \frac{\partial}{\partial \boldsymbol{\lambda}} - \frac{\partial F_{H^\circ}}{\partial \mathbf{v}} \frac{\partial}{\partial \boldsymbol{\mu}} \; . \end{aligned}$$

From this definition of $\mathbf{Y}_{H^\circ}$, it is easy to see that

$$[\mathbf{X}_{H^\circ}, \mathbf{Y}_{H^\circ}] = 0 \; . \tag{6.2}$$

Furthermore, we have

$$\mathbf{Y}_{H^o} \lrcorner\, \omega = -dF_{H^o}, \quad \mathcal{L}_{\mathbf{Y}_{H^o}}\omega = 0\,,$$

$$\mathcal{L}_{\mathbf{Y}_{H^o}} H^o = \omega(\mathbf{Y}_{H^o}, \mathbf{X}_{H^o}) = 0\,, \quad \text{and} \quad \mathcal{L}_{\mathbf{Y}_{H^o}} F_{H^o} = 0\,.$$

The construction of $\mathbf{Y}_{H^o}$ is unique due to the fact that ω is nondegenerate.

6.2 Reduction of Dimension

6.2.1 Reduction of Dimension by one First Integral

The dimension of the Hamiltonian System on K can be reduced by 2 with the help of one First Integral.

Let $F_{H^o} : K \to \mathbb{R}$ be a First Integral. To F_{H^o} there is associated a symmetry vector field $\mathbf{Y}_{H^o} = \operatorname{sgrad} F_{H^o}$, with flow Υ^t. We introduce a symplectic map $\Phi : K \to K$ that can be considered as a coordinate transformation on K. Let

$$\Phi^{-1} : (\mathbf{x}, \mathbf{v}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \mapsto (\mathbf{y}, \mathbf{w}, \boldsymbol{\varphi}, \boldsymbol{\psi})$$

where $(\mathbf{y} = (y^1, \dots, y^n), \mathbf{w} = (w^1, \dots, w^n), \boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n), \boldsymbol{\psi} = (\psi_1, \dots, \psi_n))$ is a new coordinate system on K. Therefore we have

$$\bar{H}^o(\mathbf{y}, \mathbf{w}, \boldsymbol{\varphi}, \boldsymbol{\psi}) = H^o \circ \Phi$$

where $\bar{H}^o$ is the transformed Hamiltonian. Since Φ is symplectic, the transformed system is still a Hamiltonian system with $\dot{\mathbf{y}} = \partial\bar{H}^o/\partial\boldsymbol{\varphi}$, $\dot{\mathbf{w}} = \partial\bar{H}^o/\partial\boldsymbol{\psi}$, $\dot{\boldsymbol{\varphi}} = -\partial\bar{H}^o/\partial\mathbf{y}$, and $\dot{\boldsymbol{\psi}} = -\partial\bar{H}^o/\partial\mathbf{w}$.
We want Φ to have the properties

$$F_{H^o} \circ \Phi = \varphi_1 \tag{6.3}$$

and

$$\bar{\mathbf{Y}}_{H^o} = \Phi_*^{-1}\mathbf{Y}_{H^o} = 1\frac{\partial}{\partial y^1}\,.$$

In other words, we want φ_1 to become the transformed First Integral and the transformed symmetry vector field to be "parallel" to the coordinate y^1. Let Θ^t be the flow of $\bar{\mathbf{Y}}_{H^o}$ which yields

$$\Theta^t(\mathbf{y}, \mathbf{w}, \boldsymbol{\varphi}, \boldsymbol{\psi}) = (y^1 + t\frac{\partial}{\partial y_1}, \mathbf{y}', \mathbf{w}, \boldsymbol{\varphi}, \boldsymbol{\psi}) \tag{6.4}$$

for small t where $\mathbf{y}' = (y^2, \dots, y^n)$. Furthermore, Φ has to satisfy

$$\Upsilon^t \circ \Phi = \Phi \circ \Theta^t\,. \tag{6.5}$$

Since we are largely free in the choice of Φ, when restricted to the hypersurface $y^1 = 0$, we could attempt to set Φ equal to the identity. However, this would not be compatible with (6.3). Therefore we require for $y^1 = 0$ only

$$\begin{array}{rclcrcl} x^i &=& y^i & & & & \\ v^i &=& w^i & \qquad & i &=& 1,\ldots,n \\ \lambda_j &=& \varphi_j & & j &=& 2,\ldots,n\,. \\ \mu_i &=& \psi_i & & & & \end{array}$$

We set

$$\lambda_1 = b_1(\mathbf{y}', \mathbf{w}, \varphi', \psi)$$

with $\varphi' = (\varphi_2, \ldots, \varphi_n)$ and define b_1 to be

$$F_{H^o}(0, \mathbf{y}', \mathbf{w}, b_1, \varphi', \psi) = \varphi_1\,. \tag{6.6}$$

If we set $y^1 = 0$, with (6.5) we find from (6.4)

$$\Upsilon^t(0, \mathbf{y}', \mathbf{w}, b_1, \varphi', \psi) = \Phi(t, \mathbf{y}', \mathbf{w}, \varphi_1, \varphi', \psi)\,.$$

We will thus define $\Phi(\mathbf{y}, \mathbf{w}, \varphi, \psi)$ as

$$\Phi(\mathbf{y}, \mathbf{w}, \varphi, \psi) = \Upsilon^{y^1}(0, \mathbf{y}', \mathbf{w}, b_1, \varphi', \psi) \tag{6.7}$$

whereby b_1 is defined by (6.6).

We have to verify that Φ possesses the required properties. For example Φ satisfies (6.5) since

$$\Upsilon^t \circ \Phi = \Upsilon^t \circ \Upsilon^{y^1}(0, \mathbf{y}', \mathbf{w}, b_1, \varphi', \psi) = \Upsilon^{t+y^1}(0, \mathbf{y}', \mathbf{w}, b_1, \varphi', \psi) = \Phi \circ \Theta^t$$

(Figure 18).

Due to its length, we refrain from performing the proof here. We suggest to look it up in ([18], p. 65). We would like to emphasize that [18] is the only work where we learned how to execute the reduction step. We did not succeed, on the other hand, in performing the reduction step with the help of a generating function ([3], p. 259) for the symplectic transformation Φ.

In Figure 18, t and $y^1_{(1)}$ are assumed to be small, therefore the coordinate axes are chosen to be straight lines.

We summarize this tricky construction: We introduce the time t elapsed since passing the hypersurface $y^1 = 0$ and the Hamiltonian F_{H^o} as independent variables (Figure 18).

We recognize that $\bar{H}^o$ is no longer dependent on y^1 and $\dot{\varphi}_1 = \partial \bar{H}^o / \partial y^1 = 0$.

Whenever we have several First Integrals, further reduction steps can be performed. The First Integrals have to be in involution, that is $\{F_i, F_j\} = 0$, where F_i and F_j are two First Integrals. For more information see ([11], p. 107).

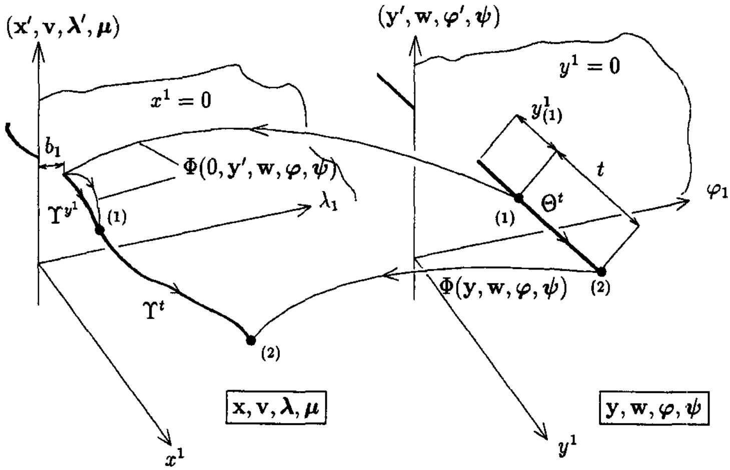

Figure 18: Reduction by canonical transformation Φ

7 Symmetries on the Setting

7.1 Spray and Symmetries of the Lagrangian system on TM

We are trying to find symmetries of our system in Lagrangian form on our setting. In a first step we will look for symmetries of the uncontrolled system without forces. We assume this system to be described by a homogeneous Lagrange function $L : TM \to \mathbb{R}$ (Definition 5.1). This system therefore is described by a spray $\boldsymbol{\xi}_L$ (Definition 4.3). We then look for symmetries of this spray.

A vector field $\mathbf{W}$ is an ***infinitesimal symmetry*** of the spray $\boldsymbol{\xi}_L$ iff

$$\mathcal{L}_{\mathbf{W}}\boldsymbol{\xi}_L = [\mathbf{W}, \boldsymbol{\xi}_L] = 0 \,. \tag{7.1}$$

Proposition: 7.1 $\mathbf{Y}^C$, *the complete lift of a vector field* $\mathbf{Y}$ *on* M, *is an infinitesimal symmetry of the spray* $\boldsymbol{\xi}_L$ *if and only if* $\mathbf{Y}$ *is an* ***infinitesimal affine transformation*** *of the corresponding affine connection.*

Proof: 7.1 We will first have to find a criterion that expresses the fact that $\mathbf{Y}$ generates an infinitesimal affine transformation. This condition will then be formulated in terms of lifts of the corresponding vector fields. This will lead us to Equation (7.11). In what follows the different parts of this equation will be investigated. Therefore, we apply the connection $\mathbf{\Gamma} = -\mathcal{L}_{\boldsymbol{\xi}_L}\mathbf{S}$ on $(TM, \mathbf{S})$ which is entirely determined by the spray $\boldsymbol{\xi}_L$. With the help of this connection and the related projections (4.7) we will be able to show that some parts of (7.11) are horizontal and others are vertical vector fields. We then will see that what we proposed is only true if $[\mathbf{Y}^C, \boldsymbol{\xi}_L] = 0$ without contradiction. Once and for all, we fix a Riemannian metric $\mathbf{g}$ on M and a connection $\mathbf{\Gamma}$ which is compatible with $\mathbf{g}$. With $\nabla = \nabla^{\mathbf{\Gamma}}$ we denote the associated covariant derivative. An affine map of M onto itself which is a diffeomorphism is called an affine transformation of M. A one-parameter group ϕ_t of diffeomorphisms of M is called **affine** if ϕ_t is an affine transformation of M for each t. The generator $\mathbf{Y}$ of a one-parameter group of affine transformations is called an infinitesimal affine transformation. Since the corresponding one-parameter group consists of affine transformations, for every pair of vector fields $\mathbf{V}$, $\mathbf{W} \in M$ it must be that

(7.2)
$$\phi_{t*}(\nabla_{\mathbf{V}}\mathbf{W}) = \nabla_{(\phi_{t*}\mathbf{V})}\phi_{t*}\mathbf{W} = \nabla_{(\phi_{t*}\mathbf{V}-\mathbf{V})}\phi_{t*}\mathbf{W} + \nabla_{\mathbf{V}}(\phi_{t*}\mathbf{W} - \mathbf{W}) + \nabla_{\mathbf{V}}\mathbf{W}$$

so that

$$\phi_{t*}(\nabla_{\mathbf{V}}\mathbf{W}) - \nabla_{\mathbf{V}}\mathbf{W} = \nabla_{(\phi_{t*}\mathbf{V}-\mathbf{V})}\phi_{t*}\mathbf{W} + \nabla_{\mathbf{V}}(\phi_{t*}\mathbf{W} - \mathbf{W})\,. \tag{7.3}$$

Dividing by t and taking the limits as $t \to 0$ we obtain

$$\boxed{\mathcal{L}_{\mathbf{Y}}\nabla_{\mathbf{V}}\mathbf{W} - \nabla_{\mathbf{V}}\mathcal{L}_{\mathbf{Y}}\mathbf{W} - \nabla_{[\mathbf{Y},\mathbf{V}]}\mathbf{W} = 0}\,. \tag{7.4}$$

This is the condition for $\mathbf{Y}$ to be an infinitesimal affine transformation. In the next step we will formulate this expression in terms of **lifts**.

A calculation yields

$$[\mathbf{V}^C, \mathbf{W}^V] = [\mathbf{V}, \mathbf{W}]^V \qquad \text{and} \qquad [\mathbf{V}^H, \mathbf{W}^V] = (\nabla_{\mathbf{V}}\mathbf{W})^V\,. \tag{7.5}$$

Furthermore the Jacobi identity

$$[\mathbf{U}, [\mathbf{V}, \mathbf{W}]] + [\mathbf{V}, [\mathbf{W}, \mathbf{U}]] + [\mathbf{W}, [\mathbf{U}, \mathbf{V}]] = 0 \tag{7.6}$$

is considered. From (7.4) it follows naturally that

$$(\mathcal{L}_{\mathbf{Y}}\nabla_{\mathbf{V}}\mathbf{W})^V - (\nabla_{\mathbf{V}}[\mathbf{Y}, \mathbf{W}])^V - \left(\nabla_{[\mathbf{Y},\mathbf{V}]}\mathbf{W}\right)^V = 0 \tag{7.7}$$

which, together with (7.5), yields

$$[\mathbf{Y}^C, [\mathbf{V}^H, \mathbf{W}^V]] - (\nabla_{\mathbf{V}}[\mathbf{Y}, \mathbf{W}])^V - \left(\nabla_{[\mathbf{Y},\mathbf{V}]}\mathbf{W}\right)^V = 0 \tag{7.8}$$

and

$$[\mathbf{Y}^C,[\mathbf{V}^H,\mathbf{W}^V]]-[\mathbf{V}^H,[\mathbf{Y}^C,\mathbf{W}^V]]-[[\mathbf{Y},\mathbf{V}]^H,\mathbf{W}^V]=0\,. \tag{7.9}$$

From the Jacobi identity, for the first two terms of (7.9), follows that

$$[\mathbf{Y}^C[\mathbf{V}^H,\mathbf{W}^V]]+[\mathbf{V}^H,[\mathbf{W}^V,\mathbf{Y}^C]]=-[\mathbf{W}^V,[\mathbf{Y}^C,\mathbf{V}^H]] \tag{7.10}$$

and from (7.9), together with (7.10), we obtain

$$-[\mathbf{W}^V,[\mathbf{Y}^C,\mathbf{V}^H]]-[[\mathbf{Y},\mathbf{V}]^H,\mathbf{W}^V]=0\,,$$

$$[[\mathbf{Y}^C,\mathbf{V}^H],\mathbf{W}^V]]-[[\mathbf{Y},\mathbf{V}]^H,\mathbf{W}^V]=0$$

and

$$\boxed{[[\mathbf{Y}^C,\mathbf{V}^H]-[\mathbf{Y},\mathbf{V}]^H,\mathbf{W}^V]=0}\,. \tag{7.11}$$

Hence (7.11) gives us the representation of (7.4) in terms of lifts.

Now we will focus on (7.11).

Proposition: 7.2 *The vector field*

$$[\mathbf{Y}^C,\mathbf{V}^H]-[\mathbf{Y},\mathbf{V}]^H \tag{7.12}$$

of (7.11) *is vertical.*

Proof: 7.2 With $\tau_{M*}:T(TM)\to TM$ and $\tau_{M*}\mathbf{Z}^V=0$, we get

$$\begin{aligned}\tau_{M*}([\mathbf{Y}^C,\mathbf{V}^H]-[\mathbf{Y},\mathbf{V}]^H)&=[\mathbf{Y},\mathbf{V}]-\tau_{M*}([\mathbf{Y},\mathbf{V}]^H)\\&=[\mathbf{Y},\mathbf{V}]-[\mathbf{Y},\mathbf{V}]=0\,,\end{aligned}$$

since $\tau_{M*}\mathbf{Z}^H=\mathbf{Z}$. □

The continuation of the discussion of (7.11) requires the following proposition.

Proposition: 7.3 *A vertical vector field on TM which commutes with every vertical lift* ***must*** *itself be a vertical lift.*

Proof: 7.3 Let $\bar{\mathbf{Z}}$ be a vector field in $T(TM)$ with local representation

$$\bar{\mathbf{Z}}=Z_x^i\frac{\partial}{\partial x^i}+Z_v^i\frac{\partial}{\partial v^i}\,. \tag{7.13}$$

For the projection of $\bar{\mathbf{Z}}$ onto the vertical subspace V of $T(TM)$, according to (4.7), Proposition 4.1, and (4.23) we get

$$\begin{aligned}
\mathrm{v}(\bar{\mathbf{Z}}) &= = \frac{1}{2}(\mathrm{Id}_{TTM} - \mathbf{\Gamma})\left(Z_x^i \frac{\partial}{\partial x^i} + Z_v^i \frac{\partial}{\partial v^i}\right) \\
&= \frac{1}{2}\left(Z_x^i \frac{\partial}{\partial x^i} + Z_v^i \frac{\partial}{\partial v^i} - Z_x^i \frac{\partial}{\partial x^i} - Z_x^k \frac{\partial \xi_L^i}{\partial v^k}\frac{\partial}{\partial v^i} + Z_v^i \frac{\partial}{\partial v^i}\right) \\
&= \frac{1}{2}\left(2Z_v^i - Z_x^k \frac{\partial \xi_L^i}{\partial v^k}\right)\frac{\partial}{\partial v^i} \\
&= \mathbf{Z}\,.
\end{aligned}$$

Now, calculating the Lie bracket between $\mathbf{Z}$ and $\mathbf{W}^V$, for $\mathbf{W}$ on M yields

$$\begin{aligned}
\left[\mathbf{Z}, W^i \frac{\partial}{\partial v^i}\right] &= \left[\left(Z_v^i - \frac{1}{2} Z_x^k \frac{\partial \xi_L^i}{\partial v^k}\right)\frac{\partial}{\partial v^i}, W^i \frac{\partial}{\partial v^i}\right] \\
&= Z_v^l \frac{\partial W^i}{\partial v^l}\frac{\partial}{\partial v^i} - \frac{1}{2} Z_x^k \frac{\partial \xi_L^l}{\partial v^k}\frac{\partial W^i}{\partial v^l}\frac{\partial}{\partial v^i} - W^l \frac{\partial Z_v^i}{\partial v^l}\frac{\partial}{\partial v^i} + W^l \frac{1}{2}\frac{\partial\left(Z_x^k \frac{\partial \xi_L^i}{\partial v^k}\right)}{\partial v^l}\frac{\partial}{\partial v^i}\,.
\end{aligned}$$

Clearly, this expression is zero if and only if $Z_x^i = 0$ and $Z_v^i = Z_v^i(\mathbf{x})$ which, on the other hand, implies that $\bar{\mathbf{Z}}$ is a vertical lift.

Therefore, $\mathbf{Y}$ is an infinitesimal affine transformation of M if and only if, for all vector fields V on M,

$$[\mathbf{Y}^C, \mathbf{V}^H] - [\mathbf{Y}, \mathbf{V}]^H \tag{7.14}$$

is a vertical lift. □

Proposition: 7.4 $\mathbf{Y}$ *is an infinitesimal affine transformation of* M *if and only if* $[\mathbf{Y}^C, \mathbf{V}^H] = [\mathbf{Y}, \mathbf{V}]^H$ *for every* $\mathbf{V}$.

Proof: 7.4 With (11.16) and (11.17) we get

$$[\mathbf{C}, [\mathbf{Y}^C, \mathbf{V}^H] - [\mathbf{Y}, \mathbf{V}]^H] = [\mathbf{C}, \mathbf{Y}^C] + [\mathbf{C}, \mathbf{V}^H] - [\mathbf{C}, [\mathbf{Y}, \mathbf{V}]^H] = 0\,.$$

However, according to (11.15)

$$[\mathbf{C}, \mathbf{Z}^V] = -\mathbf{Z}^V$$

for any $\mathbf{Z} \in TM$ and $[\mathbf{Y}^C, \mathbf{V}^H] - [\mathbf{Y}, \mathbf{V}]^H$ is a vertical lift by Proposition 7.3, we obtain

$$[\mathbf{Y}^C, \mathbf{V}^H] - [\mathbf{Y}, \mathbf{V}]^H = 0\,. \tag{7.15}$$

Hence $[\mathbf{Y}^C, \mathbf{V}^H] = [\mathbf{Y}, \mathbf{V}]^H$. □

Next we will evaluate $\mathcal{L}_{\xi_L}\mathbf{S}$ on vertical and horizontal vector fields. This leads us to the following two propositions:

Proposition: 7.5 *For any two vector fields* $\mathbf{V}$ *and* $\mathbf{W}$ *on* TM *we have*

$$\left(\mathcal{L}_{\mathbf{V}^C}(\mathcal{L}_{\xi_L}\mathbf{S})\right)(\mathbf{W}^V) = 0\,. \tag{7.16}$$

Proof: 7.5 From (4.3) we deduce

$$\begin{aligned}\left(\mathcal{L}_{\mathbf{V}^C}(\mathcal{L}_{\xi_L}\mathbf{S})\right)(\mathbf{W}^V) &= \mathcal{L}_{\mathbf{V}^C}((\mathcal{L}_{\xi_L}\mathbf{S}(\mathbf{W}^V)) - (\mathcal{L}_{\xi_L}\mathbf{S})(\mathcal{L}_{\mathbf{V}^C}\mathbf{W}^V)\\ = \mathcal{L}_{\mathbf{V}^C}\left(\mathbf{W}^V\right) - (\mathcal{L}_{\xi_L}\mathbf{S})[\mathbf{V}^C,\mathbf{W}]^V &= [\mathbf{V}^C,\mathbf{W}]^V - [\mathbf{V}^C,\mathbf{W}]^V = 0\,,\end{aligned}$$

and we are done. □

Proposition: 7.6 *For any two vector fields* $\mathbf{V}$ *and* $\mathbf{W}$ *on* TM *we get*

$$\left(\mathcal{L}_{\mathbf{V}^C}(\mathcal{L}_{\xi_L}\mathbf{S})\right)(\mathbf{W}^H) = -2\mathbf{v}([\mathbf{V}^C,\mathbf{W}^H]) \tag{7.17}$$

where $\mathbf{v}$ *is the vertical projection of* (4.7).

Proof: 7.6 Now

$$\left(\mathcal{L}_{\mathbf{V}^C}(\mathcal{L}_{\xi_L}\mathbf{S})\right)(\mathbf{W}^H) = \mathcal{L}_{\mathbf{V}^C}((\mathcal{L}_{\xi_L}\mathbf{S})(\mathbf{W}^H)) - (\mathcal{L}_{\xi_L}\mathbf{S})(\mathcal{L}_{\mathbf{V}^C}\mathbf{W}^H)\,,$$

together with $\mathbf{v} = \frac{1}{2}(\mathrm{Id}_{TTM} + \mathcal{L}_{\xi_L}\mathbf{S})$ and $\mathcal{L}_{\xi_L}\mathbf{S} = 2\mathbf{v} - \mathrm{Id}_{TTM}$ follows that

$$\begin{aligned}(\mathcal{L}_{\mathbf{V}^C}(2\mathbf{v} - \mathrm{Id}_{TTM}))(\mathbf{W}^H) &= \mathcal{L}_{\mathbf{V}^C}((2\mathbf{v} - \mathrm{Id}_{TTM})(\mathbf{W}^H)) - (2\mathbf{v} - \mathrm{Id}_{TTM})(\mathcal{L}_{\mathbf{V}^C}\mathbf{W}^H)\\ &= \mathcal{L}_{\mathbf{V}^C}(2\mathbf{v}(\mathbf{W}^H) - \mathbf{W}^H) - 2\mathbf{v}(\mathcal{L}_{\mathbf{V}^C}\mathbf{W}^H) + \mathcal{L}_{\mathbf{V}^C}\mathbf{W}^H\\ &= \mathcal{L}_{\mathbf{V}^C}(2\mathbf{v}\mathbf{W}^H) - \mathcal{L}_{\mathbf{V}^C}\mathbf{W}^H - 2\mathbf{v}(\mathcal{L}_{\mathbf{V}^C}\mathbf{W}^H) + \mathcal{L}_{\mathbf{V}^C}\mathbf{W}^H\\ &= \mathcal{L}_{\mathbf{V}^C}(2\mathbf{v}(\mathbf{W}^H) - 2\mathbf{v}(\mathcal{L}_{\mathbf{V}^C}\mathbf{W}^H)\\ &= -2\mathbf{v}([\mathbf{V}^C,\mathbf{W}^H]\,,\end{aligned}$$

as claimed. □

Therefore $\mathcal{L}_{\mathbf{Y}^C}(\mathcal{L}_{\xi_L}\mathbf{S}) = 0$ if and only if $[\mathbf{Y}^C,\mathbf{V}^H]$ is horizontal for every $\mathbf{V}$. It also follows that $\mathcal{L}_{\mathbf{Y}^C}(\mathcal{L}_{\xi_L}\mathbf{S}) = 0$ if and only if $[\mathbf{Y}^C,\mathbf{V}^H] = [\mathbf{Y},\mathbf{V}]^H$ for any $\mathbf{V}$ and $\mathcal{L}_{\mathbf{Y}^C}(\mathcal{L}_{\xi_L}\mathbf{S}) = 0$ if and only if $\mathbf{Y}$ is an infinitesimal affine transformation on M. To finish the proof of Proposition 7.1, we have to express this with a symmetry, i.e., with an affine transformation.

Proposition: 7.7 *For every vector field* $\mathbf{V}$ *on* M *we have*

$$\mathcal{L}_{\mathbf{V}^C}(\mathcal{L}_{\xi_L}\mathbf{S}) = \mathcal{L}_{[\mathbf{V}^C,\xi_L]}\mathbf{S} \ .$$

Proof: 7.7 From (11.14) we conclude, including (11.8), that

$$\mathcal{L}_{\mathbf{V}^C}(\mathcal{L}_{\xi_L}\mathbf{S}) = \mathcal{L}_{\xi_L}(\mathcal{L}_{\mathbf{V}^C}\mathbf{S}) + \mathcal{L}_{[\mathbf{V}^C,\xi_L]}\mathbf{S} = \mathcal{L}_{[\mathbf{V}^C,\xi_L]}\mathbf{S} \ . \quad \Box$$

Since

$$\mathcal{L}_{\mathbf{Y}^C}(\mathcal{L}_{\xi_L}\mathbf{S}) = \mathcal{L}_{\xi_L}(\mathcal{L}_{\mathbf{Y}^C}\mathbf{S}) + \mathcal{L}_{[\mathbf{Y}^C,\xi_L]}\mathbf{S} = \mathcal{L}_{[\mathbf{Y}^C,\xi_L]}\mathbf{S} = 0$$

including (11.1), it follows that $[\mathbf{Y}^C, \xi_L]$ is a vertical lift.

Proposition: 7.8 *For any vector field* $\mathbf{V}$ *on* M *there is*

$$[\mathbf{C}, [\mathbf{V}^C, \xi_L]] = [\mathbf{V}^C, \xi_L] \ .$$

Proof: 7.8 Recall the Jacobi identity

$$[\mathbf{C}, [\mathbf{V}^C, \xi_L]] + [\mathbf{V}^C, [\xi_L, \mathbf{C}]] + [\xi_L, [\mathbf{C}, \mathbf{V}^C]] = 0 \ .$$

From (11.16) we see that

$$[\mathbf{C}, [\mathbf{V}^C, \xi_L]] + [\mathbf{V}^C, -\xi_L] + 0 = 0 \ ,$$

hence

$$[\mathbf{C}, [\mathbf{V}^C, \xi_L]] = [\mathbf{V}^C, \xi_L] \ . \quad \Box$$

Since $[\mathbf{Y}^C, \xi_L]$ is a vertical lift, from (11.15) it follows that $[\mathbf{C}, [\mathbf{Y}^C, \xi_L]] = -[\mathbf{Y}^C, \xi_L]$. However, in Proposition 7.8 we have seen that $[\mathbf{C}, [\mathbf{Y}^C, \xi_L]] = [\mathbf{Y}^C, \xi_L]$, hence

$$[\mathbf{Y}^C, \xi_L] = 0 \tag{7.18}$$

if $\mathbf{Y}^C$ is the **generator of an affine transformation.** $\Box$

Proposition 7.1 is thus proved.

7.2 Hamiltonian System on $T^*(TM)$ and its Symmetries

In Proposition 7.1 we have discovered symmetries of the spray ξ_L describing a homogeneous Lagrangian system. This leads us to the following question: With these symmetries, is there any chance of finding a symmetry of the Hamiltonian system on $T^*(TM)$?

In what follows we assume the system's Lagrange function to be given by $(1/2)g_{ij}v^iv^j$, where g_{ij} is a Riemannian metric on M. In Example 5.2 we see that the spray which stands for the Lagrange equation with Lagrangian $L = (1/2)g_{ij}v^iv^j$ is given by

$$\boldsymbol{\xi}_L = v^i\frac{\partial}{\partial x^i} - v^jv^k\Gamma^i_{jk}\frac{\partial}{\partial v^i}$$

where Γ^i_{jk} are the Christoffel symbols, hence the connection compatible with the metric $\mathbf{g}$. In ([9], p. 317) it is shown that the Lagrange equation of a holonomic system with homogeneous Lagrange function can be written in the form

$$\dot{v}^i = f^i - \Gamma^i_{jk}v^jv^k$$

where $f^i : M \to \mathbb{R}$ represent the forces. We recognize that many mechanical systems are of this form and hence L corresponds to the kinetic energy. Considering the equations for the Hamiltonian system on $T^*(TM)$ (Section 5.3), we are able to construct the Hamiltonian system with Hamiltonian function $S : T^*TM \to \mathbb{R}$,

$$S = v^i\lambda_i + f^i\mu_i - \Gamma^i_{jk}v^jv^k\mu_i\ .$$

The Hamiltonian system vector field $\mathbf{X}_S$ is given by

$$\mathbf{X}_S = \operatorname{sgrad} S$$

or in canonical form as

$$\begin{aligned}
\dot{x}^i &= \frac{\partial S}{\partial \lambda_i} = v^i \\
\dot{v}^i &= \frac{\partial S}{\partial \mu_i} = f^i - \Gamma^i_{jk}v^jv^k \\
\dot{\lambda}_i &= -\frac{\partial S}{\partial x^i} = -\left(\frac{\partial f^l}{\partial x^i} - \frac{\partial \Gamma^l_{jk}}{\partial x^i}v^jv^k\right)\mu_l \\
\dot{\mu}_i &= -\frac{\partial S}{\partial v^i} = -\lambda_i + 2\Gamma^l_{ij}v^j\mu_l\ .
\end{aligned}$$

It is easy to verify that $\mathbf{X}_S \lrcorner\, \boldsymbol{\omega}_S = \mathbf{X}_S \lrcorner\, (d\boldsymbol{\lambda} \wedge d\mathbf{x} + d\boldsymbol{\mu} \wedge d\mathbf{v}) = -dS$.

Let $\mathbf{Y} = Y^i(\partial/\partial x^i)$ be a vector field in TM that generates an affine transformation on M.

Proposition: 7.9 *The vector field* $\mathbf{Y}_S = \operatorname{sgrad} F_S$, *with* $F_S = Y^i\lambda_i + (\partial Y^i/\partial x^j)v^j\mu_i$, *a smooth function from* $T^*(TM)$ *to* M, *is symmetric to* $\mathbf{X}_S$ *if and only if*

$$[\mathbf{Y}, \mathbf{f}] = 0$$

where $f = (f^1, \ldots, f^n)$ *are the forces.*

Then F_S is a First Integral of the Hamiltonian system and

$$\begin{aligned}
\mathbf{Y}_S = \operatorname{sgrad} F_S &= \frac{\partial F_S}{\partial \lambda_s}\frac{\partial}{\partial x^s} + \frac{\partial F_S}{\partial \mu_s}\frac{\partial}{\partial v^s} - \frac{\partial F_S}{\partial x^s}\frac{\partial}{\partial \lambda_s} - \frac{\partial F_S}{\partial v^s}\frac{\partial}{\partial \mu_s} \\
&= Y^s\frac{\partial}{\partial x^s} + \frac{\partial Y^s}{\partial x^j}v^j\frac{\partial}{\partial v^s} - \left(\frac{\partial Y^i}{\partial x^s}\lambda_i + \frac{\partial^2 Y^i}{\partial x^j \partial x^s}v^j\mu_i\right)\frac{\partial}{\partial \lambda_s} \\
&\quad - \frac{\partial Y^i}{\partial x^s}\mu_i\frac{\partial}{\partial \mu_s} .
\end{aligned} \tag{7.19}$$

Moreover, as we know, the complete lift of $\mathbf{Y}$ is given by

$$\mathbf{Y}^C = \frac{\partial F_S}{\partial \lambda_s}\frac{\partial}{\partial x^s} + \frac{\partial F_S}{\partial \mu_s}\frac{\partial}{\partial v^s} ,$$

and it equals the first part of (7.19).

Proof: 7.9 For F_S to be a First Integral we have to verify that

$$\{F_S, S\} = \frac{\partial F_S}{\partial \lambda_s}\frac{\partial S}{\partial x^s} + \frac{\partial F_S}{\partial \mu_s}\frac{\partial S}{\partial v^s} - \frac{\partial F_S}{\partial x^s}\frac{\partial S}{\partial \lambda_s} - \frac{\partial F_S}{\partial v^s}\frac{\partial S}{\partial \mu_s} = 0 . \tag{7.20}$$

Now $\boldsymbol{Y}_S = \operatorname{sgrad} F_S$,

$$\begin{aligned}
\mathbf{X}_S = \operatorname{sgrad} S &= \frac{\partial S}{\partial \lambda_s}\frac{\partial}{\partial x^s} + \frac{\partial S}{\partial \mu_s}\frac{\partial}{\partial v^s} - \frac{\partial S}{\partial x^s}\frac{\partial}{\partial \lambda_s} - \frac{\partial S}{\partial v^s}\frac{\partial}{\partial \mu_s} \\
&= v^s\frac{\partial}{\partial x^s} + (f^s - \Gamma^s_{jk}v^j v^k)\frac{\partial}{\partial v^s} - \left(\frac{\partial f^i}{\partial x^s} - \frac{\partial \Gamma^i_{jk}}{\partial x^s}v^i v^k\right)\mu_i\frac{\partial}{\partial \lambda_s} \\
&\quad - \left(\lambda_s - \Gamma^i_{sk}v^k\mu_i - \Gamma^i_{js}v^j\mu_i\right)\frac{\partial}{\partial \mu_s} ,
\end{aligned} \tag{7.21}$$

(7.20), together with (7.19) and (7.21), then yields

$$\begin{aligned}
\{F_S, S\} &= Y^s\left(\frac{\partial f^i}{\partial x^s} - \frac{\partial \Gamma^i_{jk}}{\partial x^s}v^j v^k\right)\mu_i + \frac{\partial Y^s}{\partial x^j}v^j\left(\lambda_s - \Gamma^i_{sk}v^k\mu_i - \Gamma^i_{js}v^j\mu_i\right) - \\
&\quad - \left(\frac{\partial Y^i}{\partial x^s}\lambda_i + \frac{\partial^2 Y^i}{\partial x^j \partial x^s}v^j\mu_i\right)v^s - \frac{\partial Y^i}{\partial x^s}\mu_i\left(f^s - \Gamma^s_{jk}v^j v^k\right) \\
&= Y^s\frac{\partial f^i}{\partial x^s}\mu_i - Y^s\frac{\partial \Gamma^i_{jk}}{\partial x^s}v^j v^k\mu_i - \frac{\partial Y^s}{\partial x^j}v^j\Gamma^i_{sk}v^k\mu_i - \frac{\partial Y^s}{\partial x^j}v^j\Gamma^i_{js}v^j\mu_i \\
&\quad + \frac{\partial Y^s}{\partial x^j}v^j\lambda_s - \frac{\partial Y^i}{\partial x^s}\lambda_i v^s - \frac{\partial^2 Y^i}{\partial x^j\partial x^s}v^j\mu_i v^s - \frac{\partial Y^i}{\partial x^s}\mu_i f^s + \frac{\partial Y^i}{\partial x^s}\Gamma^s_{jk}v^j v^k\mu_i \\
&= -\frac{\partial^2 Y^i}{\partial x^j\partial x^k}v^j\mu_i v^k - Y^s\frac{\partial \Gamma^i_{jk}}{\partial x^s}v^j v^k\mu_i - \frac{\partial Y^s}{\partial x^j}v^j\Gamma^i_{sk}v^k\mu_i \\
&\quad - \frac{\partial Y^s}{\partial x^k}v^k\Gamma^i_{js}v^j\mu_i + \frac{\partial Y^i}{\partial x^s}\Gamma^s_{jk}v^j v^k\mu_i + Y^s\frac{\partial f^i}{\partial x^s}\mu_i - \frac{\partial Y^i}{\partial x^s}\mu_i f^s \\
&= [\mathbf{Y},\mathbf{f}]^i\mu_i - \left(\frac{\partial^2 Y^i}{\partial x^j\partial x^k} + Y^s\frac{\partial \Gamma^i_{jk}}{\partial x^s} + \frac{\partial Y^s}{\partial x^j}\Gamma^i_{sk}\right. \\
&\quad \left. + \frac{\partial Y^s}{\partial x^k}\Gamma^i_{js} - \frac{\partial Y^i}{\partial x^s}\Gamma^s_{jk}\right)v^j v^k\mu_i\,.
\end{aligned}$$

Proposition: 7.10 *The contents of the last bracket are equal to* $\mathcal{L}_{\mathbf{Y}}\Gamma^i_{jk}$.

Proof: 7.10 Let Λ^t be the flow associated to the vector field $\mathbf{Y}$. Λ^t defines a map on M which can be interpreted as a coordinate transformation from $x^i(t)$ to $x'^i(t)$. If Γ^a_{bc} are the components of the connection relative to $\mathbf{x}$, then its components relative to $\mathbf{x}'$ are given by [9]

$$\Gamma'^a_{bc} = \Lambda^t\,\Gamma^a_{bc} = \frac{\partial x'^a}{\partial x^d}\left(\Gamma^d_{ef}(x(t))\frac{\partial x^e}{\partial x'^b}\frac{\partial x^f}{\partial x'^c} + \frac{\partial^2 x^d}{\partial x'^b\partial x'^c}\right)\,. \tag{7.22}$$

If the transformation is given by

$$\mathbf{x} = \mathbf{x}' + t\mathbf{Y} \qquad \text{and} \qquad \mathbf{x}' = \mathbf{x} - t\mathbf{Y} \tag{7.23}$$

for small t, then

$$\frac{\partial x^j}{\partial x'^k} = \delta^j_k + t\frac{\partial Y^j}{\partial x'^k}(\mathbf{x}(t)) \qquad \text{and} \qquad \frac{\partial x'^n}{\partial x^m} = \delta^n_m - t\frac{\partial Y^n}{\partial x^m}(\mathbf{x}(t)) \tag{7.24}$$

and

$$\mathcal{L}_{\mathbf{Y}}\Gamma^a_{bc} = \frac{d}{dt}\Lambda^t\Gamma^a_{bc}\bigg|_{t=0}\,.$$

(7.22) then, together with (7.23) and (7.24), becomes

$$\begin{aligned}
\Gamma'^a_{bc} &= \left(\delta^a_d - t\frac{\partial Y^a}{\partial x^d}\right)\left[\Gamma^d_{ef}\left(\delta^e_b + t\frac{\partial Y^e}{\partial x'^b}\right)\left(\delta^f_c + t\frac{\partial Y^f}{\partial x'^c}\right) + t\frac{\partial^2 Y^d}{\partial x'^c \partial x'^b}\right] \\
&= \delta^a_d\left[\left(\Gamma^d_{ef}\delta^e_b + \Gamma^d_{ef} t\frac{\partial Y^e}{\partial x'^b}\right)\left(\delta^f_c + t\frac{\partial Y^f}{\partial x'^c}\right) + t\frac{\partial^2 Y^d}{\partial x'^c \partial x'^b}\right] \\
&\quad - t\frac{\partial Y^a}{\partial x^d}\left[\left(\Gamma^d_{ef}\delta^e_b + \Gamma^d_{ef} t\frac{\partial Y^e}{\partial x'^b}\right)\left(\delta^f_c + t\frac{\partial Y^f}{\partial x'^c}\right) + t\frac{\partial^2 Y^d}{\partial x'^c \partial x'^b}\right].
\end{aligned}$$

Continuing the calculation yields

$$\begin{aligned}
\Gamma'^a_{bc} &= \delta^a_d\left[\left(\Gamma^d_{bf} + \Gamma^d_{ef} t\frac{\partial Y^e}{\partial x'^b}\right)\left(\delta^f_c + t\frac{\partial Y^f}{\partial x'^c}\right) + t\frac{\partial^2 Y^d}{\partial x'^c \partial x'^b}\right] \\
&\quad - t\frac{\partial Y^a}{\partial x^d}\left[\left(\Gamma^d_{bf} + \Gamma^d_{ef} t\frac{\partial Y^e}{\partial x'^b}\right)\left(\delta^f_c + t\frac{\partial Y^f}{\partial x'^c}\right) + t\frac{\partial^2 Y^d}{\partial x'^c \partial x'^b}\right] \\
&= \Gamma^a_{bc} + \Gamma^a_{bf} t\frac{\partial Y^f}{\partial x'^c} + \Gamma^a_{ec} t\frac{\partial Y^e}{\partial x'^b} + t\frac{\partial^2 Y^a}{\partial x'^c \partial x'^b} - t\frac{\partial Y^a}{\partial x^d}\Gamma^d_{bc} + o(t^2) \\
&= \Gamma^a_{bc} + t\left[\Gamma^a_{bf}\frac{\partial Y^f}{\partial x'^c} + \Gamma^a_{ec}\frac{\partial Y^e}{\partial x'^b} + -\Gamma^d_{bc}\frac{\partial Y^a}{\partial x^d} + \frac{\partial^2 Y^a}{\partial x'^c \partial x'^b}\right] + o(t^2)\,. \qquad (7.25)
\end{aligned}$$

Differentiating (7.25) by t and setting $t = 0$ and $\mathbf{x} = \mathbf{x}'$ yields

$$(7.26)\quad \mathcal{L}_{\mathbf{Y}}\Gamma^a_{bc} = \frac{\partial \Gamma^a_{bc}}{\partial x^s}Y^s + \Gamma^a_{bf}\frac{\partial Y^f}{\partial x^c} + \Gamma^a_{ec}\frac{\partial Y^e}{\partial x^b} - \Gamma^d_{bc}\frac{\partial Y^a}{\partial x^d} + \frac{\partial^2 Y^a}{\partial x^c \partial x^b}\,. \quad \square$$

If $\mathbf{Y}$ is the generator of an affine transformation, then $\mathcal{L}_{\mathbf{Y}}\Gamma^i_{jk} = 0$. This can easily be seen by considering equation (7.4). When we write it down in coordinates we will see that it is equal to (7.26). However, according to (7.4) it is zero. $\square$

Thus Proposition 7.9 is proved.

7.3 Hamiltonian Systems and Control

The previous chapter describes how to find certain symmetries in the Hamiltonian system on $T^*(TM)$. We now want to find symmetries in the adjoint system, which is the optimally controlled system. We first have to introduce the control $\mathbf{u}$, which we assume to be a section of the bundle E. As we introduce the control forces, our system becomes a controlled system. Although we have seen that this system is not a Hamiltonian system, we can speak of a Hamiltonian function $H : E \to \mathbb{R}$,

$$H = v^i\lambda_i + f^i\mu_i - \Gamma^i_{jk}v^jv^k\mu_i + u^i\mu_i\,.$$

Since we treat concrete systems, it does not make sense to perform time-optimal control without constraints for the norm of the control $\mathbf{u}$. Implicitly we assume to have an Euclidean norm on the fibre of E which therefore can be considered as a vector bundle over $T^*(TM)$. Let us separate $\mathbf{u}$ into a unit vector $\boldsymbol{\alpha}$ that gives the direction of $\mathbf{u}$ and a scalar $s : K \to \mathbb{R}$ that performs a stretching of $\boldsymbol{\alpha}$. We set

$$\mathbf{u} = s\boldsymbol{\alpha}$$

with $0 \leq s \leq s_{max}$ where we have taken into account the constraints for $\mathbf{u}$. Then account. Then H becomes

$$H = v^i\lambda_i + f^i\mu_i - \Gamma^i_{jk}v^jv^k\mu_i + s\alpha^i\mu_i \ ,$$

and the system on E (Section 5.3) is described by the system of differential equations

$$\begin{aligned}
\dot{x}^i &= \frac{\partial H}{\partial \lambda_i} = v^i \\
\dot{v}^i &= \frac{\partial H}{\partial \mu_i} = f^i - \Gamma^i_{jk}v^jv^k + s\alpha^i \\
\dot{\lambda}_i &= -\frac{\partial H}{\partial x^i} = -\left(\frac{\partial f^l}{\partial x^i} - \frac{\partial \Gamma^l_{jk}}{\partial x^i}v^jv^k\right)\mu_l \\
\dot{\mu}_i &= -\frac{\partial H}{\partial v^i} = -\lambda_i + 2\Gamma^l_{ij}v^j\mu_l \ .
\end{aligned}$$

7.4 Optimal Control and Symmetries

Let us now construct the adjoint system on K (Section 5.3). According to Pontryagin's minimum principle [20], we have to construct a Hamiltonian function $H' : E \to \mathbb{R}$ by

$$H' = L_{perf} + v^i\lambda_i + f^i\mu_i - \Gamma^i_{jk}v^jv^k\mu_i + s\alpha^i\mu_i \ ,$$

where $L_{perf}(\mathbf{x}, \mathbf{u})$ is the performance index (Chapter 1). In the case of time-optimal control, L_{perf} becomes 1. Then we have to minimize H' with respect to the control $\mathbf{u}$ to arrive at

$$H^\circ = \min_{\mathbf{u}} H' \ .$$

Clearly, the Hamiltonian H' will become minimal if the vector $\boldsymbol{\alpha}$ runs the opposite direction of the vector $\boldsymbol{\mu}$ and $s = s_{max}$.

$$\boldsymbol{\alpha} = -\frac{\boldsymbol{\mu}}{|\boldsymbol{\mu}|} = -\frac{\boldsymbol{\mu}}{\sqrt{<\boldsymbol{\mu}, \boldsymbol{\mu}>}} \ .$$

From the definition of the Riemannian metric on M, (i.e., $< \cdot, \cdot >$)

$$< \boldsymbol{\eta}, \boldsymbol{\xi} >= g_{ij}\eta^i\xi^j$$

we obtain for $\boldsymbol{\alpha}$

$$\alpha^i = -\frac{\mu^i}{|\boldsymbol{\mu}|} = -\frac{g^{il}\mu_l}{\sqrt{g^{kl}\mu_k\mu_l}} \, .$$

To arrive at the expression for μ^i we performed a raising of the index ([9], p. 169). g^{ij} being the inverse of g_{ij}, i.e., $g^{lk}g_{kn} = \delta^l_n$, we obtain for the Hamiltonian H^o

$$\begin{aligned} H^o &= 1 + v^i\lambda_i + f^i\mu_i - \Gamma^i_{jk}v^jv^k\mu_i - s_{max}\frac{\mu^i}{\sqrt{g^{kl}\mu_k\mu_l}}\mu_i \\ &= 1 + v^i\lambda_i + f^i\mu_i - \Gamma^i_{jk}v^jv^k\mu_i - s_{max}\frac{g^{kj}\mu_k\mu_j}{\sqrt{g^{kj}\mu_k\mu_j}} \\ &= 1 + v^i\lambda_i + f^i\mu_i - \Gamma^i_{jk}v^jv^k\mu_i - s_{max}\sqrt{g^{kj}\mu_k\mu_j} \, . \end{aligned}$$

The optimally controlled system on K then becomes (Section 5.3)

$$\begin{aligned} \dot{x}^s &= \frac{\partial H^o}{\partial \lambda_s} = v^s - \frac{\partial s_{max}}{\partial \lambda_s}\sqrt{g^{kj}\mu_k\mu_j} \\ \dot{v}^s &= \frac{\partial H^o}{\partial \mu_s} = f^s - \Gamma^s_{jk}v^jv^k - \frac{\partial s_{max}}{\partial \mu_s}\sqrt{g^{kj}\mu_k\mu_j} - s_{max}\frac{1}{2}\frac{g^{sn}\mu_n}{\sqrt{g^{kj}\mu_k\mu_j}} \\ &\quad - s_{max}\frac{1}{2}\frac{g^{ms}\mu_m}{\sqrt{g^{kj}\mu_k\mu_j}} \\ \dot{\lambda}_s &= -\frac{\partial H^o}{\partial x^s} = -\left(\frac{\partial f^i}{\partial x^s} - \frac{\partial \Gamma^i_{jk}}{\partial x^s}v^jv^k\right)\mu_i + \frac{\partial s_{max}}{\partial x^s}\sqrt{g^{kj}\mu_k\mu_j} \\ &\quad + s_{max}\frac{1}{2}\frac{\partial g^{nm}}{\partial x^s}\frac{\mu_n\mu_m}{\sqrt{g^{kj}\mu_k\mu_j}} \\ \dot{\mu}_s &= -\frac{\partial H^o}{\partial v^s} = -\lambda_s + \left(\Gamma^i_{sk}v^k + \Gamma^i_{js}v^j\right)\mu_i + \frac{\partial s_{max}}{\partial v^s}\sqrt{g^{kj}\mu_k\mu_j} \, . \end{aligned}$$

Now we look for conditions for the vector field $\mathbf{Y}_{H^o}$ on K to be a symmetry of the system with Hamiltonian H^o. In the following we have to take a few things into account concerning the constraints for the control $\mathbf{u}$. We discovered that $\mathbf{u}$ is such that $s = s_{max}$, hence $\mathbf{u}^o$ may be understood as a section into a sphere bundle over $T^*(TM)$ with spheres of constant radius. Furthermore, the radius is assumed to be independent of the base point (Figure 19), from which follows that

$$\frac{\partial s_{max}}{\partial \mathbf{x}} = \frac{\partial s_{max}}{\partial \boldsymbol{\lambda}} = \frac{\partial s_{max}}{\partial \boldsymbol{\mu}} = \frac{\partial s_{max}}{\partial \mathbf{v}} = 0 \, .$$

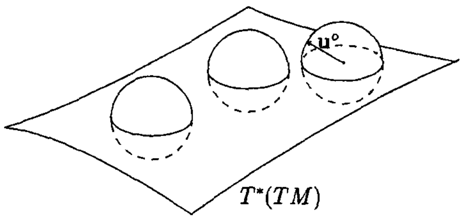

Figure 19: Spheres on $T^*(TM)$

Let $\mathbf{Y} = Y^i(\partial/\partial x^i)$ be a vector field on M.

Proposition: 7.11 $\mathbf{Y}_{H^\circ} = \operatorname{sgrad} F_{H^\circ}$ *with* $F_{H^\circ} = Y^i\lambda_i + (\partial Y^i/\partial x^j)v^j\mu_i$ *is a symmetry of the system with Hamiltonian* H° *if and only if* $\mathcal{L}_{\mathbf{Y}}\mathbf{f} = \mathcal{L}_{\mathbf{Y}}g^{ij} = 0$.

Proof: 7.11 For F_{H° to be a First Integral we have to verify that

$$\{F_{H^\circ}, H^\circ\} = \frac{\partial F_{H^\circ}}{\partial \lambda_s}\frac{\partial H^\circ}{\partial x^s} + \frac{\partial F_{H^\circ}}{\partial \mu_s}\frac{\partial H^\circ}{\partial v^s} - \frac{\partial F_{H^\circ}}{\partial x^s}\frac{\partial H^\circ}{\partial \lambda_s} - \frac{\partial F_{H^\circ}}{\partial v^s}\frac{\partial H^\circ}{\partial \mu_s} = 0\ .$$

From

$$\begin{aligned}
\{F_{H^\circ}, H^\circ\} &= Y^s\left(\frac{\partial f^i}{\partial x^s}\mu_i - \frac{\partial \Gamma^i_{jk}}{\partial x^s}v^jv^k\mu_i - s_{max}\frac{1}{2}\frac{\partial g^{nm}}{\partial x^s}\frac{\mu_n\mu_m}{\sqrt{g^{kj}\mu_k\mu_j}}\right)\\
&\quad+\frac{\partial Y^s}{\partial x^l}v^l\left(\lambda_s - \Gamma^i_{sk}v^k\mu_i - \Gamma^i_{js}v^j\mu_i\right) - \left(\frac{\partial Y^i}{\partial x^s}\lambda_i + \frac{\partial^2 Y^i}{\partial x^j\partial x^s}v^j\mu_i\right)v^s\\
&\quad-\frac{\partial Y^i}{\partial x^s}\mu_i\left(f^s - \Gamma^s_{jk}v^jv^k - s_{max}\frac{1}{2}\frac{g^{sn}\mu_n}{\sqrt{g^{kj}\mu_k\mu_j}} - s_{max}\frac{1}{2}\frac{g^{ms}\mu_m}{\sqrt{g^{kj}\mu_k\mu_j}}\right)
\end{aligned}$$

we get

$$\begin{aligned}
\{F_{H^\circ}, H^\circ\} &= Y^s\frac{\partial f^i}{\partial x^s}\mu_i - Y^s\frac{\partial \Gamma^i_{jk}}{\partial x^s}v^jv^k\mu_i - Y^s s_{max}\frac{\partial g^{nm}}{\partial x^s}\frac{1}{2}\frac{\mu_n\mu_m}{\sqrt{g^{kj}\mu_k\mu_j}}\\
&\quad+\frac{\partial Y^s}{\partial x^l}v^l\lambda_s - \frac{\partial Y^s}{\partial x^l}v^l\Gamma^i_{sk}v^k\mu_i - \frac{\partial Y^s}{\partial x^l}v^l\Gamma^i_{js}v^j\mu_i - \frac{\partial Y^i}{\partial x^s}\lambda_i v^s\\
&\quad-\frac{\partial^2 Y^i}{\partial x^s\partial x^j}v^j\mu_i v^s - \frac{\partial Y^c}{\partial x^s}\mu_c f^s + \frac{\partial Y^c}{\partial x^s}\mu_c\Gamma^s_{jk}v^jv^k\\
&\quad+\frac{\partial Y^c}{\partial x^s}\mu_c s_{max}\frac{1}{2}\frac{g^{sn}\mu_n}{\sqrt{g^{kj}\mu_k\mu_j}} + \frac{\partial Y^c}{\partial x^s}\mu_c s_{max}\frac{1}{2}\frac{g^{ms}\mu_m}{\sqrt{g^{kj}\mu_k\mu_j}}\ .
\end{aligned}$$

Comparing this result with the proof of Proposition 7.9, we arrive at

$$\begin{aligned} \{F_{H^o}, H^o\} &= [\mathbf{Y}, \mathbf{f}]^i \mu_i - \mathcal{L}_{\mathbf{Y}} \Gamma^i_{jk} v^j v^k \mu_i \\ &\quad - \frac{1}{2} \frac{\mu_n \mu_m}{\sqrt{g^{kj} \mu_k \mu_j}} s_{max} \left(Y^s \frac{\partial g^{nm}}{\partial x^s} - \frac{\partial Y^m}{\partial x^s} g^{sn} - \frac{\partial Y^n}{\partial x^s} g^{ms} \right) . \end{aligned}$$

According to the Lie derivative of a two-covariant tensor

$$\mathcal{L}_{\mathbf{Y}} g^{ij} = Y^s \frac{\partial g^{ij}}{\partial x^s} - g^{lj} \frac{\partial Y^i}{\partial x^l} - g^{il} \frac{\partial Y^j}{\partial x^l} ,$$

we obtain

$$\{F_{H^o}, H^o\} = [\mathbf{Y}, \mathbf{f}]^i \mu_i - \mathcal{L}_{\mathbf{Y}} \Gamma^i_{jk} v^j v^k \mu_i - \frac{1}{2} s_{max} \frac{\mu_n \mu_m}{\sqrt{g^{kj} \mu_k \mu_j}} \mathcal{L}_{\mathbf{Y}} g^{nm} . \quad \square$$

While a transformation with generating vector field $\mathbf{Y}$ with $\mathcal{L}_{\mathbf{Y}} g^{ij} = 0$ is an isometry, the vector field satisfying this condition is called a ***Killing*** vector field.

8 A Class of Systems with Symmetries

We look for a class of systems which admit symmetries as given by Proposition 7.11. Two conditions are therefore of importance: The Lie bracket between $\mathbf{Y}$ and the forces $\mathbf{f}$, and the Killing equation, i.e.,

$$[\mathbf{Y}, \mathbf{f}] = 0 \qquad \text{and} \qquad \mathcal{L}_{\mathbf{Y}}\mathbf{g} = 0$$

where $g_{ij}v^iv^j$ corresponds to the Lagrange function of our system.

8.1 Systems with a Potential

We consider systems which have a **potential** $u(\mathbf{x})$, i.e.,

$$f_i = -\frac{\partial u(\mathbf{x})}{\partial x^i} .$$

We focus on systems which have a potential that is invariant with respect to $\mathbf{Y}$, that is

$$\mathcal{L}_{\mathbf{Y}}u(\mathbf{x}) = Y^i(-f_i) = 0 . \tag{8.1}$$

We make the

Proposition: 8.1 *If a holonomic system has a potential $u(\mathbf{x})$ with $\mathcal{L}_{\mathbf{Y}}u(\mathbf{x}) = 0$, with $\mathbf{Y}$ being a Killing vector field, then this system has symmetries as described in Proposition* 7.11.

Proof: 8.1 We assume to have an affine coordinate system with $\nabla_i j\,(\mathbf{x}) = 0$. Equation (8.1) yields

$$\begin{aligned} 0 &= \nabla_l(Y^k f_k) = \nabla_l(Y_r g^{rk} f_n) = \frac{\partial Y_r}{\partial x^l} g^{rk} f_k + Y_r \frac{\partial g^{rk}}{\partial x^l} f_k + Y_r g^{rk} \frac{\partial f_k}{\partial x^l} \\ &= \frac{\partial Y_r}{\partial x^l} g^{rk} f_k + Y_r g^{rk} \frac{\partial f_k}{\partial x^l} . \end{aligned} \tag{8.2}$$

Since $\mathbf{Y}$ is a Killing vector field, based on ([13], p. 43), it follows that

$$\nabla_l Y_r + \nabla_r Y_l = 0 \tag{8.3}$$

where $Y_r = Y^i g_{ir}$. Equation (8.2) with (8.3) yields

$$0 = -\frac{\partial Y_l}{\partial x^r} g^{rk} f_k + Y_r g^{rk} \frac{\partial f_k}{\partial x^l} . \tag{8.4}$$

In accordance with the **integrability condition** of **f**, i.e.,

$$\frac{\partial f_k}{\partial x^l} = \frac{\partial^2 u}{\partial x^l \partial x^k} = \frac{\partial f_l}{\partial x^k}$$

(8.4) yields

$$\begin{aligned} 0 &= -\frac{\partial Y_l}{\partial x^r} g^{rk} f_k + Y_r g^{rk} \frac{\partial f_l}{\partial x^k} = -\frac{\partial (g_{ls} Y^s)}{\partial x^r} g^{rk} f_k + g_{rs} Y^s g^{rk} \frac{\partial f_l}{\partial x^k} \\ &= -g_{ls} \frac{\partial Y^s}{\partial x^r} g^{rk} f_k + Y^k \frac{\partial f_l}{\partial x^k} . \end{aligned}$$

If we multiply the last expression with g^{lt}, it becomes

$$\begin{aligned} 0 &= -g_{ls} g^{lt} \frac{\partial Y^s}{\partial x^r} g^{rk} f_k + g^{lt} Y^k \frac{\partial f_l}{\partial x^k} \\ &= -\delta^t_s \frac{\partial Y^s}{\partial x^r} g^{rk} f_k + g^{lt} Y^k \frac{\partial f_l}{\partial x^k} = -\frac{\partial Y^t}{\partial x^r} g^{rk} f_k + g^{lt} Y^k \frac{\partial f_l}{\partial x^k} \\ &= -\frac{\partial Y^t}{\partial x^r} g^{rk} f_k + Y^k \frac{\partial f_l}{\partial x^k} g^{lt} \\ &= [\mathbf{Y}, \mathbf{f}]^t . \quad \square \end{aligned}$$

8.2 Solutions of the Killing Equation

In what follows we look for conditions for the Killing equation to have solutions. Whether the Killing equation has solutions or not is entirely determined by the metric, hence by the Lagrange function of the system. To answer this question we cite two theorems due to ([22], p. 37) who in turn cited them from ([10], p. 45):

1. *In order that a Riemannian space admits a one-parameter group of motions (isometries), it is necessary and sufficient that there exists a coordinate system with respect to which the components of the Riemannian tensor (metric) are homogeneous of degree -2 of the coordinates.*

2. *If and only if the Riemannian space is a space of constant curvature, the Killing equation has $n(n+1)/2$ solutions; in all other cases there are fewer solutions.*

From the last theorem we conclude that a complete reduction of a time-optimally controlled system (by the above symmetries) is possible for systems with constant curvature and with dimension $n = 3$. This is due to the fact that we must have $2n$ First Integrals to perform a complete reduction and therefore n has to satisfy the relation $2n = n(n+1)/2$. This reduction corresponds to a solution of the Hamilton-Bellman-Jacobi (HBJ) equation of the control problem. We know that the solution of the HBJ equation can be interpreted as a generating function for a symplectic transformation that transforms the Hamiltonian system with Hamiltonian H^o into an equilibrium point.

What kinds of systems can a Riemannian tensor with constant curvature have? According to ([9], p. 296) the Riemannian curvature tensor is given by

$$-R^i_{qkl} = \frac{\partial \Gamma^i_{ql}}{\partial x^k} - \frac{\partial \Gamma^i_{qk}}{\partial x^l} + \Gamma^i_{pk}\Gamma^p_{ql} - \Gamma^i_{pl}\Gamma^p_{qk} .$$

We also know (2.2) that the Christoffel symbols are calculated from the entries of the Riemannian tensor; they are equal to zero if g_{ij} are constant. Hence a metric with constant coefficients leads to a curvature tensor which is zero and hence to a curvature that is constant.

Concluding from the above we find some classes of systems that admit symmetries when time-optimally controlled. We list some of them in the following table.

Lagrangian L	forces $\mathbf{f}$
$L = T = m_{ij}v^iv^j$ or	none
$L = g_{ij}v^iv^j$ g_{ij} homogeneous of degree -2	$f_i = -\partial u/\partial x^i$ $Y^if_i = 0$

Here, T denotes the kinetic energy of a mechanical system. Considering mechanical systems, the matrix m_{ij}, i.e., the mass matrix, is always positive-definite and symmetric. It therefore corresponds to a Riemannian metric with constant curvature.

9 Examples

9.1 Example 1

Let us solve a relatively simple example. Nevertheless, to be able to handle the formulas one is forced to apply *Maple V*, a symbolic computation program, because it took us about half an hour to calculate a Lie bracket. In this example the results given in Chapter 7 are verified and the reduction procedure described in Chapter 6 is performed.

The routines we programmed to apply *Maple* can be found in Section 11.2.

We consider a Lagrangian system with Lagrangian

$$L = \frac{1}{2} g_{ij} v^i v^j ,$$

with

$$\mathbf{g} = \begin{bmatrix} \frac{1}{\left(x^1\right)^2} & 0 \\ 0 & 1 \end{bmatrix}$$

as metric. From this metric it is easy to determine the Christoffel symbols according to (2.2)

$$\Gamma^m_{ij} = \frac{1}{2} g^{km} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) ,$$

with $i, j, k = 1, \ldots, n$ and g^{km} being the inverse of g_{lk} with $g_{lk} g^{km} = \delta^m_l$. This yields

$$\Gamma^1_{11} = -\frac{1}{x^1}$$
$$\Gamma^1_{21} = \Gamma^1_{12} = \Gamma^1_{22} = \Gamma^2_{11} = \Gamma^2_{21} = \Gamma^2_{12} = \Gamma^2_{22} = 0 .$$

We wrote the *Maple* routine *comp_gamma* (cf. appendix) which computes the Christoffel symbols out of a given metric.

The **spray** $\boldsymbol{\xi}_L$ on M (5.3) is given by $\boldsymbol{\xi}_L = v^i(\partial/\partial x^i) + \xi^i_L(\partial/\partial v^i)$. From Example 5.2 we know that

$$\xi^i_L = \ddot{x}^i = -v^j v^k \Gamma^i_{jk} .$$

Therefore $\xi^1_L = (1/x^1)\left(v^1\right)^2$ and $\xi^2_L = 0$, which yields

$$\boldsymbol{\xi}_L = v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2} + \frac{1}{x^1}\left(v^1\right)^2 \frac{\partial}{\partial v^1} + 0\,\frac{\partial}{\partial v^2} .$$

It is easy to find a vector field $\mathbf{Y}$ on M with $\mathcal{L}_{\mathbf{Y}}\mathbf{g} = 0$. This is done by writing $\mathcal{L}_{\mathbf{Y}}\mathbf{g} = 0$ in coordinates.

$$\mathcal{L}_{\mathbf{Y}} g_{ij} = 0 = Y^s \frac{\partial g_{ij}}{\partial x^s} + g_{kj} \frac{\partial Y^k}{\partial x^i} + g_{ik} \frac{\partial Y^k}{\partial x^j} .$$

This is a differential-equation for Y^i, the components of $\mathbf{Y}$, and it can be solved with a standard routine of *Maple*. As a result we obtain

$$\mathbf{Y} = x^1 \frac{\partial}{\partial x^1} .$$

Based on this and including (4.2) we determine the complete lift of $\mathbf{Y}$

$$\mathbf{Y}^C = x^1 \frac{\partial}{\partial x^1} + v^1 \frac{\partial}{\partial v^1} .$$

To verify (7.1) we have to calculate the Lie bracket $[\mathbf{Y}^C, \boldsymbol{\xi}_L]$. After a tedious calculation we find that $[\mathbf{Y}^C, \boldsymbol{\xi}_L] = 0$.

Since the calculation of the Lie bracket for these relatively simple vector fields required a lot of time (and about three pages of discussion) we decided to write a routine called *Lie Bracket* in *Maple* to calculate the Lie bracket between two vector fields (Section 11.2). The bracket $[\mathbf{Y}^C, \boldsymbol{\xi}_L]$ is given as an example in the description of the routine *Lie Bracket.*

Out of $\boldsymbol{\xi}_L$ we are able to construct the Hamiltonian system on $T^*(TM)$ (Section 7.2) with the Hamiltonian function

$$S = v^1\lambda_1 + v^2\lambda_2 + \frac{1}{x^1}\left(v^1\right)^2 \mu_1 .$$

The system is stated in canonical form as

$$\begin{aligned} \dot{x}^1 &= v^1 & \dot{v}^1 &= \frac{1}{x^1}\left(v^1\right)^2 \\ \dot{x}^2 &= v^2 & \dot{v}^2 &= 0 \\ \dot{\lambda}_1 &= \frac{1}{(x^1)^2}\left(v^1\right)^2 \mu_1 & \dot{\mu}_1 &= -\lambda_1 - 2\,\frac{1}{x^1} v^1 \mu_1 \\ \dot{\lambda}_2 &= 0 & \dot{\mu}_2 &= -\lambda_2 . \end{aligned}$$

According to Proposition 7.9 we have given a First Integral of the Hamiltonian system on $T^*(TM)$ as

$$F_S = x^1\lambda_1 + v^1\mu_1$$

and the symmetry vector field

$$\mathbf{Y}_S = \operatorname{sgrad} F_S = x^1 \frac{\partial}{\partial x^1} + v^1 \frac{\partial}{\partial v^1} - \lambda_1 \frac{\partial}{\partial \lambda_1} - \mu_1 \frac{\partial}{\partial \mu_1} .$$

With the *Maple* routine *Lie Bracket* it is then easy to verify that with $\mathbf{X}_S = \operatorname{sgrad} S$, $[\mathbf{Y}_S, \mathbf{X}_S] = 0$.

Now, we verify Proposition 7.11 which is concerned with the symmetries of the optimally controlled system. As stated in Section 7.4,

$$H^\circ = 1 + v^1\lambda_1 + v^2\lambda_2 + \frac{1}{x^1}\left(v^1\right)^2\mu_1 - s_{max}\sqrt{\left(\mu_1\right)^2\left(x^1\right)^2 + \left(\mu_2\right)^2}$$

from which results the optimally controlled system in canonical form

$$\begin{aligned}
\dot{x}^1 &= v^1 \\
\dot{x}^2 &= v^2 \\
\dot{v}^1 &= \frac{\left(v^1\right)^2\sqrt{\left(\mu_1\right)^2\left(x^1\right)^2 + \left(\mu_2\right)^2} - s_{max}\,\mu_1\left(x^1\right)^3}{x^1\sqrt{\left(\mu_1\right)^2\left(x^1\right)^2 + \left(\mu_2\right)^2}} \\
\dot{v}^2 &= \frac{-s_{max}\,\mu_2}{\sqrt{\left(\mu_1\right)^2\left(x^1\right)^2 + \left(\mu_2\right)^2}} \\
\dot{\lambda}_1 &= \frac{\left(v^1\right)^2\mu_1}{\left(x^1\right)^2} + \frac{s_{max}\left(\mu_1\right)^2 x^1}{\sqrt{\left(\mu_1\right)^2\left(x^1\right)^2 + \left(\mu_2\right)^2}} \\
\dot{\lambda}_2 &= 0 \\
\dot{\mu}_1 &= -\lambda_1 - 2\,\frac{1}{x^1}v^1\mu_1 \\
\dot{\mu}_2 &= -\lambda_2\,.
\end{aligned}$$

According to Proposition 7.11 the vector field $\mathbf{Y}_{H^\circ} = \operatorname{sgrad}\left(Y^1\lambda_1 + v^1\mu_1\right)$ is a symmetry of $\mathbf{X}_{H^\circ}$. This can easily be verified by calculating $[\mathbf{Y}_{H^\circ}, \mathbf{X}_{H^\circ}]$ with the routine *Lie Bracket*.

9.1.1 Reduction Procedure

After finding a First Integral and a symmetry vector field of the optimally controlled system, we now look for a symplectic transformation Φ (Chapter 6) that transforms the Hamiltonian system with Hamiltonian H° into a Hamiltonian $\bar{H}^\circ$ such that $\bar{F}_{H^\circ}$ will become the new coordinate φ_1 and the transformed symmetry vector field $\bar{\mathbf{Y}}_{H^\circ}$ is parallel to the new coordinate y^1. The coordinate transformation,

$$\Phi : (\mathbf{y}, \mathbf{w}, \boldsymbol{\varphi}, \boldsymbol{\psi}) \mapsto (\mathbf{x}, \mathbf{v}, \boldsymbol{\lambda}, \boldsymbol{\mu})\,,$$

is given by (6.7) and, with a slight modification, by

$$\Phi(\mathbf{y}, \mathbf{w}, \boldsymbol{\varphi}, \boldsymbol{\psi}) = \Upsilon^{y^1-1}(1, \mathbf{y}', \mathbf{w}, b_1, \boldsymbol{\varphi}', \boldsymbol{\psi})$$

where Υ^t is the flow of $\mathbf{Y}_{H^o}$. By this method we get the transformation

$$\begin{aligned} x^1 &= e^{y^1-1} & v^1 &= w^1 e^{y^1-1} \\ x^2 &= y^2 & v^2 &= w^2 \\ \lambda_1 &= (\varphi_1 - w^1\psi_1)e^{-y^1+1} & \mu_1 &= \psi_1 e^{1-y^1} \\ \lambda_2 &= \varphi_2 & \mu_2 &= \psi_2 \end{aligned}$$

and the transformed Hamiltonian becomes

$$\tilde{H}^o = 1 + w^1\varphi_1 + w^2\varphi_2 - s_{max}\sqrt{(\psi_1)^2 + (\psi_2)^2}\,.$$

The transformed Hamiltonian system thus is given by

$$\begin{aligned} \dot{y}^1 &= w^1 \\ \dot{y}^2 &= w^2 \\ \dot{w}^1 &= \frac{-s_{max}\,\psi_1}{\sqrt{(\psi_1)^2 + (\psi_2)^2}} \\ \dot{w}^2 &= \frac{-s_{max}\,\psi_2}{\sqrt{(\psi_1)^2 + (\psi_2)^2}} \\ \dot{\varphi}_1 &= 0 \\ \dot{\varphi}_2 &= 0 \\ \dot{\psi}_1 &= -\varphi_1 \\ \dot{\psi}_2 &= -\varphi_2\,. \end{aligned}$$

We see that y^1 no longer appears in $\tilde{H}^o$. Due to the fact that $\varphi_1 = \tilde{F}_{H^o}$, $\dot{\varphi}_1 = 0$ where $\tilde{F}_{H^o}$ is the transformed First Integral F_{H^o}.

9.2 Example 2

In this example we consider a rocket circling the earth (Figure 20). The controls of the rocket are the thrust u, $0 \leq u \leq u_{max}$, and the angle of the rockets axes θ to the line to the center of the earth. The rocket's mass is m. The aim of the control is to reach an orbit from an other orbit in the shortest possible time. The kinetic energy T of the rocket is given by

$$T = \frac{1}{2}\left(m(\dot{x}^1)^2 + m(\dot{x}^2)^2\right)\,.$$

From this expression the Riemannian metric in matrix form follows

$$\mathbf{g} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\,.$$

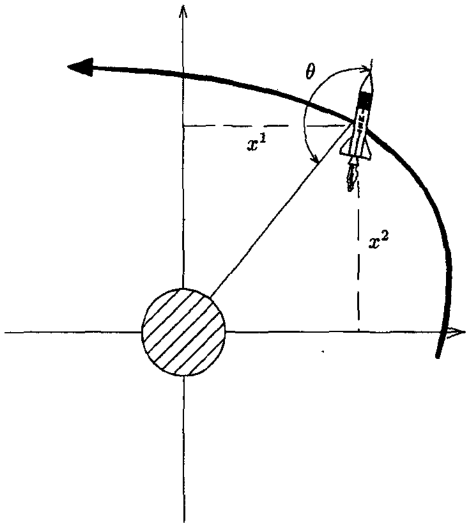

Figure 20: Rocket and Earth

We now calculate the forces $\mathbf{f}$. There is a potential given

$$U(x^1, x^2) = -\frac{K}{\sqrt{(x^1)^2 + (x^2)^2}}$$

where x^1, x^2 are the coordinates of the center of mass of the rocket. From the potential we derive the forces. We have

$$f_1 = -\frac{\partial U}{\partial x^1} = -\frac{x^1}{\sqrt{(x^1)^2 + (x^2)^2}} \frac{K}{(x^1)^2 + (x^2)^2}$$

and

$$f_2 = -\frac{\partial U}{\partial x^2} = -\frac{x^2}{\sqrt{(x^2)^2 + (x^2)^2}} \frac{K}{(x^1)^2 + (x^2)^2} \, .$$

Since we must have f^1, f^2 rather than f_1, f_2 we must raise the index of the vector $\mathbf{f}$. With the inverse of $\mathbf{g}$ with components g^{ij}

$$\mathbf{g}_{inv} = \begin{bmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{bmatrix}$$

it follows that $f^i = f_j g^{ji}$, hence $f^1 = f_1/m$ and $f^2 = f_2/m$. For the Hamiltonian of the optimally controlled system we obtain

$$\begin{aligned} H^o &= 1 + v^1\lambda_1 + v^2\lambda_2 - \frac{x^1}{\sqrt{(x^1)^2+(x^2)^2}} \frac{K}{(x^1)^2+(x^2)^2} \frac{\mu_1}{m} \\ &\quad - \frac{x^2}{\sqrt{(x^1)^2+(x^2)^2}} \frac{K}{(x^1)^2+(x^2)^2} \frac{\mu_2}{m} - u_{max}\sqrt{\frac{1}{m}\left((\mu_1)^2+(\mu_2)^2\right)} \,. \end{aligned}$$

In the next step we look for vector fields that satisfy

$$\mathcal{L}_{\mathbf{Y}} U(x^1, x^2) = 0 \,,$$

hence those that let the potential $U(x^1, x^2)$ be invariant (Equation (8.1)). This leads us to the equation

$$\mathcal{L}_{\mathbf{Y}} U(x^1, x^2) = Y^1 f^1 + Y^2 f^2 = 0 \tag{9.1}$$

where $\mathbf{Y} = Y^1(\partial/\partial x^1) + Y^2(\partial/\partial x^2)$. In order to satisfy (9.1) $Y^1/Y^2 = -x^2/x^1$ must hold.

Now we are going to look for conditions for $\mathbf{Y}$ to satisfy the Killing equation $\mathcal{L}_{\mathbf{Y}}\mathbf{g} = 0$. This equation is given in local coordinates by

$$\mathcal{L}_{\mathbf{Y}}\mathbf{g} = 0 = Y^s \frac{\partial g_{ij}}{\partial x^s} + g_{kj}\frac{\partial Y^k}{\partial x^i} + g_{ik}\frac{\partial Y^k}{\partial x^j} \,.$$

This corresponds to the following system of partial differential equations

$$\begin{aligned} m\frac{\partial Y^1}{\partial x^1} + m\frac{\partial Y^1}{\partial x^1} &= 0 \,, \qquad m\frac{\partial Y^2}{\partial x^1} + m\frac{\partial Y^1}{\partial x^2} = 0 \\ m\frac{\partial Y^1}{\partial x^2} + m\frac{\partial Y^2}{\partial x^1} &= 0 \,, \qquad m\frac{\partial Y^2}{\partial x^2} + m\frac{\partial Y^2}{\partial x^2} = 0 \,. \end{aligned}$$

From these equations it is easy to deduce that $Y^1 = -x^2$ and $Y^2 = x^1$. These conditions are the same as those that let the potential be invariant! It is obvious that the path of $\mathbf{Y}$ is a **circle** with origin of the coordinate system (x^1, x^2) as center. This is not surprising since the angle of the vector from the rocket to the center of the earth is of no importance for our considerations. According to Proposition 7.11, the First Integral

$$F_{H^o} = -x^2\lambda_1 + x^1\lambda_2 - v^2\mu_1 - v^1\mu_2 \,.$$

The symmetry vector field

$$\begin{aligned} \mathbf{Y}_{H^o} &= \operatorname{sgrad} F_{H^o} \\ &= -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2} - v^2\frac{\partial}{\partial v^1} + v^1\frac{\partial}{\partial v^2} \\ &\quad -\lambda_2\frac{\partial}{\partial \lambda_1} + \lambda_1\frac{\partial}{\partial \lambda_2} - \mu_2\frac{\partial}{\partial \mu_1} + \mu_1\frac{\partial}{\partial \mu_2} \,. \end{aligned}$$

9.2.1 Transformation Φ

In the transformed system we want φ_2 to become a First Integral, hence

$$F_{H^\circ} \circ \Phi = \varphi_2 \tag{9.2}$$

and the flow Θ^t of the transformed symmetry vector field should satisfy

$$\Theta^t(\mathbf{y}, \mathbf{w}, \boldsymbol{\varphi}, \boldsymbol{\psi}) = (y^1, y^2 + t\frac{\partial}{\partial y^2}, \mathbf{w}, \boldsymbol{\varphi}, \boldsymbol{\psi})$$

(Equation (6.4)). From Equations (9.2) and (6.6) it follows that

$$F_{H^\circ}(y^1, 0, w^1, w^2, \varphi_1, b_2, \psi_1, \psi_2) = \varphi_2 \,.$$

We conclude that

$$b_2 = \frac{\varphi_2 - w^1\psi_2 + w^2\psi_1}{y^1} \,.$$

To determine transformation Φ we must determine the flow Υ^t of $\mathbf{Y}_{H^\circ}$. This leads us to the integration of the system of first-order differential equations

$$\begin{aligned} \dot{x}^1 &= -x^2 & \dot{\lambda}_1 &= -\lambda_2 \\ \dot{x}^2 &= x^1 & \dot{\lambda}_2 &= \lambda_1 \\ \dot{v}^1 &= -v^2 & \dot{\mu}_1 &= -\mu_2 \\ \dot{v}^2 &= v^1 & \dot{\mu}_2 &= \mu_1 \,. \end{aligned}$$

These equations must be solved with boundary conditions (Equation (6.7))

$$\begin{aligned} x^1(0) &= y^1 & \lambda_1(0) &= \varphi_1 \\ x^2(0) &= 0 & \lambda_2(0) &= b_2 \\ v^1(0) &= w^1 & \mu_1(0) &= \psi_1 \\ v^2(0) &= w^2 & \mu_2(0) &= \psi_2 \,. \end{aligned}$$

By integrating the differential equations for the flow of $\mathbf{Y}_{H^\circ}$ we obtain $\Phi = \Upsilon^{y^2}$ by

$$\begin{aligned} x^1 &= y^1\cos(y^2) \\ x^2 &= y^1\sin(y^2) \\ v^1 &= -w^2\sin(y^2) + w^1\cos(y^2) \\ v^2 &= w^1\sin(y^2) + w^2\cos(y^2) \end{aligned}$$

and

$$
\begin{aligned}
\lambda_1 &= -\frac{\sin(y^2)(\varphi_2 - w^1\psi_2 + w^2\psi_1)}{y^1} + \varphi_1\cos(y^2) \\
\lambda_2 &= \sin(y^2)\varphi_1 + \frac{(\varphi_2 - w^1\psi_2 + w^2\psi_1)\cos(y^2)}{y^1} \\
\mu_1 &= -\sin(y^2)\psi_2 + \psi_1\cos(y^2) \\
\mu_2 &= \sin(y^2)\psi_1 + \psi_2\cos(y^2)\,.
\end{aligned}
$$

The transformed Hamiltonian function becomes

$$
\begin{aligned}
\bar{H}^\circ &= -\frac{(y^1)^2m - y^1mw^2\varphi_2 + y^1mw^2w^1\psi_2 - y^1m(w^2)^2\psi_1 - (y^1)^2mw^1\varphi_1}{(y^1)^2m} \\
&\quad -\frac{K\psi_1 + u_{max}\sqrt{\psi_1)^2 + (\psi_2)^2}\,(y^1)^2\sqrt{m}}{(y^1)^2m}\,.
\end{aligned}
$$

We recognize that $\bar{H}^\circ$ no longer depends on y^2. If we have a closer look onto transformation Φ we recognize that it corresponds more ore less to the transformation of a cartesian to a polar coordinate system!

10 Conclusions

The problem of time-optimal control was formulated in the language of differential geometry. All the mathematical tools necessary were introduced.

By studying the second-order (Euler-Lagrange equation) differential equation of the uncontrolled system one was able to determine a rule to obtain First Integrals of the time-optimally controlled systems. We were able to show that these First Integrals can be recovered from symmetries of the uncontrolled system. We then performed a reduction step to reduce the dimension of the adjoint system to allow a numerical treatment. We finally described certain systems that admit the reduction procedure. We recognized that the existence of a solution of the Killing equation, $\mathcal{L}_{\mathbf{Y}}\mathbf{g} = 0$ with $g_{ij}v^iv^j$ equal to the Lagrange function of the system, is essential.

Two *Maple* routines were supplied that are needed for the calculation of the First Integrals.

Questions

The following questions remain open and would be interesting to pursue in further research:

- The symmetry of the optimally controlled system is constructed from a complete lift of a Killing vector field. Are there any other constructions that would yield a symmetry from a Killing vector field?
- Is it possible to describe a Lagrangian system with Lagrange function L and external forces f^i by a homogeneous system with new Lagrange function $\bar{L} = L + \chi(\mathbf{x})$? By this construction, is it possible to obtain further First Integrals of the time-optimally controlled system?
- When the reduction is performed, the symmetry vector field has to be integrated to determine its flow. Is there any possibility to determine the reduction if the flow is determined numerically?
- Is it possible to treat the problem of energy minimal control with the same kind of symmetry?

11 Appendix

11.1 Calculus

11.1.1 Some relations concerning the almost tangent structure

In this section, $\mathbf{S}$ always denotes the canonical almost tangent structure on TM.

Proposition: 11.1

(11.1)
$$\mathcal{L}_{\mathbf{U}^V}\mathbf{S} = 0 \qquad \textit{with} \qquad \mathbf{U} = U^i\frac{\partial}{\partial x^i} \qquad \textit{and} \qquad \mathbf{U}^V = U^i\frac{\partial}{\partial v^i},$$

where $\mathbf{U}^V$ is the ***vertical lift*** *of a vector field $\mathbf{U}$ on M.*

Proof: 11.1 By the Leibnitz-Rule we have

$$\begin{aligned}
\mathcal{L}_{\mathbf{U}^V}\mathbf{S} &= \mathcal{L}_{\mathbf{U}^V}\left(\frac{\partial}{\partial v^a}\otimes dx^a\right) = \left(\mathcal{L}_{\mathbf{U}^V}\frac{\partial}{\partial v^a}\right)\otimes dx^a + \frac{\partial}{\partial v^a}\otimes(\mathcal{L}_{\mathbf{U}^V}dx^a) \\
&= \left[U^l\frac{\partial}{\partial v^l}, \frac{\partial}{\partial v^a}\right]\otimes dx^a + \frac{\partial}{\partial v^a}\otimes d\left(U^l\frac{\partial}{\partial v^l}x^a\right) \\
&= \left(U^l\frac{\delta^i_a}{\partial v^l}\frac{\partial}{\partial v^i} - \delta^l_a\frac{\partial U^i}{\partial v^l}\frac{\partial}{\partial v^i}\right)\otimes dx^a + \frac{\partial}{\partial v^a}\otimes d(\mathbf{U}^V x^a) \\
&= 0\,. \quad \square
\end{aligned}$$

Proposition: 11.2 *Let $\mathbf{C} = v^i(\partial/\partial v^i)$ be the Liouville vector field on M. Then*

(11.2)
$$\mathbf{S}(\boldsymbol{\xi}) = \mathbf{C}\,,$$

where $\boldsymbol{\xi}$ is a semispray on M.

Proof: 11.2

$$\mathbf{S}(\boldsymbol{\xi}) = \frac{\partial}{\partial v^a}\otimes dx^a\left(v^i\frac{\partial}{\partial x^i} + \xi^l\frac{\partial}{\partial v^l}\right) = v^i\frac{\partial}{\partial v^i} = \mathbf{C}\,. \quad \square$$

Proposition: 11.3 *For a vertical lift $\mathbf{U}^V$ of a vector field U on M we get*

(11.3)
$$\mathbf{S}(\mathbf{U}^V) = 0.$$

Proof: 11.3

$$\mathbf{S}(\mathbf{U}^V) = \frac{\partial}{\partial v^a} \otimes dx^a \left(U^i \frac{\partial}{\partial v^i} \right) = 0 \,. \quad \square$$

Proposition: 11.4 *Let* $\mathbf{U} = U^i(\partial/\partial x^i)$ *be a vector field on* M. *Then*

(11.4)
$$\mathbf{S}(\mathbf{U}^C) = \mathbf{U}^V$$

where

$$\mathbf{U}^C = U^i \frac{\partial}{\partial x^i} + v^b \frac{\partial U^a}{\partial x^b} \frac{\partial}{\partial v^a} \,,$$

is the complete lift of $\mathbf{U}$ *and* $\mathbf{U}^V$ *is the vertical lift of* $\mathbf{U}$.

Proof: 11.4

$$\begin{aligned} \mathbf{S}(\mathbf{U}^C) &= \frac{\partial}{\partial v^a} \otimes dx^a \left(U^i \frac{\partial}{\partial x^i} + v^b \frac{\partial U^a}{\partial x^b} \frac{\partial}{\partial v^a} \right) \\ &= U^i \frac{\partial}{\partial v^a} \delta^i_a = U^i \frac{\partial}{\partial v^i} = \mathbf{U}^V \,. \quad \square \end{aligned}$$

Proposition: 11.5 *For*

$$\boldsymbol{\xi} = v^i \frac{\partial}{\partial x^i} + \xi^l \frac{\partial}{\partial v^l} \,,$$

we have

(11.5)
$$\mathcal{L}_{\mathbf{U}^C} \mathbf{S}(\boldsymbol{\xi}) = 0,$$

where $\mathbf{U}^C$ *is the complete lift of* $\mathbf{U}$, *and* $\boldsymbol{\xi}$ *is a semispray or second-order equation.*

Proof: 11.5

$$\mathcal{L}_{\mathbf{U}^C} \mathbf{S}(\boldsymbol{\xi}) \overset{(11.12)}{=} \mathcal{L}_{\mathbf{U}^C} \mathbf{C} = [\mathbf{U}^C, \mathbf{C}] \overset{(11.16)}{=} 0 \,. \quad \square$$

Proposition: 11.6

(11.6)
$$(\mathcal{L}_{\boldsymbol{\xi}} \mathbf{S})(\mathbf{U}^C) = [\boldsymbol{\xi}, \mathbf{U}^V]$$

Proof: 11.6 With (11.4) we get

$$\mathcal{L}_{\boldsymbol{\xi}} \mathbf{S}(\mathbf{U}^C) = [\boldsymbol{\xi}, \mathbf{S}(\mathbf{U}^C)] - \mathbf{S}([\boldsymbol{\xi}, \mathbf{U}^C]) = [\boldsymbol{\xi}, \mathbf{U}^V] + \mathbf{S}([\mathbf{U}^C, \boldsymbol{\xi}]) \,.$$

By (11.2) and (11.16) we have

$$\begin{aligned} 0 &= \mathcal{L}_{\mathbf{U}^C} \mathbf{S}(\boldsymbol{\xi}) = [\mathbf{U}^C, \mathbf{S}(\boldsymbol{\xi})] - \mathbf{S}([\mathbf{U}^C, \xi]) \\ &= [\mathbf{U}^C, \mathbf{C}] - \mathbf{S}([\mathbf{U}^C, \boldsymbol{\xi}]) = -\mathbf{S}([\mathbf{U}^C, \boldsymbol{\xi}]) \,. \quad \square \end{aligned}$$

Proposition: 11.7

$$\mathbf{S} = -\mathcal{L}_{\mathbf{C}}\mathbf{S} \tag{11.7}$$

where $\mathbf{C}$ *is the Liouville vector field.*

Proof: 11.7

$$\begin{aligned}
\mathcal{L}_{\mathbf{C}}\left(\frac{\partial}{\partial v^a} \otimes dx^a\right) &= \mathcal{L}_{\mathbf{C}}\frac{\partial}{\partial v^a} \otimes dx^a + \frac{\partial}{\partial v^a} \otimes \mathcal{L}_{\mathbf{C}} dx^a \\
&= \left[v^l \frac{\partial}{\partial v^l}, \frac{\partial}{\partial v^a}\right] \otimes dx^a + \frac{\partial}{\partial v^a} \otimes d\left(v^l \frac{\partial}{\partial v^l} x^a\right) \\
&= \left(v^l \frac{\delta^i_a}{\partial v^l}\frac{\partial}{\partial v^i} - \delta^l_a \frac{\partial v^i}{\partial v^l}\frac{\partial}{\partial v^i}\right) \otimes dx^a + \frac{\partial}{\partial v^a} \otimes d(\mathbf{C}x^a) \\
&= -\delta^l_a \delta^i_l \frac{\partial}{\partial v^i} \otimes dx^a \\
&= -\frac{\partial}{\partial v^a} \otimes dx^a \,. \quad \square
\end{aligned}$$

Proposition: 11.8

$$\mathcal{L}_{\mathbf{W}^C}\mathbf{S} = 0\,, \tag{11.8}$$

with

$$\mathbf{W}^C = W^l \frac{\partial}{\partial x^l} + v^l \frac{\partial W^a}{\partial x^l}\frac{\partial}{\partial v^a}\,.$$

Proof: 11.8

$$\begin{aligned}
\mathcal{L}_{\mathbf{W}^C}\left(\frac{\partial}{\partial v^a} \otimes dx^a\right) &= \left(\mathcal{L}_{\mathbf{W}^C}\frac{\partial}{\partial v^a}\right) \otimes dx^a + \frac{\partial}{\partial v^a} \otimes \mathcal{L}_{\mathbf{W}^C} dx^a \\
&= \left[\mathbf{W}^C, \frac{\partial}{\partial v^a}\right] \otimes dx^a + \frac{\partial}{\partial v^a} \otimes d\left(W^l \frac{\partial x^a}{\partial x^l}\right) \\
&= \left[W^l \frac{\partial}{\partial x^l}, \frac{\partial}{\partial v^a}\right] \otimes dx^a + \left[v^b \frac{\partial W^a}{\partial x^b}\frac{\partial}{\partial v^a}, \frac{\partial}{\partial v^a}\right] \otimes dx^a + \frac{\partial}{\partial v^a} \otimes d(W^a) \\
&= (0) \otimes dx^a + \left(-1 \frac{\partial\left(v^b \frac{\partial W^j}{\partial x^b}\right)}{\partial v^a}\frac{\partial}{\partial v^j}\right) \otimes dx^a + \frac{\partial}{\partial v^a} \otimes \left(\frac{\partial W^j}{\partial x^a} dx^a\right) \\
&= 0\,. \quad \square
\end{aligned}$$

11.1.2 Some Rules for the Lie Derivative

Leibnitz Rule:

$$\mathcal{L}_\xi(\mathbf{T} \otimes \mathbf{R}) = \mathcal{L}_\xi \mathbf{T} \otimes \mathbf{R} + \mathbf{T} \otimes \mathcal{L}_\xi \mathbf{R} \tag{11.9}$$

($\mathbf{T}, \mathbf{R}$ tensor fields). Lie derivatives of forms:

$$\mathcal{L}_\xi(df) = d(\mathcal{L}_\xi f) \tag{11.10}$$

$$\mathcal{L}_\xi dx^a = d(\boldsymbol{\xi} x^a) = d\xi^a, \text{ where } \boldsymbol{\xi} = \xi^i \frac{\partial}{\partial x^i}\,. \tag{11.11}$$

$$(\mathcal{L}_\xi \alpha)(\mathbf{W}) = \boldsymbol{\xi}(\alpha(\mathbf{W})) - \alpha([\boldsymbol{\xi}, \mathbf{W}])\,. \tag{11.12}$$

(11.13)
$$(\mathcal{L}_\xi \omega)(\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_p) = \boldsymbol{\xi}(\omega(\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_p) - \sum_{r=1}^{p} \omega(\mathbf{W}_1, \cdots, [\boldsymbol{\xi}, \mathbf{W}_r], \dots, \mathbf{W}_p)\,.$$

$$\boldsymbol{\xi}(\mathcal{L}_\chi \omega) = \mathcal{L}_\chi(\mathcal{L}_\xi \omega) + \mathcal{L}_{[\xi,\chi]}\omega\,. \tag{11.14}$$

11.1.3 Liouville Vector Field and Lifts

Proposition: 11.9 *For the vertical lift* $\mathbf{W}^V = W^i(\partial/\partial v^i)$ *of* W, *we get*

$$[\mathbf{C}, \mathbf{W}^V] = -\mathbf{W}^V\,, \tag{11.15}$$

where $\mathbf{C} = v^i(\partial/\partial v^i)$ *is the Liouville vector field.*

Proof: 11.9

$$\begin{aligned}
[\mathbf{C}, \mathbf{W}^V] &= \left[v^a \frac{\partial}{\partial v^a}, W^i \frac{\partial}{\partial v^i}\right] \\
&= v^a \frac{\partial W^i}{\partial v^a}\frac{\partial}{\partial v^i} - W^l \frac{\partial v^i}{\partial v^l}\frac{\partial}{\partial v^i} = -W^l \frac{\partial}{\partial v^i}\delta^i_l \\
&= -W^i \frac{\partial}{\partial v^i} = -\mathbf{W}^V\,. \quad \square
\end{aligned}$$

Proposition: 11.10

$$[\mathbf{C}, \mathbf{W}^C] = 0 \,, \tag{11.16}$$

with $\mathbf{W}^C = W^i(\partial/\partial x^i) + v^b(\partial W^a/\partial x^b)(\partial/\partial v^a)$, *the complete lift of* $\mathbf{W}$.

Proof: 11.10

$$\begin{aligned}
[\mathbf{C}, \mathbf{W}^C] &= \left[v^i\frac{\partial}{\partial v^i}, W^i\frac{\partial}{\partial x^i} + v^b\frac{\partial W^a}{\partial x^b}\frac{\partial}{\partial v^a}\right] \\
&= v^l\frac{\partial W^k}{\partial v^l}\frac{\partial}{\partial x^k} - W^l\frac{\partial v^i}{\partial x^l}\frac{\partial}{\partial v^i} + v^l\frac{\partial}{\partial v^l}\left(v^b\frac{\partial W^i}{\partial x^b}\right)\frac{\partial}{\partial v^a} - v^b\frac{\partial W^a}{\partial x^b}\frac{\partial v^i}{\partial v^a}\frac{\partial}{\partial v^i} \\
&= v^l\left[\frac{\partial W^i}{\partial x^b}\delta^b_l\right]\frac{\partial}{\partial v^a} - v^b\frac{\partial W^a}{\partial x^b}\delta^i_a = v^b\frac{\partial W^i}{\partial x^b} - v^b\frac{\partial W^i}{\partial x^b} = 0 \,. \quad \square
\end{aligned}$$

(11.15) and (11.16) mean that $\mathbf{W}^V$ is homogeneous of degree (-1) in $\mathbf{v}$ and $\mathbf{W}^C$ is homogeneous of degree 0 in $\mathbf{v}$.

Proposition: 11.11

$$[\mathbf{C}, \mathbf{W}^H] = 0 \,, \tag{11.17}$$

with $\mathbf{W} = W^i(\partial/\partial x^i)$, *and* $\mathbf{W}^H = W^i(\partial/\partial x^i) - W^c\Gamma^a_{bc}(\mathbf{x})v^b(\partial/\partial v^a)$ *is the horizontal lift of* $\mathbf{W}$.

Proof: 11.11

$$\begin{aligned}
[\mathbf{C}, \mathbf{W}^H] &= \left[v^i\frac{\partial}{\partial v^i}, W^i\frac{\partial}{\partial x^i} - W^c\Gamma^a_{bc}v^b\frac{\partial}{\partial v^a}\right] \\
&= v^l\frac{\partial W^i}{\partial v^l}\frac{\partial}{\partial x^i} - W^l\frac{\partial v^i}{\partial x^l}\frac{\partial}{\partial v^i} - v^l\frac{\partial}{\partial v^l}\left(W^c\Gamma^i_{bc}v^b\frac{\partial}{\partial v^i}\right) + W^c\Gamma^l_{bc}v^l\frac{\partial v^i}{\partial v^l}\frac{\partial}{\partial v^i} \\
&= -v^lW^c\Gamma^i_{bc}\frac{\partial}{\partial v^i}\delta^l_b + W^c\Gamma^l_{bc}v^b\delta^i_l\frac{\partial}{\partial v^i} \\
&= -v^bW^c\Gamma^i_{bc}\frac{\partial}{\partial v^i} + v^bW^c\Gamma^i_{bc}\frac{\partial}{\partial v^i} = 0 \,. \quad \square
\end{aligned}$$

11.2 Maple Routines

Maple is a product of Waterloo Software. It was installed on a work station Sun sparc 1+. This software is very common at universities. The description of the routines is given in the source code. To apply the routines, the *Maple* command `with(linalg);` has to be carried out first.

11.2.1 Routine *comp_gamma*

```
comp_gamma := proc (g,g_inv) local i, j, k, l, n;
####################################################################
# This Maple routine computes the Christoffel symbols
#                     gamma^k_{ij}
# Input:
# g:  Metric in matrix form.  g_inv is the inverse of matrix g
# Output:
# gamma[k,i,j]:  Contains the Christoffel symbols.
####################################################################
#
n := coldim(g);
#
for i to n do
   for j to n do
      for k to n do
         gamma[k,i,j] := 0;
         for l to n do
            gamma[k,i,j] := gamma[k,i,j] +
            g_inv[k,l]*(diff(g[l,j],x[i])+
            diff(g[i,l],x[j]) - diff(g[i,j],x[l]))/2
         od
      od
   od
od;
op(gamma);
end;
#
save ('comp_gamma.m');
#Example:
#g:=matrix([[1/x[1]^2,0],[0,1]]);
#g_inv:=inverse(g);
#comp_gamma(g,g_inv);
```

Processing the example given in the source code we get the following output on the screen:

```
                 [   1          ]
                 [ -----    0   ]
                 [     2        ]
      g :=       [  x[1]        ]
                 [              ]
                 [   0      1   ]

>
                 [     2        ]
                 [  x[1]    0   ]
      g_inv :=   [              ]
                 [   0      1   ]

>
           table([
                 (2, 2, 2) = 0
                 (1, 2, 1) = 0
                                  1
                 (1, 1, 1) = - ----
                               x[1]
                 (2, 2, 1) = 0
                 (2, 1, 1) = 0
                 (1, 2, 2) = 0
                 (2, 1, 2) = 0
                 (1, 1, 2) = 0
           ])

>
```

11.2.2 Routine *Lie_Bracket*

```
lie_bracket := proc(vvxi,vveta,Basis,vvResultat) local i, j, n;
######################################################################
# This Maple routine computes the Lie Bracket:
#                     [vvxi,vveta]
# Input:
# Basis:  Is a vector containing the basis with respect to which
# vvxi and vveta are written.  Vectors vvxi and vveta.
# Output:
# vvResultat:  Contains the result.
######################################################################
#
# Getting the dimension
n := vectdim(vvxi);
#
# Setting zero of the result vector
for i to n do
   vvResultat[i]:=0;
od;
#
# Computing the Lie Bracket
for i to n do
   for j to n do
      vvResultat[j]:= vvResultat[j]+vvxi[i]*diff(vveta[j],
      Basis[i]);
      vvResultat[i]:= vvResultat[i]-vveta[j]*diff(vvxi[i],
      Basis[j])
   od
od;
op(vvResultat);
end;
#
save ('lie_bracket.m');
#
# Example:
# Basis:=vector([x1,x2,v1,v2]);
# yc:=vector([x1,0,v1,0]);
# xiL:=vector([v1,v2,1/x1*v1*v1,0]);
# lie_bracket(yc,xiL,Basis,Z);
```

The example given in the routine produces the following output:

```
                    Basis := [ x1, x2, v1, v2 ]

>
                    yc := [ x1, 0, v1, 0 ]

>
                                        2
                                      v1
                    xiL := [ v1, v2, ---, 0 ]
                                      x1

>
                    table([
                        4 = 0
                        1 = 0
                        2 = 0
                        3 = 0
                    ])

>
```

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12 Curriculum Vitae

I was born on March 17, 1961 in Aarau (Switzerland) where I grew up with a brother and a sister. In 1981 I graduated with the Type C Maturity Certificate at the Kantonsschule Aarau. In the same year I enrolled as a mechanical engineering student at the ETH (Federal Institute of Technology) in Zurich. My master's thesis, in 1987, was entitled "PC-Software für den Entwurf von Mehrgrössen Reglern". After having earned the rank of a Lieutenant in the Swiss Army Service I entered my position as an assistant at the Measurement and Control Laboratory of the ETH Zurich. While pursuing my thesis I simultaneously accomplished the optimization of the sophisticated heating system of the ETH.