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Arakelov Theory of Noncommutative Arithmetic Curves and Surfaces

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presented by
Thomas Borek
dipl. math., Universität Zürich
born December 19, 1975
citizen of Kreuzlingen TG

accepted on the recommendation of
Prof. Dr. Gisbert Wüstholz, examiner
Prof. Dr. Jürg Kramer, co-examiner

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Contents

Abstract

The purpose of this thesis is to initiate Arakelov theory in a noncommutative setting. The first chapter is concerned with Arakelov theory of noncommutative arithmetic curves. A noncommutative arithmetic curve is the spectrum of a $\mathbb{Z}$-order $\mathcal{O}$ in a finite dimensional semisimple $\mathbb{Q}$-algebra. Our first main result is an arithmetic Riemann-Roch formula in this setup. We proceed with introducing the Grothendieck group $\widehat{K}_0(\mathcal{O})$ of arithmetic vector bundles on a noncommutative arithmetic curve $\operatorname{Spec}\mathcal{O}$ and show that there is a uniquely determined degree map $\widehat{\deg}_{\mathcal{O}} : \widehat{K}_0(\mathcal{O}) \to \mathbb{R}$, which we use to define a height function $H_{\mathcal{O}}$. We prove that the height $H_{\mathcal{O}}$ satisfies duality whenever the order $\mathcal{O}$ is maximal. As an application of the height $H_{\mathcal{O}}$, we establish a version of "Siegel's Lemma" over division algebras of finite dimension over $\mathbb{Q}$.

Arakelov theory of noncommutative arithmetic surfaces is subject of the second chapter. Roughly speaking, a noncommutative arithmetic surface is a noncommutative projective scheme of cohomological dimension 1 of finite type over $\operatorname{Spec}\mathbb{Z}$. An important example is the category of coherent $\mathcal{O}$-modules, where $\mathcal{O}$ is a coherent sheaf of $\mathcal{O}_X$-algebras and $\mathcal{O}_X$ is the structure sheaf of a commutative arithmetic surface X. Since smooth hermitian metrics are not available in our noncommutative setting, we have to adapt the definition of arithmetic vector bundles on noncommutative arithmetic surfaces. We consider pairs $(\mathcal{E}, \beta)$ consisting of a coherent sheaf $\mathcal{E}$ and an automorphism β of the real sheaf $\mathcal{E}_{\mathbb{R}}$ induced by $\mathcal{E}$. We define the intersection of two such objects via the determinant of the cohomology and prove a Riemann-Roch theorem for arithmetic line bundles on noncommutative arithmetic surfaces.

Zusammenfassung

Die Absicht der vorliegenden Arbeit ist, Arakelov-Theorie in einem nichtkommutativen Rahmen einzuführen. Das erste Kapitel befasst sich mit der Arakelov-Theorie nichtkommutativer arithmetischer Kurven. Darunter verstehen wir das Spektrum einer $\mathbb{Z}$-Ordnung in einer endlich-dimensionalen halbeinfachen $\mathbb{Q}$-Algebra. Das erste zentrale Ergebnis ist eine Riemann-Roch Formel über nichtkommutativen arithmetischen Kurven. Des weiteren definieren wir die Grothendieck-Gruppe $\widehat{K}_0(\mathcal{O})$ arithmetischer Vektorbündel auf einer nichtkommutativen arithmetischen Kurve $\operatorname{Spec}\mathcal{O}$ und beweisen, dass es eine eindeutig bestimmte Gradabbildung $\widehat{\deg}_{\mathcal{O}} : \widehat{K}_0(\mathcal{O}) \to \mathbb{R}$ gibt, die wir benutzen, um eine Höhenfunktion $H_{\mathcal{O}}$ einzuführen. Wir zeigen, dass die Höhe $H_{\mathcal{O}}$ die Dualitätseigenschaft erfüllt, falls die Ordnung $\mathcal{O}$ maximal ist. Als eine Anwendung der Höhe $H_{\mathcal{O}}$ beweisen wir eine Version von "Siegel's Lemma" über Divisionsalgebren von endlicher Dimension über $\mathbb{Q}$.

Arakelov-Theorie nichtkommutativer arithmetischer Flächen ist das Thema des zweiten Kapitels. Grob gesagt, ist eine nichtkommutative arithmetische Fläche ein nichtkommutatives projektives Schema, das kohomologische Dimension 1 hat und von endlichem Typ über $\operatorname{Spec}\mathbb{Z}$ ist. Ein wichtiges Beispiel ist die Kategorie der koherenten $\mathcal{O}$-Moduln, wobei $\mathcal{O}$ eine koherente Garbe von $\mathcal{O}_X$-Algebren und $\mathcal{O}_X$ die Strukturgarbe einer kommutativen arithmetischen Fläche X ist. Da in unserem nichtkommutativen Fall glatte hermitsche Metriken nicht zur Verfügung stehen, müssen wir die Definition arithmetischer Vektorbündel auf nichtkommutativen Flächen anpassen. Wir betrachten Paare $(\mathcal{E}, \beta)$, die aus einer koherenten Garbe $\mathcal{E}$ und einem Automorphismus β der von $\mathcal{E}$ induzierten reellen Garbe $\mathcal{E}_{\mathbb{R}}$ bestehen. Wir definieren die Schnittvielfachheit zweier solcher Objekte mittels der Determinante der Kohomologie und beweisen ein Riemann-Roch Theorem für arithmetische Geradenbündel auf nichtkommutativen arithmetischen Flächen.

Introduction

Back in 1974, Arakelov [A1; A2] founded a new geometry which allows to attack problems in number theory with geometric tools. The crucial idea is to complete an algebraic variety over the spectrum of the ring of integers of an algebraic number field by including the points at infinity, that is, the archimedean places. This is done by considering hermitian structures in addition to sheaf structures, so analysis becomes number theory at infinity. Analogues of many important results from algebraic geometry are nowadays established in Arakelov geometry; for instance the Adjunction Formula, the Hodge Index Theorem, and the Riemann-Roch Theorem. For arithmetic surfaces this was mainly done by Faltings [F2]; see also [La], [So] and [Sz]. The higher dimensional cases were established by Gillet and Soulé [GS1; GS2]. A nice textbook on higher dimensional Arakelov geometry is [ABKS]. The main achievement of Arakelov geometry is the proof of the Mordell conjecture [F1; Vo]: a smooth projective curve of genus greater than one has only finitely many rational points.

In recent years, number theory and arithmetic geometry have been enriched by new techniques from noncommutative geometry. For instance, Consani and Marcolli show how noncommutative (differential) geometry à la Connes provides a unified description of the archimedean and the totally split degenerate fibres of an arithmetic surface. We refer the interested reader to [CM1; CM2; CM3] and [Ma]. Until now, only ideas and methods of noncommutative geometry in the form developed by Connes [Co] have been applied to number theory and arithmetic geometry. But as Marcolli mentions in the last chapter of her book [Ma], in which she addresses the question "where do we go from here?", it would be interesting to consider more algebraic versions of noncommutative geometry in the context of number theory and arithmetic geometry. This is exactly what we have done in this thesis.

The lowest dimensional arithmetic varieties are arithmetic curves. These are simply the spectra of the rings of integers of algebraic number fields. In turns out that Arakelov theory of arithmetic curves is mainly a reformulation of Minkowski's Geometry of Numbers. One of the basic questions in Geometry of Numbers is, if one can control the "size" of a small integer solution of a system of homogeneous linear equations with coefficients in a number field. Solutions to this problem are usually called "Siegel's Lemma", as Siegel [Si] was the first who formally stated a result in this direction. In her thesis [Li1; Li2], Liebendörfer studied the question whether it is possible to obtain some kind of Siegel's Lemma also in a situation where the coefficients are taken from a (possibly) noncommutative division algebra of finite dimension over $\mathbb{Q}$. She establishes a version of Siegel's Lemma when the coefficients lie in a positive definite rational quaternion algebra D and the solution vectors belong to a maximal order in D.

Our thesis was initially motivated by the question: can we establish some kind of Arakelov theory of noncommutative arithmetic curves which enables us to reformulate the results of Liebendörfer? To answer this question, first of all we have to give a definition of noncommutative arithmetic curves. We did this by asking what conditions we have to impose on a possible candidate in order to obtain a nice Arakelov theory for it. It turns out that for our purposes semisimplicity is the right assumption. This has mainly two reasons. Firstly, given a finite dimensional algebra A over a number field K, the trace form $\mathrm{tr}_{A|K} : A \times A \to K$, $(a, b) \mapsto \mathrm{tr}_{A|K}(ab)$ is nondegenerate if and only if A is semisimple. The non-degeneracy of the trace form is crucial in order to get a reasonable measure on the associated real algebra $A_\mathbb{R} = A \otimes_\mathbb{Q} \mathbb{R}$. This so-called canonical measure on $A_\mathbb{R}$ is essential in Arakelov theory. Secondly, semisimple algebras over number fields enjoy the important property that their orders are one-dimensional in the sense that prime ideals are maximal. This enables us to prove a product formula in this setting. Without some kind of product formula it is probably impossible to establish a concise Arakelov theory. To cut a long story short, a noncommutative arithmetic curve is the spectrum of a $\mathbb{Z}$-order in a finite dimensional semisimple $\mathbb{Q}$-algebra.

Since the generalisation for arithmetic curves has worked so smoothly, we have been wondering whether we could go one step further and establish some kind of Arakelov theory of noncommutative arithmetic surfaces. This is the topic of the second chapter of our thesis. To describe the sheaf-theoretic (finite) part, we may use the well developed theory of noncommutative projective schemes. The standard reference for noncommutative projective geometry is the article [AZ] of Artin and Zhang. Noncommutative algebraic geometry was mainly developed by Artin, Tate and van den

Bergh [ATV], by Manin [M], and by Kontsevich and Rosenberg [Ro]. An informal guided tour to some of the literature in noncommutative algebraic geometry was written recently by Mahanta [Mah]. The general philosophy of noncommutative algebraic geometry is that noncommutative spaces are made manifest by the modules that live on them in the same way that the properties of a commutative scheme X are manifested by the category of (quasi)-coherent $\mathcal{O}_X$-modules. The modules over a noncommutative space form, by definition, an abelian category. The category is the basic object of study in noncommutative algebraic geometry. In short, a noncommutative space is an abelian category.

Noncommutative *projective* schemes are determined by specific abelian categories. They are obtained as follows. Let A be an $\mathbb{N}$-graded right noetherian algebra over a commutative noetherian ring k, let $\operatorname{Gr} A$ be the category of graded right A-modules and $\operatorname{Tors} A$ its full subcategory generated by the right bounded modules. The noncommutative projective scheme associated to A is the pair $\operatorname{Proj} A = (\operatorname{QGr} A, \mathcal{A})$, where $\operatorname{QGr} A$ is the quotient category $\operatorname{Gr} A / \operatorname{Tors} A$ and $\mathcal{A}$ the image of A in $\operatorname{QGr} A$. This definition is justified by a theorem of Serre [Se, Prop. 7.8], which asserts that if A is a commutative graded algebra generated in degree 1 and $X = \operatorname{Proj} A$ the associated projective scheme, then the category of quasi-coherent $\mathcal{O}_X$-modules is equivalent to the quotient category $\operatorname{QGr} A$. Thus the objects in $\operatorname{QGr} A$ are the noncommutative geometric objects analogous to sheaves of $\mathcal{O}_X$-modules.

Working with the Grothendieck category $\operatorname{QGr} A$ allows to define a lot of useful tools known from the theory of commutative projective schemes, such as ample sheaves, twisting sheaf, cohomology, dualizing object, Serre duality and so on. However equating noncommutative schemes with abelian categories has drawbacks: most notably there are no points, so the powerful tool of localisation is not available. This forces to search for equivalent global definitions of objects which are usually defined locally. The main example are locally free sheaves. Secondly, at present there is almost no connection between noncommutative algebraic geometry considered in the just described way and noncommutative geometry developed by Connes. In particular it is not clear what should be the (noncommutative) differential structure on a noncommutative algebraic variety over $\mathbb{C}$. The lack of a differential structure makes it impossible to define *smooth* hermitian metrics on locally free sheaves, which is a serious problem because hermitian vector bundles are essential in Arakelov theory. Fortunately, we found a substitute for the metrics. Namely, we consider pairs $(\mathcal{E}, \beta)$ consisting of a coherent sheaf $\mathcal{E}$ and an automorphism β of the real sheaf $\mathcal{E}_{\mathbb{R}}$ induced by $\mathcal{E}$. It turns out that we can formulate a concise theory using such pairs instead of hermitian vector bundles.

We proceed now with an outline of the chapters and present our main results and methods. Chapter 1 is concerned with Arakelov theory of noncommutative arithmetic curves. Let $\mathcal{O}$ be a $\mathbb{Z}$-order in a finite dimensional semisimple $\mathbb{Q}$-algebra A. To any $\mathcal{O}$-module M of finite length and any prime ideal $\mathfrak{p}$ of $\mathcal{O}$ we associate a number $\operatorname{ord}_{\mathfrak{p}}(M)$, which is a natural generalisation of the valuation function $v_{\mathfrak{p}}$ known for prime ideals in a Dedekind domain. Actually, $\operatorname{ord}_{\mathfrak{p}}$ defines a group homomorphism from the unit group of A onto the integers. This enables us to define the (first) Chow group $CH(\mathcal{O})$ of $\mathcal{O}$. As well as in the commutative case we can define ideal classes of $\mathcal{O}$, but unlike the commutative case the set of all ideal classes does not admit a natural group structure. This forces us to introduce a new equivalence relation on the set of all $\mathcal{O}$-ideals in A, which leads to the group $LFC(\mathcal{O})$ consisting of all locally free ideal classes of $\mathcal{O}$. This group is related to the Chow group by the homomorphism div. It is crucial in Arakelov theory not only to deal with the finite places, i.e. the prime ideals, but also to take the infinite places into account. Since every semisimple algebra A which is not a division algebra has nontrivial zero divisors, it does not admit a valuation in the usual sense, and hence there is no well-defined notion of infinite places of A. This is the reason why we have to find a substitute to describe the infinite part. Recall that the Minkowski space $K_{\mathbb{R}}$ of an algebraic number field K is isomorphic to $K \otimes_{\mathbb{Q}} \mathbb{R}$ and that the logarithm induces a homomorphism $\log : (K_{\mathbb{R}})^{\times} \to \bigoplus_{v|\infty} \mathbb{R}v$, where the sum is taken over all infinite places of K. Hence a possible candidate to describe the infinite part is the unit group $(A_{\mathbb{R}})^{\times}$ of the real algebra $A_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} A$, and we use it to define arithmetic divisors and complete ideal classes of $\mathcal{O}$.

The main results in the second section are Corollary 1.2.3 and an arithmetic Riemann-Roch formula for noncommutative arithmetic curves. Corollary 1.2.3 makes a statement about the elements of the unit group $A^{\times}$, which generalises the product formula satisfied by the nonzero elements of a number field. In Section 3 we introduce arithmetic vector bundles on a noncommutative arithmetic curve $\operatorname{Spec} \mathcal{O}$ and define the arithmetic Grothendieck group $\widehat{K}_0(\mathcal{O})$ of $\mathcal{O}$. We show that there exists a uniquely determined homomorphism $\widehat{\deg}_{\mathcal{O}} : \widehat{K}_0(\mathcal{O}) \to \mathbb{R}$, called the arithmetic degree, which behaves exactly like the corresponding map for hermitian vector bundles on commutative arithmetic curves. We use the arithmetic degree in Section 4 to define the height $H_{\mathcal{O}}(V)$ of a free A-submodule V of A^n. It is an important property of height functions that they satisfy duality, i.e. that the height of a subspace equals the height of its orthogonal complement. We establish duality for the height $H_{\mathcal{O}}$ under the assumption that $\mathcal{O}$ is a maximal order in A. If the order $\mathcal{O}$ is not maximal, then duality does not hold anymore, but we can still prove some statements relating the height of a free A-submodule of A^n and the height of its orthogonal complement. The results in Section 4 generalise the corresponding results of Liebendörfer and Rémond [LR2] over rational division algebras, which themselves are generalisations of statements valid over number fields.

The last section of the first chapter is concerned with Minkowski's Second Theorem and Siegel's Lemma over division algebras of finite dimension over $\mathbb{Q}$. Let $\mathcal{O}$ be a $\mathbb{Z}$-order in a finite dimensional rational division algebra D, and let L be an $\mathcal{O}$-lattice. Since a division algebra has no zero divisors, it follows that out of $\dim_{\mathbb{Q}}(D) + 1$ $\mathbb{Q}$-linearly independent elements of L there are at least two D-linearly independent, hence we can deduce an upper bound for the product of the successive minima of L over D directly from Minkowski's Second Theorem. As an application, we obtain a version of Siegel's Lemma over division algebras. A similar result was established by Liebendörfer and Rémond [LR2]. Unlike the rest of the first chapter, the statements in the last section only hold over division algebras. This is because they all rely on the generalisation of Minkowski's Second Theorem, which we are only able to prove under the assumption that the algebra in question has no zero divisors.

Arakelov theory of noncommutative arithmetic surfaces is the subject of the second chapter. In Section 1 we review the definition and construction of noncommutative projective schemes and provide the results which are needed in the following sections. As already mentioned above, a noncommutative projective scheme is determined by the quotient category $\operatorname{QGr} A$ of the category of graded modules over a graded algebra A. We analyse which objects of $\operatorname{QGr} A$ play the role of locally free sheaves and which the one of invertible sheaves, and we give a characterisation of those objects in terms of their associated graded modules. The proof of a base change lemma finishes our study of arbitrary noncommutative projective schemes.

In the second section, we introduce noncommutative arithmetic surfaces. Following the usual strategy in noncommutative algebraic geometry, we first look at commutative arithmetic surfaces and try to isolate the conditions that we want to be satisfied by our noncommutative analogues. In this vein we define a noncommutative arithmetic surface to be a noncommutative projective scheme of cohomological dimension 1, which has a dualizing object and satisfies some finiteness conditions. We provide a few examples of noncommutative arithmetic surfaces. The main example is the category of coherent $\mathcal{O}$-modules, where $\mathcal{O}$ is a coherent sheaf of $\mathcal{O}_X$-algebras and $\mathcal{O}_X$ is the structure sheaf of a commutative arithmetic surface X. More specific examples are noncommutative plane projective curves over $\mathbb{Z}$. They are obtained as follows. Let R be a finitely generated $\mathbb{Z}$-algebra and $p \in R[T_0, T_1, T_2]$ be a homogenous normal polynomial of positive degree such that $(p) \cap \mathbb{Z}[T_0, T_1, T_2]$ is a prime ideal. Then the noncommutative projective scheme associated to the graded algebra $R[T_0, T_1, T_2]/(p)$ is a noncommutative arithmetic surface. At the end of the second section we introduce arithmetic vector bundles on noncommutative arithmetic surfaces. As mentioned before, the definition of an arithmetic vector bundle on a commutative arithmetic surface $\operatorname{Proj} A$ cannot be generalised directly to our noncommutative situation because smooth hermitian metrics are not available in this framework. As a substitute we take an automorphism of the induced real bundle to describe the infinite part. This leads to pairs $(\mathcal{E}, \beta)$ consisting of a noetherian object $\mathcal{E}$ of $\operatorname{QGr} A$ and an automorphism β of the object $\mathcal{E}_{\mathbb{R}}$ induced by $\mathcal{E}$ on the associated noncommutative real variety $\operatorname{Proj} A_{\mathbb{R}}$.

The third section is concerned with the intersection on noncommutative arithmetic surfaces. Since Faltings published his article [F2], it is well-known that the intersection of two arithmetic line bundles on a commutative arithmetic surface can be computed via the determinant of the cohomology. We follow this approach and firstly define the determinant of the cohomology of an arithmetic vector bundle on a noncommutative arithmetic surface. Then we use the determinant of the cohomology to define the intersection of an arithmetic line bundle with an arithmetic vector bundle. This yields an arithmetic line bundle on $\operatorname{Spec} \mathbb{Z}$ and its degree equals the intersection number of the two bundles in question. We show that the determinant of the cohomology is compatible with Serre duality and apply this fact to prove a Riemann-Roch theorem. As a corollary, we obtain a Riemann-Roch formula which looks exactly like the one known for arithmetic line bundles on commutative arithmetic surfaces.

In the appendix, we briefly resume the construction of quotient categories and list some of their important properties.

1 Noncommutative Arithmetic Curves

1.1 Definitions and notations

In this section, we fix the notation which we will use throughout the chapter. For a group G with neutral element e, we let $G^* = G\backslash\{e\}$. The unit group of a ring B with 1 is denoted by $B^\times$. A *prime ideal* of a ring B is a proper nonzero two-sided ideal $\mathfrak{p}$ in B such that $aBb \not\subset \mathfrak{p}$ for all $a, b \in B \backslash \mathfrak{p}$. The set of all prime ideals of B is denoted by $\operatorname{Spec} B$ and called the (prime) spectrum of B. Given an integral domain R with quotient field K, an *R-lattice* is a finitely generated torsionfree R-module L. We call L a *full* R-lattice in a finite dimensional K-vector space V, if L a finitely generated R-submodule in V such that $KL = V$, where KL is the K-subspace generated by L. An *R-order* in a finite dimensional K-algebra A is a subring $\mathcal{O}$ of A such that $\mathcal{O}$ is a full R-lattice in A. By a left (right) *$\mathcal{O}$-lattice* we mean a left (right) $\mathcal{O}$-module which is an R-lattice. Specifically, a *full left $\mathcal{O}$-ideal* in A is a full left $\mathcal{O}$-lattice in A.

From now on we let K denote an algebraic number field and R its ring of integers. If $\mathcal{O}$ is an R-order in a finite dimensional semisimple K-algebra A, then we call $\operatorname{Spec}\mathcal{O}$ a *noncommutative arithmetic curve.* By [Re, (22.3)], the prime ideals $\mathfrak{p}$ of $\mathcal{O}$ coincide with the maximal two-sided ideals of $\mathcal{O}$. Therefore $\mathcal{O}/\mathfrak{p}$ is a simple ring, hence it is isomorphic to a matrix algebra $M_{\kappa_\mathfrak{p}}(S)$ over a (skew)field S. The natural number $\kappa_\mathfrak{p}$ is uniquely determined by $\mathfrak{p}$ and is called the *capacity* of the prime ideal $\mathfrak{p}$.

1.2 A Riemann-Roch formula

1.2.1 Chow group and ideal classes

Chow group

Let $\operatorname{Spec}\mathcal{O}$ be a noncommutative arithmetic curve. Analogously to the commutative case, we define the *divisor group* of $\mathcal{O}$ to be the free abelian group

$$Div(\mathcal{O}) = \bigoplus_{\mathfrak{p}} \mathbb{Z}\mathfrak{p}$$

over the set of all prime ideals $\mathfrak{p}$ of $\mathcal{O}$. The construction of principal divisors is a little bit more complicated as in the standard case. It is based on the following construction. By the Jordan-Hölder Theorem, every left $\mathcal{O}$-module M of finite length has an $\mathcal{O}$-decomposition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_l = M,$$

where the composition factors $S_i = M_i/M_{i-1}$, $i = 1, \ldots, l$, are simple left $\mathcal{O}$-modules and where $l = l_{\mathcal{O}}(M)$ is the length of M. The set of composition factors S_i is uniquely determined by M. We claim that the annihilator

$$\mathfrak{p}_i = \mathrm{ann}_{\mathcal{O}}(S_i) = \{x \in \mathcal{O} \mid xS_i = 0\}$$

of S_i is a prime ideal of $\mathcal{O}$. Obviously it is a proper nonzero two-sided ideal of $\mathcal{O}$, and for any two elements $x, y \in \mathcal{O} \setminus \mathfrak{p}_i$ we have $xS_i \neq 0$ and $yS_i \neq 0$, which implies $\mathcal{O}yS_i = S_i$ because S_i is a simple left $\mathcal{O}$-module. Hence $x\mathcal{O}yS_i = xS_i \neq 0$, therefore $x\mathcal{O}y \not\subset \mathfrak{p}_i$, which shows that $\mathfrak{p}_i$ is indeed a prime ideal of $\mathcal{O}$.

To every left $\mathcal{O}$-module M of finite length and every prime ideal $\mathfrak{p}$ of $\mathcal{O}$, we may thus associate a natural number $\mathrm{ord}_{\mathfrak{p}}(M)$, which is defined to be the number of composition factors S in the Jordan-Hölder decomposition series of M for which $\mathrm{ann}_{\mathcal{O}}(S) = \mathfrak{p}$. Note that $\mathrm{ord}_{\mathfrak{p}}$ generalise the valuation functions associated to prime ideals of Dedekind domains. Similar functions are also introduced by Neukirch for orders in number fields, cf. [N, I.12]. A way to compute the number $\mathrm{ord}_{\mathfrak{p}}$ is given in the next lemma. We will use the following standard notation. Given a prime ideal $\mathfrak{p}$ of $\mathcal{O}$ and a prime ideal P of R, we write $\mathfrak{p} \mid P$ to indicate that $\mathfrak{p}$ lies above P which means $P = \mathfrak{p} \cap R$. For every prime ideal P of R, we let R_P denote the localisation of R at P. If M is an R-module, then M_P denotes the R_P-module $M \otimes_R R_P$. Especially, $\mathcal{O}_P$ is an R_P-algebra and an $\mathcal{O}$-bimodule.

Lemma 1.2.1. *Let M be a left $\mathcal{O}$-module of finite length. Then for every prime ideal P of R, the $\mathcal{O}_P$-module M_P is of finite length as well and its length can be computed as*

$$l_{\mathcal{O}_P}(M_P) = \sum_{\mathfrak{p}|P} \mathrm{ord}_{\mathfrak{p}}(M). \tag{1.1}$$

Moreover, if $\mathfrak{p}$ *lies above* P *then* $\mathfrak{p}\mathcal{O}_P$ *is a prime ideal of* $\mathcal{O}_P$ *and*

$$\mathrm{ord}_{\mathfrak{p}\mathcal{O}_P}(M_P) = \mathrm{ord}_{\mathfrak{p}}(M). \tag{1.2}$$

Proof. The localisation S_P of a simple left $\mathcal{O}$-module S at a prime ideal P of R is either zero or a simple left $\mathcal{O}_P$-module. To see this, suppose that $U' \subsetneqq U \subset S_P$ is an inclusion of $\mathcal{O}_P$-modules and $u = \frac{s}{r} \in U \setminus U'$ with $s \in S, r \in R \setminus P$. Then $s \notin U' \cap S$, because otherwise $u = \frac{1}{r}s$ would also lie in U' which contradicts the assumption. This shows $U' \cap S \subsetneqq U \cap S$. Applying the same argument to a chain $0 \subsetneqq U \subsetneqq S_P$ twice, yields $0 \subsetneqq U \cap S \subsetneqq S$ which is ruled out by the simplicity of S.

Let $0 \subset M_1 \subset \cdots \subset M_l = M$ be an $\mathcal{O}$-decomposition series of M. Then $0 \subset M_{1,P} \subset \cdots \subset M_{l,P} = M_P$, but there may be some indices i where $M_{i,P} = M_{i+1,P}$. Nevertheless, if for some index j, $M_{j,P} \neq M_{j-1,P}$ then $S_{j,P} = M_{j,P}/M_{j-1,P}$ is a simple left $\mathcal{O}_P$-module as we have just seen above. This means that in order to compute the length of the $\mathcal{O}_P$-module M_P, we have to examine which $S_{i,P}$ are zero and which are not.

Let S be a simple module in the $\mathcal{O}$-decomposition series of M, let $\mathfrak{p} = \mathrm{ann}_{\mathcal{O}}(S)$, and let P be a prime ideal of R. If the prime ideal $\mathfrak{p}$ of $\mathcal{O}$ lies above the prime ideal P, then $S_P \neq 0$. This holds because $x \notin \mathfrak{p}$ whenever $x \in R \setminus P$, hence for all $x \in R \setminus P$ there is some $s \in S$ such that $xs \neq 0$, which implies $S_P \neq 0$. On the other hand, if $\mathfrak{p}$ does not lie above P, then $(R \setminus P) \cap \mathfrak{p} \neq \emptyset$. Thus for all $s \in S$ there is some $x \in R \setminus P$ such that $xs = 0$, whence $S_P = 0$. This shows that S_P is a simple $\mathcal{O}_P$-module if and only if $\mathrm{ann}_{\mathcal{O}}(S) \mid P$ and thus establishes (1.1).

If $\mathfrak{p} \mid P$ then $\mathfrak{p}\mathcal{O}_P \subset \mathrm{ann}_{\mathcal{O}_P}(S_P) \neq \mathcal{O}_P$. But $\mathfrak{p}\mathcal{O}_P$ is a maximal two-sided ideal of $\mathcal{O}_P$, so $\mathfrak{p}\mathcal{O}_P = \mathrm{ann}_{\mathcal{O}_P}(S_P)$. This shows (1.2). $\square$

Suppose we have a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of left $\mathcal{O}$-modules of finite length. Then the composition factors of M are those of M' together with those of M'', therefore $\mathrm{ord}_{\mathfrak{p}}$ behaves additively on short exact sequences.

We apply this fact to the following situation. Let $\mathfrak{a} \subset \mathcal{O}$ be an ideal of $\mathcal{O}$ such that $K\mathfrak{a} = A$, and let $x \in \mathcal{O}$ be no zero divisor. Then $\mathcal{O}x$ is a full R-lattice in A and we have the following exact sequence of left $\mathcal{O}$-modules of finite length

$$0 \to \mathcal{O}x/\mathfrak{a}x \to \mathcal{O}/\mathfrak{a}x \to \mathcal{O}/\mathcal{O}x \to 0.$$

Since x is not a zero divisor, it follows $\mathcal{O}x/\mathfrak{a}x \cong \mathcal{O}/\mathfrak{a}$, thus

$$\mathrm{ord}_{\mathfrak{p}}(\mathcal{O}/\mathfrak{a}x) = \mathrm{ord}_{\mathfrak{p}}(\mathcal{O}/\mathcal{O}x) + \mathrm{ord}_{\mathfrak{p}}(\mathcal{O}/\mathfrak{a}). \tag{1.3}$$

In particular, $\mathrm{ord}_{\mathfrak{p}}$ can be extended to a group homomorphism

$$\mathrm{ord}_{\mathfrak{p}} : A^{\times} \to \mathbb{Z},\ u \mapsto \mathrm{ord}_{\mathfrak{p}}(u) = \mathrm{ord}_{\mathfrak{p}}(\mathcal{O}/\mathcal{O}x) - \mathrm{ord}_{\mathfrak{p}}(\mathcal{O}/\mathcal{O}r),$$

where $r \in R$ and $x \in \mathcal{O}$ are such that $ru = x$. To see that this map is well-defined, we let $x, y \in \mathcal{O}$, $r, s \in R$ such that $u = \frac{x}{r} = \frac{y}{s}$. Since u is a unit, it follows that x and y are no zero divisors. The same is true for $r, s \in R^* \subset A^{\times}$. Hence both, $\mathcal{O}/\mathcal{O}x$ and $\mathcal{O}/\mathcal{O}r$ are $\mathcal{O}$-module of finite length and applying (1.3) yields

$$\mathrm{ord}_{\mathfrak{p}}(\mathcal{O}/\mathcal{O}xs) = \mathrm{ord}_{\mathfrak{p}}(\mathcal{O}/\mathcal{O}x) + \mathrm{ord}_{\mathfrak{p}}(\mathcal{O}/\mathcal{O}s)$$

and

$$\mathrm{ord}_{\mathfrak{p}}(\mathcal{O}/\mathcal{O}yr) = \mathrm{ord}_{\mathfrak{p}}(\mathcal{O}/\mathcal{O}y) + \mathrm{ord}_{\mathfrak{p}}(\mathcal{O}/\mathcal{O}r).$$

But $xs = yr$, thus

$$\mathrm{ord}_{\mathfrak{p}}(\mathcal{O}/\mathcal{O}x) + \mathrm{ord}_{\mathfrak{p}}(\mathcal{O}/\mathcal{O}s) = \mathrm{ord}_{\mathfrak{p}}(\mathcal{O}/\mathcal{O}y) + \mathrm{ord}_{\mathfrak{p}}(\mathcal{O}/\mathcal{O}r),$$

which shows that $\mathrm{ord}_{\mathfrak{p}}(u)$ does not depend on the choice of x and r with $ru = x$.

Likewise one sees that $\mathrm{ord}_{\mathfrak{p}} : A^{\times} \to \mathbb{Z}$ is additive. The homomorphisms $\mathrm{ord}_{\mathfrak{p}}$ provide us with a further homomorphism

$$\mathrm{div} : A^{\times} \to Div(\mathcal{O}),\ u \mapsto \mathrm{div}(u) = \left(\mathrm{ord}_{\mathfrak{p}}(u)\mathfrak{p}\right)_{\mathfrak{p}}.$$

The elements $\mathrm{div}(u)$ are called *principal divisors* and they form a subgroup of $Div(\mathcal{O})$ which we denote by $\mathcal{P}(\mathcal{O})$. The factor group

$$CH(\mathcal{O}) = Div(\mathcal{O})/\mathcal{P}(\mathcal{O})$$

is called the (first) *Chow group* of $\mathcal{O}$.

Ideal classes

Let $\mathcal{O}$ be an R-order in a finite dimensional semisimple K-algebra A. We may partition the set $J(\mathcal{O})$ of all full left $\mathcal{O}$-ideals in A into ideal classes by placing two ideals $\mathfrak{a}, \mathfrak{b}$ in the same class if $\mathfrak{a} \cong \mathfrak{b}$ as left $\mathcal{O}$-modules. Each such isomorphism extends to a left A-isomorphism $A = K\mathfrak{a} \cong K\mathfrak{b} = A$, hence is given by right multiplication by some $u \in A^\times$. Thus the ideal class containing $\mathfrak{a}$ consists of all ideals $\{\mathfrak{a}u \mid u \in A^\times\}$. We denote the set of ideal classes of $\mathcal{O}$ by $Cl(\mathcal{O})$. Given any two full left $\mathcal{O}$-ideals $\mathfrak{a}$, $\mathfrak{b}$ in A, their product $\mathfrak{a}\mathfrak{b}$ is also a full left $\mathcal{O}$-ideal in A. However when A is noncommutative, the ideal class of $\mathfrak{a}\mathfrak{b}$ is not necessarily determined by the ideal class of $\mathfrak{a}$ and $\mathfrak{b}$. Specifically for each $u \in A^\times$, the ideal $\mathfrak{a}u$ is in the same class as $\mathfrak{a}$, but possibly $\mathfrak{a}u\mathfrak{b}$ and $\mathfrak{a}\mathfrak{b}$ are in different classes. Hence, if A is noncommutative, the multiplication of $\mathcal{O}$-ideals does not induce a multiplication on $Cl(\mathcal{O})$.

However there is still a connection between the set $Cl(\mathcal{O})$ and the Chow group of $\mathcal{O}$. Namely, since $\mathcal{O}$ is an R-lattice in A, for every full left $\mathcal{O}$-ideal $\mathfrak{a}$ in A there is a nonzero $r \in R$ such that $\mathfrak{a}r$ is a left ideal in $\mathcal{O}$. As above, (1.3) implies that the map

$$\operatorname{ord}_\mathfrak{p} : J(\mathcal{O}) \to \mathbb{Z}, \quad \operatorname{ord}_\mathfrak{p}(\mathfrak{a}) = \operatorname{ord}_\mathfrak{p}(\mathcal{O}/\mathfrak{a}r) - \operatorname{ord}_\mathfrak{p}(\mathcal{O}/\mathcal{O}r)$$

does not depend on the choice of the element $r \in R$ with $\mathfrak{a}r \subset \mathcal{O}$ and thus is well-defined. This yields a mapping

$$\operatorname{div} : J(\mathcal{O}) \to Div(\mathcal{O}),\ \mathfrak{a} \mapsto \operatorname{div}(\mathfrak{a}) = \left(-\operatorname{ord}_\mathfrak{p}(\mathfrak{a})\mathfrak{p}\right)_\mathfrak{p}.$$

For all $\mathfrak{a} \in J(\mathcal{O})$ and all $u \in A^\times$, we have $\operatorname{div}(\mathfrak{a}u) = \operatorname{div}(\mathfrak{a}) - \operatorname{div}(u)$, so the map div actually lives on ideal classes and we may write

$$\operatorname{div} : Cl(\mathcal{O}) \to CH(\mathcal{O}).$$

The fact that $Cl(\mathcal{O})$ does not admit a natural group structure forces us to consider only a subset of $J(\mathcal{O})$ and to introduce a new equivalence relation.

Locally free ideal classes

Following Reiner [Re, (38)], we call a full left $\mathcal{O}$-ideal $\mathfrak{a}$ in A *locally free* if for all prime ideals P of R, the $\mathcal{O}_P$-module $\mathfrak{a}_P$ is free. We may partition the set of all locally free $\mathcal{O}$-ideals into *locally free ideal classes* by placing two locally free $\mathcal{O}$-ideals $\mathfrak{a}$, $\mathfrak{b}$ in the same class if there is a unit $u \in A^\times$ such that for all prime ideals $\mathfrak{p}$ of $\mathcal{O}$,

$$\operatorname{ord}_\mathfrak{p}(\mathfrak{a}) = \operatorname{ord}_\mathfrak{p}(\mathfrak{b}) + \operatorname{ord}_\mathfrak{p}(u). \tag{1.4}$$

It is easily verified that this defines an equivalence relation. Clearly, if two locally free $\mathcal{O}$-ideals are isomorphic then they are also equivalent. We let $[\mathfrak{a}]$ denote the equivalence class of a locally free $\mathcal{O}$-ideal $\mathfrak{a}$ and $LFC(\mathcal{O})$ the set of all these classes. Our aim is to define a multiplication on this set in such a way that $\mathrm{div} : LFC(\mathcal{O}) \to CH(\mathcal{O})$ becomes a group homomorphism.

Given two locally free $\mathcal{O}$-ideals $\mathfrak{a}$ and $\mathfrak{b}$, for every prime ideal P of R there are some $a_P, b_P \in A^\times$ such that $\mathfrak{a}_P = \mathcal{O}_P a_P$ and $\mathfrak{b}_P = \mathcal{O}_P b_P$. We set $N = \bigcap_P \mathcal{O}_P a_P b_P$ and define a multiplication in $LFC(\mathcal{O})$ by

$$[\mathfrak{a}][\mathfrak{b}] = [N]. \tag{1.5}$$

Of course we have to prove that this definition makes sense. Firstly, we show that N is a locally free $\mathcal{O}$-ideal. Since $\mathfrak{a}$ and $\mathfrak{b}$ are full R-lattices in A there is a nonzero $s \in R$ such that $s\mathcal{O} \subset \mathfrak{a}$ and $s\mathcal{O} \subset \mathfrak{b}$. Let $sR = P_1 \cdots P_n$ be the decomposition of the principal ideal sR into a product of prime ideals. Then $\mathfrak{a}_P = \mathfrak{b}_P = \mathcal{O}_P$ whenever $P \notin \{P_1, \ldots, P_n\}$. In this situation, [Re, (4.22)] ensures that N is a full R-lattice in A and $N_P = \mathcal{O}_P a_P b_P$ for all P. Hence N is indeed a locally free $\mathcal{O}$-ideal.

Secondly, we have to show that $[N]$ does not depend on the choice of the bases a_P and b_P of $\mathfrak{a}_P$ and $\mathfrak{b}_P$, respectively. Let $r \in R^*$ such that $r\mathfrak{a}, r\mathfrak{b} \subset \mathcal{O}$ and consider the short exact sequence

$$0 \to r\mathcal{O}_P b_P / r^2 \mathcal{O}_P a_P b_P \to \mathcal{O}_P / r^2 \mathcal{O}_P a_P b_P \to \frac{\mathcal{O}_P / r^2 \mathcal{O}_P a_P b_P}{r\mathcal{O}_P b_P / r^2 \mathcal{O}_P a_P b_P} \to 0$$

of $\mathcal{O}_P$-modules of finite length. We have

$$r\mathcal{O}_P b_P / r^2 \mathcal{O}_P a_P b_P \cong \mathcal{O}_P / r\mathcal{O}_P a_P \quad \text{and} \quad \frac{\mathcal{O}_P / r^2 \mathcal{O}_P a_P b_P}{r\mathcal{O}_P b_P / r^2 \mathcal{O}_P a_P b_P} \cong \mathcal{O}_P / r\mathcal{O}_P b_P,$$

thus

$$\mathrm{ord}_{\mathfrak{p}\mathcal{O}_P}(\mathcal{O}_P / r^2 \mathcal{O}_P a_P b_P) = \mathrm{ord}_{\mathfrak{p}\mathcal{O}_P}(\mathcal{O}_P / r\mathcal{O}_P a_P) + \mathrm{ord}_{\mathfrak{p}\mathcal{O}_P}(\mathcal{O}_P / r\mathcal{O}_P b_P) \tag{1.6}$$

whenever $\mathfrak{p} \mid P$. On the other hand,

$$\begin{aligned}
\mathrm{ord}_{\mathfrak{p}\mathcal{O}_P}(N_P) &= \mathrm{ord}_{\mathfrak{p}\mathcal{O}_P}\left(\mathcal{O}_P / r^2 \mathcal{O}_P a_P b_P\right) - \mathrm{ord}_{\mathfrak{p}\mathcal{O}_P}(r^2) \\
\mathrm{ord}_{\mathfrak{p}\mathcal{O}_P}(\mathfrak{a}_P) &= \mathrm{ord}_{\mathfrak{p}\mathcal{O}_P}\left(\mathcal{O}_P / r\mathcal{O}_P a_P\right) - \mathrm{ord}_{\mathfrak{p}\mathcal{O}_P}(r) \\
\mathrm{ord}_{\mathfrak{p}\mathcal{O}_P}(\mathfrak{b}_P) &= \mathrm{ord}_{\mathfrak{p}\mathcal{O}_P}\left(\mathcal{O}_P / r\mathcal{O}_P b_P\right) - \mathrm{ord}_{\mathfrak{p}\mathcal{O}_P}(r).
\end{aligned}$$

Combining this with (1.6) and (1.2) yields

$$\mathrm{ord}_{\mathfrak{p}}(N) = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{a}) + \mathrm{ord}_{\mathfrak{p}}(\mathfrak{b}), \tag{1.7}$$

which shows that $\operatorname{ord}_{\mathfrak{p}}(N)$ does not depend on the choice of the bases a_P and b_P of $\mathfrak{a}_P$ and $\mathfrak{b}_P$, respectively, and therefore $[N]$ is also independent of this choice. However, when A is noncommutative, the isomorphism class of N depends on the choice of the bases of $\mathfrak{a}_P$ and $\mathfrak{b}_P$. For this reason we cannot simply use isomorphism classes of locally free $\mathcal{O}$-ideals but really have to work with the equivalence relation defined in (1.4).

Thirdly, we have to show that $[N]$ does not depend on the choice of the representatives $\mathfrak{a}$ and $\mathfrak{b}$ of the locally free ideal classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$. Let $\mathfrak{a}'$ and $\mathfrak{b}'$ be other representatives of $[\mathfrak{a}]$ and $[\mathfrak{b}]$, respectively. Then there are two units $u_1, u_2 \in A^\times$ such that for all $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}$, $\operatorname{ord}_{\mathfrak{p}}(\mathfrak{a}) = \operatorname{ord}_{\mathfrak{p}}(\mathfrak{a}') + \operatorname{ord}_{\mathfrak{p}}(u_1)$ and $\operatorname{ord}_{\mathfrak{p}}(\mathfrak{b}) = \operatorname{ord}_{\mathfrak{p}}(\mathfrak{b}') + \operatorname{ord}_{\mathfrak{p}}(u_2)$. Applying (1.7), we obtain

$$\begin{aligned}\operatorname{ord}_{\mathfrak{p}}(N) &= \operatorname{ord}_{\mathfrak{p}}(\mathfrak{a}) + \operatorname{ord}_{\mathfrak{p}}(\mathfrak{b}) \\ &= \operatorname{ord}_{\mathfrak{p}}(\mathfrak{a}') + \operatorname{ord}_{\mathfrak{p}}(u_1) + \operatorname{ord}_{\mathfrak{p}}(\mathfrak{b}') + \operatorname{ord}_{\mathfrak{p}}(u_2) \\ &= \operatorname{ord}_{\mathfrak{p}}(N') + \operatorname{ord}_{\mathfrak{p}}(u_1 u_2),\end{aligned}$$

whence $[N] = [N']$.

All together this shows that (1.5) defines a well-defined multiplication on $LFC(\mathcal{O})$, which is obviously associative and commutative. Clearly, $[\mathcal{O}]$ is the neutral element and $\left[\bigcap_P \mathcal{O}_P a_P^{-1}\right]$ the inverse of $[\mathfrak{a}]$. Hence $LFC(\mathcal{O})$ equipped with this multiplication is an abelian group and the application

$$\operatorname{div} : LFC(\mathcal{O}) \to CH(\mathcal{O}),\ [\mathfrak{a}] \mapsto \operatorname{div}([\mathfrak{a}]) = \Big[\big(-\operatorname{ord}_{\mathfrak{p}}(\mathfrak{a})\mathfrak{p}\big)_{\mathfrak{p}}\Big]$$

is a group homomorphism.

Even when $\mathcal{O}$ is an order in a number field, $Cl(\mathcal{O})$ is a group only if $\mathcal{O}$ is the maximal order. If $\mathcal{O}$ is not the maximal order then one has to restrict to invertible $\mathcal{O}$-ideals. By [N, (I.12.4)] the invertible and the locally free $\mathcal{O}$-ideals coincide. The corresponding result for orders in separable K-algebras states that if $\mathcal{O}$ is a maximal order then every $\mathcal{O}$-ideal in A is locally free [Re, (18.10)]. This shows that our construction parallels the one for orders in number fields. We also note that Reiner [Re, (38)] partitions the set of all locally free $\mathcal{O}$-ideals into stable isomorphism classes, which defines yet another equivalence relation on the set of all locally free $\mathcal{O}$-ideals. We do not work with stable isomorphism classes because the map div does not become a group homomorphism using those classes.

1.2.2 Completion

Arithmetic divisor classes

Let $\mathcal{O}$ be an R-order in a finite dimensional semisimple K-algebra A. Until now we have only dealt with the prime ideals of $\mathcal{O}$, but it is crucial in Arakelov theory to take also the infinite places into account. Since every semisimple algebra A which is not a division algebra has nontrivial zero divisors, it does not admit a valuation in the usual sense, and hence there is no well-defined notion of infinite places of A. For this reason we have to find a substitute to describe the infinite part. Recall that the Minkowski space $K_{\mathbb{R}}$ of an algebraic number field K is isomorphic to the real vector space $K \otimes_{\mathbb{Q}} \mathbb{R}$ and that the logarithm induces a homomorphism $\log : (K_{\mathbb{R}})^{\times} \to \bigoplus_{v|\infty} \mathbb{R}v$, where the sum is taken over all infinite places of K. Hence a possible candidate to describe the infinite part is the unit group $(A_{\mathbb{R}})^{\times}$ of the associated real algebra $A_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} A$.

In this vein we define the *arithmetic divisor group* of $\mathcal{O}$ to be

$$\widehat{Div}(\mathcal{O}) = Div(\mathcal{O}) \times (A_{\mathbb{R}})^{\times} ,$$

where the group operation is defined component-wise, which makes $\widehat{Div}(\mathcal{O})$ into a non-abelian group. We write $\overline{D} = (D, D_{\infty}) = \left(\sum_{\mathfrak{p}} v_{\mathfrak{p}}\mathfrak{p}, D_{\infty}\right)$ for the elements of $\widehat{Div}(\mathcal{O})$. Clearly the homomorphism $\mathrm{div} : A^{\times} \to Div(\mathcal{O})$ extends to a homomorphism

$$\widehat{\mathrm{div}} : A^{\times} \to \widehat{Div}(\mathcal{O}),\ u \mapsto \widehat{\mathrm{div}}(u) = (\mathrm{div}(u), (1 \otimes u)) .$$

The elements $\widehat{\mathrm{div}}(u)$ are called *arithmetic principal divisors* and they form a subgroup of $\widehat{Div}(\mathcal{O})$, which we denote by $\widehat{\mathcal{P}}(\mathcal{O})$. The right cosets

$$\widehat{\mathcal{P}}(\mathcal{O})\overline{D}, \quad \overline{D} \in \widehat{Div}(\mathcal{O})$$

are called *arithmetic divisor classes* of $\mathcal{O}$, and the set of all these classes is denoted by $\widehat{CH}(\mathcal{O})$. In other words, $\widehat{CH}(\mathcal{O}) = \widehat{P}(\mathcal{O})\backslash\widehat{Div}(\mathcal{O})$. In general $\widehat{P}(\mathcal{O})$ is not a normal subgroup of $\widehat{Div}(\mathcal{O})$, so the set $\widehat{CH}(\mathcal{O})$ is usually not a group. Nevertheless, in analogy with the commutative case, there is an exact sequence

$$0 \longrightarrow (A_{\mathbb{R}})^{\times}/\mathcal{O}^{\times} \xrightarrow{a} \widehat{CH}(\mathcal{O}) \xrightarrow{\pi} CH(\mathcal{O}) \longrightarrow 0,$$

where a is given by $[x] \mapsto [(0, -x)]$ and π maps $[(D, D_{\infty})]$ to $[D]$.

Complete ideal classes

By a complete $\mathcal{O}$-ideal we understand an element of the set

$$\widehat{J}(\mathcal{O}) = J(\mathcal{O}) \times (A_{\mathbb{R}})^{\times}.$$

We write $\bar{\mathfrak{a}} = (\mathfrak{a}, \mathfrak{a}_\infty)$ for the elements of $\widehat{J}(\mathcal{O})$. There is a right-action of the unit group $A^\times$ on the set $\widehat{J}(\mathcal{O})$ given by

$$(\bar{\mathfrak{a}}, u) \mapsto \bar{\mathfrak{a}}u = \big(\mathfrak{a}u, (1 \otimes u^{-1})\mathfrak{a}_\infty\big).$$

We denote the set of orbits by $\widehat{Cl}(\mathcal{O})$ and call it the set of *complete ideal classes* of $\mathcal{O}$.

Again we have a mapping

$$\widehat{\mathrm{div}} : \widehat{J}(\mathcal{O}) \to \widehat{Div}(\mathcal{O}),\ \bar{\mathfrak{a}} = (\mathfrak{a}, \mathfrak{a}_\infty) \mapsto \widehat{\mathrm{div}}(\bar{\mathfrak{a}}) = (\mathrm{div}(\mathfrak{a}), \mathfrak{a}_\infty).$$

Since $\mathrm{div} : A^\times \to Div(\mathcal{O})$ is a group homomorphism, we have $\mathrm{div}(u^{-1}) = -\mathrm{div}(u)$, therefore $\widehat{\mathrm{div}}(\bar{\mathfrak{a}}u) = \widehat{\mathrm{div}}(u^{-1}) \cdot \widehat{\mathrm{div}}(\bar{\mathfrak{a}})$, hence we get a well-defined map

$$\widehat{\mathrm{div}} : \widehat{Cl}(\mathcal{O}) \to \widehat{CH}(\mathcal{O}).$$

1.2.3 Absolute norm, product formula and degree

The *absolute norm* of a complete $\mathcal{O}$-ideal $\bar{\mathfrak{a}} = (\mathfrak{a}, \mathfrak{a}_\infty)$ is defined to be the real number

$$\mathfrak{N}(\bar{\mathfrak{a}}) = \left|N_{A_{\mathbb{R}}|\mathbb{R}}(\mathfrak{a}_\infty)\right| \prod_{\mathfrak{p}} \mathfrak{N}(\mathfrak{p})^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})/\kappa_{\mathfrak{p}}},$$

where $\mathfrak{N}(\mathfrak{p}) = \sharp(\mathcal{O}/\mathfrak{p})$ is the absolute norm and $\kappa_{\mathfrak{p}}$ the capacity of the prime ideal $\mathfrak{p}$ of $\mathcal{O}$, and where $N_{A_{\mathbb{R}}|\mathbb{R}}$ denotes the norm map from $A_{\mathbb{R}}$ to $\mathbb{R}$. For a full left $\mathcal{O}$-ideal $\mathfrak{a}$ in A, we set $\mathfrak{N}(\mathfrak{a}) = \mathfrak{N}((\mathfrak{a}, 1))$. A way to compute this number is given in

Theorem 1.2.2. *If $\mathfrak{a}$ is a full left $\mathcal{O}$-ideal in A, then*

$$\mathfrak{N}(\mathfrak{a}) \overset{(1)}{=} \sharp(\mathcal{O}/\mathfrak{a}r)\sharp(\mathcal{O}/\mathcal{O}r)^{-1} \overset{(2)}{=} \sharp(\mathcal{O}/\mathfrak{a}r)\left|N_{K|\mathbb{Q}}(r)\right|^{-1},$$

where $r \in R$ is any nonzero element such that $\mathfrak{a}r \subset \mathcal{O}$.

Proof. Since $\mathcal{O}$ is a full R-lattice in A, there exists a nonzero $r \in R$ such that $\mathfrak{a}r \subset \mathcal{O}$. By definition, $\mathrm{ord}_\mathfrak{p}(\mathfrak{a}) = \mathrm{ord}_\mathfrak{p}(\mathfrak{a}r) - \mathrm{ord}_\mathfrak{p}(r)$ which implies

$$\begin{aligned}\mathfrak{N}(\mathfrak{a}) &= \prod_\mathfrak{p} \mathfrak{N}(\mathfrak{p})^{\mathrm{ord}_\mathfrak{p}(\mathfrak{a})/\kappa_\mathfrak{p}} \\ &= \prod_\mathfrak{p} \mathfrak{N}(\mathfrak{p})^{(\mathrm{ord}_\mathfrak{p}(\mathfrak{a}r)-\mathrm{ord}_\mathfrak{p}(r))/\kappa_p} \\ &= \prod_\mathfrak{p} \mathfrak{N}(\mathfrak{p})^{\mathrm{ord}_\mathfrak{p}(\mathfrak{a}r)/\kappa_\mathfrak{p}} \left(\prod_\mathfrak{p} \mathfrak{N}(\mathfrak{p})^{\mathrm{ord}_\mathfrak{p}(r)/\kappa_p}\right)^{-1}.\end{aligned}$$

Hence, if the formula $\mathfrak{N}(\mathfrak{b}) = \sharp(\mathcal{O}/\mathfrak{b})$ is proven for any ideal $\mathfrak{b}$ in $\mathcal{O}$, then equation (1) is established.

So let us assume that $\mathfrak{b}$ is an ideal in $\mathcal{O}$, and let S_i be the $\mathcal{O}$-composition factors in the Jordan-Hölder decomposition series of $\mathcal{O}/\mathfrak{b}$. As simple left $\mathcal{O}$-modules they are of the form $S_i \cong \mathcal{O}/\mathfrak{m}_i$ with $\mathfrak{m}_i$ a maximal left ideal of $\mathcal{O}$. If we let κ_i denote the capacity of the prime ideal $\mathfrak{p}_i = \mathrm{ann}_\mathcal{O}(S_i)$, then $\mathcal{O}/\mathfrak{m}_i$ is a minimal left ideal of $\mathcal{O}/\mathfrak{p}_i$, and the simple ring $\mathcal{O}/\mathfrak{p}_i$ is isomorphic to the ring of $\kappa_i \times \kappa_i$-matrices over the (skew)field $\mathrm{End}_{\mathcal{O}/\mathfrak{p}_i}(\mathcal{O}/\mathfrak{m}_i)$. Therefore $\mathcal{O}/\mathfrak{p}_i \cong (\mathcal{O}/\mathfrak{m}_i)^{\kappa_i}$, whence

$$\mathfrak{N}(\mathfrak{b}) = \prod_\mathfrak{p} \mathfrak{N}(\mathfrak{p})^{\mathrm{ord}_\mathfrak{p}(\mathfrak{b})/\kappa_\mathfrak{p}} = \prod_i \sharp(\mathcal{O}/\mathfrak{m}_i) = \sharp(\mathcal{O}/\mathfrak{b}).$$

To prove equation (2), it remains to show $|N_{K|\mathbb{Q}}(r)| = \sharp(\mathcal{O}/\mathcal{O}r)$. Firstly, $N_{A|\mathbb{Q}} = N_{K|\mathbb{Q}} \circ N_{A|K}$, thus $N_{K|\mathbb{Q}}(x) = N_{A|\mathbb{Q}}(x)$ for every $x \in K$. Secondly, $N_{A|\mathbb{Q}}(x) = \det_\mathbb{Q}(\rho_x)$, where $\rho_x : A \to A$ is right multiplication by x. Since $\mathcal{O}$ is a full $\mathbb{Z}$-lattice in A and $r \in \mathcal{O}$, we have $\det_\mathbb{Q}(\rho_r) = \det_\mathbb{Z}(\rho'_r)$, where $\rho'_r : \mathcal{O} \to \mathcal{O}$ is the restriction of ρ_r to $\mathcal{O}$. But it is a basic fact in the theory of finitely generated $\mathbb{Z}$-modules that $\det_\mathbb{Z}(\rho'_r) = \sharp(\mathcal{O}/\mathcal{O}r)$. □

As a corollary, we obtain a noncommutative analogue to the well-known product formula $\prod_v |x|_v = 1$, which holds for any nonzero element x in a number field. In our context the product formula reads:

Corollary 1.2.3. *Every $u \in A^\times$ satisfies*

$$\prod_\mathfrak{p} \mathfrak{N}(\mathfrak{p})^{\mathrm{ord}_\mathfrak{p}(u)/\kappa_\mathfrak{p}} = |N_{A|\mathbb{Q}}(u)|.$$

Proof. For $u \in A^\times$, we may write $u = r^{-1}x$ with $r \in R$ and $x \in \mathcal{O}$. By definition, $\mathrm{ord}_\mathfrak{p}(u) = \mathrm{ord}_\mathfrak{p}(\mathcal{O}u)$, hence applying Theorem 1.2.2 yields

$$\prod_\mathfrak{p} \mathfrak{N}(\mathfrak{p})^{\mathrm{ord}_\mathfrak{p}(u)/\kappa_\mathfrak{p}} = \prod_\mathfrak{p} \mathfrak{N}(\mathfrak{p})^{\mathrm{ord}_\mathfrak{p}(\mathcal{O}u)/\kappa_\mathfrak{p}} = \sharp(\mathcal{O}/\mathcal{O}ur)|N_{K|\mathbb{Q}}(r)|^{-1}.$$

On the other hand we have seen in the last paragraph of the proof of Theorem 1.2.2 that $\sharp(\mathcal{O}/\mathcal{O}ur) = |N_{A|\mathbb{Q}}(ur)|$. Since the norm is multiplicative and $N_{K|\mathbb{Q}}(r) = N_{A|\mathbb{Q}}(r)$, the corollary follows. □

In the light of the previous results, we see that the absolute norm defines a map

$$\mathfrak{N} : \widehat{Cl}(\mathcal{O}) \to \mathbb{R}_+^*. \tag{1.8}$$

Indeed, if $\bar{\mathfrak{a}}$ and $\bar{\mathfrak{a}}u$ are two representatives of the same complete ideal class, then

$$\begin{aligned}\mathfrak{N}(\bar{\mathfrak{a}}u) &= \left|N_{A_\mathbb{R}|\mathbb{R}}\left((1 \otimes u^{-1})\mathfrak{a}_\infty\right)\right| \mathfrak{N}(\mathfrak{a}u) \\ &= \left|N_{A_\mathbb{R}|\mathbb{R}}\left(1 \otimes u^{-1}\right)\right| \left|N_{A_\mathbb{R}|\mathbb{R}}(\mathfrak{a}_\infty)\right| \left|N_{A|\mathbb{Q}}(u)\right| \mathfrak{N}(\mathfrak{a}).\end{aligned}$$

Since $N_{A_\mathbb{R}|\mathbb{R}}(1 \otimes u^{-1}) = N_{A|\mathbb{Q}}(u^{-1}) = N_{A|\mathbb{Q}}(u)^{-1}$, cf. [Re, Ex. 1.2], it follows $\mathfrak{N}(\bar{\mathfrak{a}}u) = \mathfrak{N}(\bar{\mathfrak{a}})$, which shows that the map in (1.8) is well-defined.

Degree

There is a homomorphism $\deg : \widehat{Div}(\mathcal{O}) \to \mathbb{R}$ defined by

$$\left(\textstyle\sum_\mathfrak{p} v_\mathfrak{p}\mathfrak{p}, D_\infty\right) \mapsto \deg\left(\textstyle\sum_\mathfrak{p} v_\mathfrak{p}\mathfrak{p}, D_\infty\right) = \sum_\mathfrak{p} \tfrac{v_\mathfrak{p}}{\kappa_\mathfrak{p}} \log \mathfrak{N}(\mathfrak{p}) - \log\left|N_{A_\mathbb{R}|\mathbb{R}}(D_\infty)\right|.$$

The product formula, Corollary 1.2.3, ensures that an arithmetic principal divisor $\widehat{\mathrm{div}}(u) \in \widehat{\mathcal{P}}(\mathcal{O})$ satisfies

$$\deg(\widehat{\mathrm{div}}(u)) = \sum_\mathfrak{p} \tfrac{\mathrm{ord}_\mathfrak{p}(u)}{\kappa_\mathfrak{p}} \log \mathfrak{N}(\mathfrak{p}) - \log\left|N_{A|\mathbb{Q}}(u)\right| = 0.$$

Therefore deg induces a map

$$\deg : \widehat{CH}(\mathcal{O}) \to \mathbb{R}.$$

Putting all together, we finally obtain the commutative diagram

$$\begin{array}{ccc} \widehat{Cl}(\mathcal{O}) & \xrightarrow{\mathfrak{N}} & \mathbb{R}_+^* \\ {\scriptstyle \widehat{\mathrm{div}}}\big\downarrow & & \big\downarrow{\scriptstyle -\log} \\ \widehat{CH}(\mathcal{O}) & \xrightarrow{\deg} & \mathbb{R}. \end{array} \tag{1.9}$$

This commutative diagram generalises the one valid for orders in number fields, cf. [N, p. 192].

1.2.4 A Riemann-Roch formula

If A is a finite dimensional semisimple K-algebra, then $A_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} A$ is a finite dimensional semisimple $\mathbb{R}$-algebra, cf. [Re, (7.18)], hence it follows from [Re, (9.26)] that the reduced trace map $\mathrm{tr}_{A_{\mathbb{R}}|\mathbb{R}}$ from $A_{\mathbb{R}}$ onto $\mathbb{R}$ gives rise to the symmetric nondegenerate $\mathbb{R}$-bilinear form

$$A_{\mathbb{R}} \times A_{\mathbb{R}} \to \mathbb{R},\ (x, y) \mapsto \mathrm{tr}(xy).$$

We call this inner product the reduced trace form on $A_{\mathbb{R}}$. It provides a Haar measure on the real vector space $A_{\mathbb{R}}$, which we call the canonical measure on $A_{\mathbb{R}}$. We write $\mathrm{vol}(X)$ for the volume of a subset $X \subset A_{\mathbb{R}}$ with respect to the canonical measure on $A_{\mathbb{R}}$. For a full $\mathbb{Z}$-lattice L in $A_{\mathbb{R}}$, we put

$$\mathrm{vol}(L) = \mathrm{vol}(\Phi(L)),$$

where $\Phi(L)$ is a fundamental domain of L.

Consider now an R-order $\mathcal{O}$ in A. Every full left $\mathcal{O}$-ideal $\mathfrak{a}$ in A is mapped by the embedding $j : A \to A_{\mathbb{R}}$, $a \mapsto 1 \otimes a$, onto a full $\mathbb{Z}$-lattice $j(\mathfrak{a})$ in $A_{\mathbb{R}}$. We define

$$\mathrm{vol}(\mathfrak{a}) = \mathrm{vol}(j(\mathfrak{a})).$$

If $\overline{\mathfrak{a}} = (\mathfrak{a}, \mathfrak{a}_\infty)$ is a complete $\mathcal{O}$-ideal then $\mathfrak{a}_\infty \in (A_{\mathbb{R}})^\times$, thus $\mathfrak{a}_\infty \cdot j(\mathfrak{a})$ is a full $\mathbb{Z}$-lattice in $A_{\mathbb{R}}$. We set

$$\mathrm{vol}(\overline{\mathfrak{a}}) = \mathrm{vol}\left(\mathfrak{a}_\infty \cdot j(\mathfrak{a})\right)$$

and call the real number

$$\chi(\overline{\mathfrak{a}}) = -\log \mathrm{vol}(\overline{\mathfrak{a}})$$

the *Euler-Minkowski characteristic* of $\overline{\mathfrak{a}}$. Now we are ready to state a Riemann-Roch formula for noncommutative arithmetic curves:

Theorem 1.2.4. *Let* $\operatorname{Spec}\mathcal{O}$ *be a noncommutative arithmetic curve. Then every complete* $\mathcal{O}$*-ideal* $\overline{\mathfrak{a}}$ *satisfies the Riemann-Roch formula*

$$\chi(\overline{\mathfrak{a}}) = \deg\left(\widehat{\operatorname{div}}(\overline{\mathfrak{a}})\right) + \chi(\overline{\mathcal{O}}),$$

where $\overline{\mathcal{O}} = (\mathcal{O}, 1)$.

Proof. We already know that $\deg \circ \widehat{\operatorname{div}}$ is constant on complete ideal classes, and we claim that the same holds for the Euler-Minkowski characteristic. Indeed, if $\overline{\mathfrak{a}}u$ is another representative of the complete ideal class of $\overline{\mathfrak{a}} = (\mathfrak{a}, \mathfrak{a}_\infty)$, then

$$\operatorname{vol}(\overline{\mathfrak{a}}u) = \left|N_{A_{\mathbb{R}}|\mathbb{R}}\left((1 \otimes u^{-1})\mathfrak{a}_\infty\right)\right| \operatorname{vol}(\mathfrak{a}u) = \left|N_{A_{\mathbb{R}}|\mathbb{R}}(\mathfrak{a}_\infty)\right| \operatorname{vol}(\mathfrak{a}) = \operatorname{vol}(\overline{\mathfrak{a}}).$$

Therefore we can assume that $\mathfrak{a} \subset \mathcal{O}$. But then, $\operatorname{vol}(\mathfrak{a}) = \operatorname{vol}(\mathcal{O})\sharp(\mathcal{O}/\mathfrak{a})$, and applying Theorem 1.2.2 yields

$$\operatorname{vol}(\overline{\mathfrak{a}}) = \left|N_{A_{\mathbb{R}}|\mathbb{R}}(\mathfrak{a}_\infty)\right| \operatorname{vol}(\mathfrak{a}) = \left|N_{A_{\mathbb{R}}|\mathbb{R}}(\mathfrak{a}_\infty)\right| \operatorname{vol}(\mathcal{O})\mathfrak{N}(\mathfrak{a}) = \mathfrak{N}(\overline{\mathfrak{a}}) \operatorname{vol}(\mathcal{O}).$$

Combining this with the commutative diagram (1.9) or rather its consequence $\deg \circ \widehat{\operatorname{div}}(\overline{\mathfrak{a}}) = -\log \mathfrak{N}(\overline{\mathfrak{a}})$, establishes our Riemann-Roch formula. □

The Riemann-Roch formula finishes our study of complete $\mathcal{O}$-ideals. In the next section we will see how complete $\mathcal{O}$-ideals are embedded in the more general theory of arithmetic vector bundles on noncommutative arithmetic curves.

1.3 Arithmetic vector bundles

Let $*$ be an involution on a finite dimensional semisimple real algebra B. A $*$-*hermitian metric* on a B-module M is a $*$-hermitian form $h : M \times M \to B$ such that $\operatorname{tr}_{B|\mathbb{R}} \circ h$ is positive definite. Here $*$-hermitian means that h is B-linear in the first argument and for all $x, y \in M$, $h(x, y) = h(y, x)^*$. The involution $*$ is called *positive* if the twisted trace form $\operatorname{tr}_{B|\mathbb{R}}(xy^*)$ is positive definite. With the help of Wedderburn's Structure Theorem, it is easy to see that every finite dimensional semisimple real algebra admits a positive involution. For more details we refer to [BL, Sect. 5.5]. However we have the following:

Lemma 1.3.1. *Let B be a finite dimensional semisimple real algebra, and let $*$ be a positive involution on B. Then every $*$-hermitian metric h on B is of the form*

$$h(x, y) = x\beta\beta^* y^*, \ x, y \in B,$$

for some $\beta \in B^\times$.

Proof. It is obvious that such an h is $*$-hermitian. Since β is a unit, it is not a zero divisor, thus $x\beta \neq 0$ whenever $x \neq 0$. But $h(x, x) = x\beta(x\beta)^*$ and $*$ is positive, therefore $\operatorname{tr} \circ h$ is positive definite.

On the other hand, if h is a $*$-hermitian metric on B, then $h(x, y) = xh(1, 1)y^*$ and $h(1, 1)^* = h(1, 1)$. The positive definiteness of $\operatorname{tr} \circ h$ ensures that $h(1, 1) = \beta\beta^*$ for some $\beta \in B^\times$. □

Since every B-module is projective it follows directly from the lemma that every finitely generated B-module admits a $*$-hermitian metric.

Let us return to the study of noncommutative arithmetic curves $\operatorname{Spec} \mathcal{O}$. Recall that $\mathcal{O}$ is an R-order in a finite dimensional semisimple K-algebra A. The associated real algebra $A_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Q} A$ is also finite dimensional and semisimple [Re, (7.18)]. For the rest of the chapter, we fix a positive involution $*$ on $A_\mathbb{R}$ and we simply write hermitian instead of $*$-hermitian.

Definition 1.3.2. *An* arithmetic vector bundle *on* $\operatorname{Spec} \mathcal{O}$ *is a pair* $\overline{E} = (E, h)$, *where E is a left $\mathcal{O}$-lattice such that $A \otimes_\mathcal{O} E$ is a free A-module, and where $h : E_\mathbb{R} \times E_\mathbb{R} \to A_\mathbb{R}$ is a hermitian metric on the left $A_\mathbb{R}$-module $E_\mathbb{R} = A_\mathbb{R} \otimes_\mathcal{O} E$.*

If $A \otimes_\mathcal{O} E \cong A$ then $\overline{E}$ is called an arithmetic line bundle *on* $\operatorname{Spec} \mathcal{O}$.

Two arithmetic vector bundles $\overline{E} = (E, h)$ and $\overline{E}' = (E', h')$ on $\operatorname{Spec} \mathcal{O}$ are called *isomorphic* if there exists an isomorphism $\phi : E \to E'$ of left $\mathcal{O}$-modules which induces an isometry $\phi_\mathbb{R} : E_\mathbb{R} \to E'_\mathbb{R}$, i.e. $h'(\phi_\mathbb{R}(x), \phi_\mathbb{R}(y)) = h(x, y)$ for all $x, y \in E_\mathbb{R}$.

By a short exact sequence

$$0 \to \overline{E}' \to \overline{E} \to \overline{E}'' \to 0$$

of arithmetic vector bundles we understand a short exact sequence

$$0 \longrightarrow E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \longrightarrow 0$$

of the underlying left $\mathcal{O}$-modules which splits isometrically, that is, in the sequence

$$0 \longrightarrow E'_{\mathbb{R}} \xrightarrow{\alpha_{\mathbb{R}}} E_{\mathbb{R}} \xrightarrow{\beta_{\mathbb{R}}} E''_{\mathbb{R}} \longrightarrow 0,$$

$E'_{\mathbb{R}}$ is mapped isometrically onto $\alpha_{\mathbb{R}}(E'_{\mathbb{R}})$, and the orthogonal complement $(\alpha_{\mathbb{R}}(E'_{\mathbb{R}}))^{\perp}$ is mapped isometrically onto $E''_{\mathbb{R}}$.

1.3.1 Grothendieck group

We let

$$\widehat{F}_0(\mathcal{O}) = \bigoplus_{\{\overline{E}\}} \mathbb{Z}\{\overline{E}\}$$

be the free abelian group over the isomorphism classes $\{\overline{E}\}$ of arithmetic vector bundles on $\operatorname{Spec}\mathcal{O}$. In this group we consider the subgroup $\widehat{R}_0(\mathcal{O})$ generated by all elements $\{\overline{E}'\} + \{\overline{E}''\} - \{\overline{E}\}$, which arise in a short exact sequence

$$0 \to \overline{E}' \to \overline{E} \to \overline{E}'' \to 0$$

of arithmetic vector bundles on $\operatorname{Spec}\mathcal{O}$. The factor group

$$\widehat{K}_0(\mathcal{O}) = \widehat{F}_0(\mathcal{O})/\widehat{R}_0(\mathcal{O})$$

is called the *Grothendieck group* of arithmetic vector bundles on $\operatorname{Spec}\mathcal{O}$.

The connection to the Riemann-Roch theory of the last section is given as follows. Every complete $\mathcal{O}$-ideal $\overline{\mathfrak{a}} = (\mathfrak{a}, \mathfrak{a}_\infty)$ gives rise to the particular hermitian metric

$$h_{\overline{\mathfrak{a}}}(x, y) = x\mathfrak{a}_\infty(\mathfrak{a}_\infty)^* y^*, \ x, y \in A_{\mathbb{R}},$$

on $\mathfrak{a}_{\mathbb{R}} = A_{\mathbb{R}} \otimes_{\mathcal{O}} \mathfrak{a} = A_{\mathbb{R}}$. We thus obtain the arithmetic line bundle $(\mathfrak{a}, h_{\overline{\mathfrak{a}}})$ on $\operatorname{Spec}\mathcal{O}$ which we denote by $L(\overline{\mathfrak{a}})$.

Theorem 1.3.3. *Let* $\operatorname{Spec}\mathcal{O}$ *be a noncommutative arithmetic curve. Then there is a well-defined mapping*

$$\widehat{Cl}(\mathcal{O}) \to \widehat{K}_0(\mathcal{O}), \ [\overline{\mathfrak{a}}] \mapsto [L(\overline{\mathfrak{a}})].$$

Moreover the elements $[L(\overline{\mathfrak{a}})]$ *generate the Grothendieck group* $\widehat{K}_0(\mathcal{O})$.

Proof. The application $[\overline{\mathfrak{a}}] \mapsto [L(\overline{\mathfrak{a}})]$ is independent of the choice of the representative $\overline{\mathfrak{a}}$ of $[\overline{\mathfrak{a}}] \in \widehat{Cl}(\mathcal{O})$. Indeed, if $\overline{\mathfrak{b}} = \overline{\mathfrak{a}}u$, $u \in A^\times$, is another element in the complete ideal class of $\overline{\mathfrak{a}}$, then for all $x, y \in A_\mathbb{R}$ we have

$$\begin{aligned} h_{\overline{\mathfrak{b}}}(x(1\otimes u), y(1\otimes u)) &= x(1\otimes u)\left((1\otimes u^{-1})\mathfrak{a}_\infty\right)\left((1\otimes u^{-1})\mathfrak{a}_\infty\right)^*(y(1\otimes u))^* \\ &= x\mathfrak{a}_\infty(\mathfrak{a}_\infty)^* y^* \\ &= h_{\overline{\mathfrak{a}}}(x,y). \end{aligned}$$

Hence the two line bundles $L(\overline{\mathfrak{a}})$ and $L(\overline{\mathfrak{b}})$ are isomorphic and in particular $[L(\overline{\mathfrak{a}})] = [L(\overline{\mathfrak{b}})]$.

It remains to prove that the elements $[L(\overline{\mathfrak{a}})]$ generate the Grothendieck group $\widehat{K}_0(\mathcal{O})$. Let $[\overline{E}] \in \widehat{K}_0(\mathcal{O})$. By definition, $A \otimes_\mathcal{O} E$ is a free A-module of finite rank n say. If $(b_1, \ldots, b_n)$ is a basis of $A \otimes_\mathcal{O} E$, then for every $x \in E$, the element $1 \otimes x \in A \otimes_\mathcal{O} E$ is uniquely expressible in the form $1 \otimes x = a_1 b_1 + \cdots + a_n b_n$ with $a_i \in A$. Let $\mathfrak{a}_i$ be the set of all coefficients a_i which occur as x ranges over all elements of E. This is a finitely generated left $\mathcal{O}$-submodule of A. Since

$$Ab_1 + \cdots + Ab_n = A \otimes_\mathcal{O} E = A(\mathfrak{a}_1 b_1 + \cdots + \mathfrak{a}_n b_n) = A\mathfrak{a}_1 b_1 + \cdots + A\mathfrak{a}_n b_n,$$

it follows that $A\mathfrak{a}_i = A$ for every i, so each $\mathfrak{a}_i$ is a full left $\mathcal{O}$-ideal in A.

We write $F = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_{n-1}$ and consider the short exact sequence

$$0 \to F \to E \to \mathfrak{a}_n \to 0$$

of $\mathcal{O}$-modules. This sequence becomes an exact sequence of arithmetic vector bundles on $\operatorname{Spec}\mathcal{O}$, if we restrict the metric on $E_\mathbb{R}$ to $F_\mathbb{R}$, and if we endow $\mathfrak{a}_\mathbb{R} = A_\mathbb{R}$ with the metric, which is induced by the isomorphism $F_\mathbb{R}^\perp \cong A_\mathbb{R}$. But Lemma 1.3.1 tells us that every hermitian metric on $A_\mathbb{R}$ is of the form $h(x,y) = x\alpha\alpha^* y^*$ for some $\alpha \in (A_\mathbb{R})^\times$. Hence $[\overline{E}] = [\overline{F}] + [L((\mathfrak{a}, \alpha))]$. By induction on the rank, we get the desired decomposition of $[\overline{E}]$. □

1.3.2 Arithmetic degree

On the one hand we have the homomorphism $\mathrm{rk} : \widehat{K}_0(\mathcal{O}) \to \mathbb{Z}$ defined by $\mathrm{rk}([\overline{E}]) = \mathrm{rk}_A(A \otimes_\mathcal{O} E)$; on the other hand the inclusion $i : R \hookrightarrow \mathcal{O}$ induces a group homomorphism

$$i_* : \widehat{K}_0(\mathcal{O}) \to \widehat{K}_0(R),\ [(E,h)] \mapsto \left[\left(E, \mathrm{tr}_{A_\mathbb{R}|K_\mathbb{R}} \circ h\right)\right].$$

Both are used in

Theorem 1.3.4. *Let* $\operatorname{Spec}\mathcal{O}$ *be a noncommutative arithmetic curve. There is a unique homomorphism*

$$\widehat{\deg}_{\mathcal{O}} : \widehat{K}_0(\mathcal{O}) \to \mathbb{R}$$

which extends the map $\deg \circ \widehat{\operatorname{div}} : \widehat{Cl}(\mathcal{O}) \to \mathbb{R}$, *that is, which satisfies* $\widehat{\deg}_{\mathcal{O}}([L(\overline{\mathfrak{a}})]) = \deg(\widehat{\operatorname{div}}[\overline{\mathfrak{a}}])$ *for all* $[\overline{\mathfrak{a}}] \in \widehat{Cl}(\mathcal{O})$. *This homomorphism is given by*

$$\widehat{\deg}_{\mathcal{O}}([\overline{E}]) = \widehat{\deg}_K(i_*[\overline{E}]) - \operatorname{rk}([\overline{E}])\,\widehat{\deg}_K(i_*[\overline{\mathcal{O}}]) \qquad (1.10)$$

and is called the arithmetic degree map.

Furthermore, if $K = \mathbb{Q}$ *then* $\widehat{\deg}_K(i_*[\overline{\mathcal{O}}]) = \chi(\overline{\mathcal{O}})$.

The theorem states that $\widehat{\deg}_{\mathcal{O}}$ behaves exactly like the arithmetic degree map over number fields. Specifically, equation (1.10) generalises the Riemann-Roch formula for hermitian vector bundles on commutative arithmetic curves, cf. [N, (III.8.2)]. To prove the theorem, we need a lemma from the theory of commutative arithmetic curves, which certainly is well-known. But as we could not find a proof of it in the literature, we prove it here.

Lemma 1.3.5. *Let* $\overline{E} = (E, h)$ *be a hermitian vector bundle on a commutative arithmetic curve* $\operatorname{Spec} R$, *and let* $\phi, \psi : E_{\mathbb{R}} \to E_{\mathbb{R}}$ *be two endomorphisms. Then*

$$\widehat{\deg}\,(E, h \circ (\phi \oplus \psi)) = \widehat{\deg}(\overline{E}) - \tfrac{1}{2}\log\left|\det_{\mathbb{R}}(\phi)\right| - \tfrac{1}{2}\log\left|\det_{\mathbb{R}}(\psi)\right|. \quad (1.11)$$

Proof. Following Neukirch [N, III.7], we let $i_*\overline{E}$ denote the hermitian vector bundle on $\operatorname{Spec}\mathbb{Z}$ obtained by push-forward the bundle $\overline{E}$. The Riemann-Roch formula for hermitian vector bundles on arithmetic curves, cf. [N, (III.8.2)], asserts

$$\widehat{\deg}(\overline{E}) = \widehat{\deg}(i_*\overline{E}) - \operatorname{rk}(E)\,\widehat{\deg}(i_*\overline{R}). \qquad (1.12)$$

We recall

$$\widehat{\deg}(i_*\overline{E}) = -\tfrac{1}{2}\log\left|\det\left[\operatorname{tr}_{K_{\mathbb{R}}|\mathbb{R}} \circ h(x_i, x_j)\right]_{1\le i,j\le r}\right|, \qquad (1.13)$$

where $x_1, \ldots, x_r$ is any $\mathbb{Z}$-basis of E. On the other hand it is an easy exercise in linear algebra to show

$$\det\left[\operatorname{tr}_{K_{\mathbb{R}}|\mathbb{R}} \circ h\left(\phi(x_i), \psi(x_j)\right)\right] = \det(\phi)\det(\psi)\det\left[\operatorname{tr}_{K_{\mathbb{R}}|\mathbb{R}} \circ h(x_i, x_j)\right],$$

which in combination with (1.13) yields

$$\widehat{\deg}\left(i_*(E, h\circ(\phi\oplus\psi))\right) = -\tfrac{1}{2}\log|\det(\phi)| - \tfrac{1}{2}\log|\det(\psi)| + \widehat{\deg}\left(i_*\overline{E}\right).$$

Putting this into the Riemann-Roch formula (1.12) establishes (1.11). □

Proof of Theorem 1.3.4. By Theorem 1.3.3, the Grothendieck group $\widehat{K}_0(\mathcal{O})$ is generated by the elements $[L(\overline{\mathfrak{a}})]$, $[\overline{\mathfrak{a}}] \in \widehat{Cl}(\mathcal{O})$, hence a homomorphism $\widehat{K}_0(\mathcal{O}) \to \mathbb{R}$ for which the restriction to $\widehat{Cl}(\mathcal{O})$ coincide with the map $\deg\circ\widehat{\mathrm{div}}$, is uniquely determined.

Both, $\widehat{\deg}_K \circ i_*$ and rk are homomorphisms from $\widehat{K}_0(\mathcal{O})$ to $\mathbb{R}$, therefore their sum is a homomorphism as well. It thus remains to show that all $[\overline{\mathfrak{a}}] \in \widehat{Cl}(\mathcal{O})$ satisfy

$$\widehat{\deg}_K\left(i_*\left[L(\overline{\mathfrak{a}})\right]\right) - \widehat{\deg}_K\left(i_*\mathcal{O}\right) = \deg\left(\widehat{\mathrm{div}}\left[\overline{\mathfrak{a}}\right]\right) \tag{1.14}$$

For every nonzero $r \in R$, $[L(\overline{\mathfrak{a}}r)] = [L(\overline{\mathfrak{a}})]$, so we may assume $\mathfrak{a} \subset \mathcal{O}$. Then

$$\begin{aligned} \widehat{\deg}_K\left(i_*\left[L(\overline{\mathfrak{a}})\right]\right) &= \widehat{\deg}_K\left(\left[\mathfrak{a}, \mathrm{tr}_{A_\mathbb{R}|K_\mathbb{R}} \circ h_{\overline{\mathfrak{a}}}\right]\right) \\ &= -\log\sharp(\mathcal{O}/\mathfrak{a}) + \widehat{\deg}_K\left(\mathcal{O}, \mathrm{tr}_{A_\mathbb{R}|K_\mathbb{R}} \circ h_{\overline{\mathfrak{a}}}\right) \\ &= -\log\mathfrak{N}(\mathfrak{a}) - \log\left|N_{A_\mathbb{R}|\mathbb{R}}(\mathfrak{a}_\infty)\right| + \widehat{\deg}_K\left(i_*\overline{\mathcal{O}}\right). \end{aligned} \tag{1.15}$$

The last equality follows from Theorem 1.2.2 and Lemma 1.3.5 using that $N_{A_\mathbb{R}|\mathbb{R}}(\mathfrak{a}_\infty) = \det\left(x \mapsto x\mathfrak{a}_\infty\right)$. On the other hand the commutative diagram (1.9) tells us

$$\deg\left(\widehat{\mathrm{div}}\left[\overline{\mathfrak{a}}\right]\right) = -\log\mathfrak{N}(\overline{\mathfrak{a}}) = -\log\left|N_{A_\mathbb{R}|\mathbb{R}}(\mathfrak{a}_\infty)\mathfrak{N}(\mathfrak{a})\right|. \tag{1.16}$$

Combining (1.15) and (1.16) yields (1.14).

The formula $\widehat{\deg}_\mathbb{Q}\left(i_*[\overline{\mathcal{O}}]\right) = \chi(\overline{\mathcal{O}})$ follows from Lemma 1.3.5 and the fact $|\det(*)| = 1$. □

There is also another possibility to compute the arithmetic degree of an arithmetic vector bundle:

Proposition 1.3.6. *Let $\overline{E} = (E, h)$ be an arithmetic vector bundle on* $\mathrm{Spec}\,\mathcal{O}$ *of rank n, and let $F \subset E$ be the free $\mathcal{O}$-submodule generated by $\mathcal{O}$-linearly independent elements $x_1, \ldots, x_n$ of E. Then*

$$\widehat{\deg}_\mathcal{O}(\overline{E}) = \log\sharp(E/F) - \tfrac{1}{2}\log\left|\det\left[h(x_i, x_j)\right]_{1\le i,j\le n}\right|,$$

where by det *of a matrix in* $M_n(A_{\mathbb{R}})$ *we mean the determinant of the induced endomorphism of* $A_{\mathbb{R}}^n$ *when considered as a real vector space.*

Proof. First we consider the special case where the hermitian metric h' on $F_{\mathbb{R}} = E_{\mathbb{R}}$ is such that the basis $x_1, \ldots, x_n$ is orthonormal. If we let $\overline{E}' = (E, h')$ and $\overline{F}' = (F, h')$ then $[\overline{F}'] = n[\overline{\mathcal{O}}]$ in $\widehat{K}_0(\mathcal{O})$, hence $\widehat{\deg}_{\mathcal{O}}(\overline{F}') = n\,\widehat{\deg}_{\mathcal{O}}(\overline{\mathcal{O}}) = 0$, which leads to

$$\begin{aligned}\widehat{\deg}_{\mathcal{O}}(\overline{E}') &= \widehat{\deg}_K(i_*\overline{E}') - n\,\widehat{\deg}_K(i_*\overline{\mathcal{O}})\\ &= \log \sharp(E/F) + \widehat{\deg}_K(i_*\overline{F}') - n\,\widehat{\deg}_K(i_*\overline{\mathcal{O}})\\ &= \log \sharp(E/F).\end{aligned}$$

This proves the special case because $[h'(x_i, x_j)]_{1 \le i,j \le n}$ is the identity matrix.

To deduce the general case from the special case, we use the fact that for every hermitian metric h on $E_{\mathbb{R}}$ there exists an automorphism ϕ of $E_{\mathbb{R}}$ such that $h(x, y) = h'(x, \phi(y))$. Combining this with Lemma 1.3.5 establishes the proposition. □

We have seen in the proof of Theorem 1.3.3 that for every arithmetic vector bundle $\overline{E}$ of rank n and for every A-basis $(b_1, \ldots, b_n)$ of $A \otimes_{\mathcal{O}} E$, we find full left $\mathcal{O}$-ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ in A such that $1 \otimes E = \mathfrak{a}_1 b_1 + \cdots + \mathfrak{a}_n b_n$. This is used in

Corollary 1.3.7. *Let* $\overline{E} = (E, h)$ *be an arithmetic vector bundle on* $\operatorname{Spec}\mathcal{O}$ *of rank* n. *If* $1 \otimes E = \mathfrak{a}_1 b_1 + \cdots + \mathfrak{a}_n b_n$ *then*

$$\widehat{\deg}_{\mathcal{O}}(\overline{E}) = \sum_{k=1}^{n} \log \mathfrak{N}(\mathfrak{a}_k) - \tfrac{1}{2} \log \left|\det\, [h(b_i, b_j)]_{1 \le i,j \le n}\right|.$$

Proof. Since $\mathcal{O}$ is a full R-lattice in A, there exists a nonzero $r \in R$ such that $rb_i \in 1 \otimes E$ for all $1 \le i \le n$. Applying the preceding proposition yields

$$\widehat{\deg}_{\mathcal{O}}(\overline{E}) = \sum_{k=1}^{n} \log \sharp(\mathcal{O}/\mathfrak{a}_k r) - \tfrac{1}{2} \log |\det\, [h(rb_i, rb_j)]|\,.$$

By Theorem 1.2.2, we have $\sharp(\mathcal{O}/\mathfrak{a}_k r) = \mathfrak{N}(\mathfrak{a}_k)\left|N_{K|\mathbb{Q}}(r)\right|^{-1}$. On the other hand $\det\,[h(rb_i, rb_j)] = N_{K|\mathbb{Q}}(r)^{2n} \det\,[h(b_i, b_j)]$. Combining these three equations proves the corollary. □

1.4 Heights and duality

The height is a concept which is strongly related to the arithmetic degree. For number fields this is classical. We briefly resume the discussion in [Sc]. Let K be a number field of degree d over $\mathbb{Q}$ and let $\rho : K \to \mathbb{R}^d$ be the embedding of K in the euclidean space $\mathbb{R}^d$ given by

$$\rho(x) = (\sigma_1(x), \ldots, \sigma_r(x), \operatorname{Re}\tau_1(x), \operatorname{Im}\tau_1(x), \ldots, \operatorname{Re}\tau_s(x), \operatorname{Im}\tau_s(x)),$$

where $\sigma_1, \ldots, \sigma_r : K \to \mathbb{R}$ are the real embeddings and $\tau_1, \overline{\tau}_1, \ldots, \tau_s, \overline{\tau}_s : K \to \mathbb{C}$ the pairs of complex-conjugate embeddings of K in $\mathbb{C}$.

Let $L = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_m$ be a $\mathbb{Z}$-lattice in $\mathbb{R}^n$, and let $\langle\ ,\ \rangle$ denote the standard euclidean scalar product on $\mathbb{R}^n$. The real number $\det(L) = |\det(\langle x_i, x_j\rangle)|^{1/2}$ is called the determinant of L. It is nothing else than the m-dimensional Lebesgue volume of the fundamental domain $\Phi(L) = \{r_1x_1 + \cdots + r_mx_m \mid 0 \le r_i < 1\}$ of L.

Using this notation, Schmidt [Sc, Chap. 3] defines the height of an m-dimensional subspace $V \subset K^n$ as

$$H(V) = \left(2^s|\Delta_K|^{1/2}\right)^{-m} \det\left(\rho^n(V \cap R^n)\right), \tag{1.17}$$

where Δ_K is the discriminant and R the ring of integers of K. He shows that every $x = (x_1, \ldots, x_n) \in K^n$ satisfies the formula

$$H(x) := H(Kx) = \prod_{v \nmid \infty} \max\{|x_1|_v, \ldots, |x_n|_v\} \prod_{v \mid \infty} \left(|x_1|_v^2 + \cdots + |x_n|_v^2\right)^{1/2}.$$

Here $v \nmid \infty$ and $v \mid \infty$ denote the finite and infinite places of K, respectively, and $|\ \ |_v$ the usual valuation of K associated to v.

To describe the relation to the arithmetic degree, we let $K_\mathbb{R}$ denote the Minkowski space of K and $*$ the canonical involution on $K_\mathbb{R}$. Recall that the Minkowski space of K can be identified with the real vector space $K \otimes_\mathbb{Q} \mathbb{R}$. We call the hermitian form

$$h_n : K_\mathbb{R}^n \times K_\mathbb{R}^n \to K_\mathbb{R}^n,\ ((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto x_1y_1^* + \cdots + x_ny_n^*$$

the canonical metric on $K_\mathbb{R}^n$. With its help the height can be expressed as an arithmetic degree:

Theorem 1.4.1. *Let V be a subspace of K^n. Then*

$$\log H(V) = -\widehat{\deg}_K(V \cap R^n, h_n),$$

where h_n is the canonical metric on $K_\mathbb{R}^n$.

Proof. Suppose that the dimension of V is m. Let $\mathrm{tr} = \mathrm{tr}_{K_{\mathbb{R}}|\mathbb{R}}$ be the trace map from $K_{\mathbb{R}}$ onto $\mathbb{R}$. Applying the Riemann-Roch formula for hermitian vector bundles on $\operatorname{Spec} R$, cf.[N, (III.8.2)], together with the equality $\chi(i_*R) = -\frac{1}{2}\log|\Delta_K|$, cf. [N, (I.5.2)], yields

$$\widehat{\deg}_K(V \cap R^n, h_n) = \widehat{\deg}_{\mathbb{Q}}(V \cap R^n, \mathrm{tr} \circ h_n) - \tfrac{m}{2}\log|\Delta_K|\,.$$

Combining this equation with (1.17), we see that it remains to show

$$-\widehat{\deg}_{\mathbb{Q}}(V \cap R^n, \mathrm{tr} \circ h_n) = ms\log 2 + \log\det\left(\rho^n(V \cap R^n)\right). \tag{1.18}$$

Note that $-\widehat{\deg}_{\mathbb{Q}}(V \cap R^n, \mathrm{tr} \circ h_n)$ is nothing else than the logarithm of the volume of the lattice $V \cap R^n$ with respect to the Haar measure on $(V \cap R^n)_{\mathbb{R}}$ induced by the metric $\mathrm{tr} \circ h_n$. But the Haar measure induced by $\mathrm{tr} \circ h_n$ coincide with the product measure of the canonical measure on $K_{\mathbb{R}}$, and it is well-known, cf. [N, (I.5.1)], that the canonical measure on $K_{\mathbb{R}}$ and the pull-back of the Lebesgue measure on $\mathbb{R}^{r+2s}$ via ρ differ by 2^s. This explains why $ms\log 2$ occurs in (1.18) and thus completes the proof. $\square$

Since the arithmetic degree is also available when working with semisimple K-algebras, this observation provides a way to make a definition in a more general setting. Let $\mathcal{O}$ be an R-order in a finite dimensional semisimple K-algebra A. Fix a positive involution $*$ on $A_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} A$ and let h_n be the canonical metric on $A^n_{\mathbb{R}}$, that is

$$h_n(x, y) = x_1 y_1^* + \cdots + x_n y_n^* \quad \text{for all } x, y \in A^n_{\mathbb{R}}.$$

Then the *height* $H_{\mathcal{O}}(V)$ of a free A-submodule $V \subset A^n$ is defined by

$$\log H_{\mathcal{O}}(V) = -\widehat{\deg}_{\mathcal{O}}(V \cap \mathcal{O}^n, h_n).$$

It is important to note that the height is independent of the choice of the positive involution $*$ on $A_{\mathbb{R}}$. Indeed, the $\mathbb{R}$-bilinear form $b_n : A^n_{\mathbb{R}} \times A^n_{\mathbb{R}} \to A_{\mathbb{R}}$ given by $b_n(x, y) = h_n(x, y^*)$ is independent of the involution $*$ and the bilinear form $\mathrm{tr}_{A_{\mathbb{R}}|\mathbb{R}} \circ b_n$ induces the same Haar measure on the real vector space $A^n_{\mathbb{R}}$ as the bilinear form $\mathrm{tr}_{A_{\mathbb{R}}|\mathbb{R}} \circ h_n$ because $|\det(*)| = 1$. The height just defined is a generalisation of the height over finite dimensional rational division algebras introduced by Liebendörfer and Rémond [LR2].

Duality

It is an important property of the classical height that it satisfies duality. In the notation of the previous subsection duality means that for a subspace V of K^n the equality $H(V^\perp) = H(V)$ holds, where $V^\perp \subset K^n$ is the subspace orthogonal to V with respect to the canonical metric h_n restricted to K^n.

Liebendörfer and Rémond [LR1] have established duality for the height $H_{\mathcal{O}}$ under the assumption that $\mathcal{O}$ is a maximal order in a positive definite rational quaternion algebra. They generalised their result in a second paper [LR2] to any division algebra of finite dimension over $\mathbb{Q}$.

To formulate our duality theorem, we have to introduce some notation. As usual we let $\mathcal{O}$ denote a $\mathbb{Z}$-order in a finite dimensional semisimple $\mathbb{Q}$-algebra A. Following Reiner [Re, 25], the inverse different of $\mathcal{O}$ is defined as

$$\widetilde{\mathcal{O}} = \{x \in A \mid \operatorname{tr}_{A|\mathbb{Q}}(x\mathcal{O}) \subset \mathbb{Z}\}.$$

In general $\widetilde{\mathcal{O}}$ is no longer a ring, however it is still a two-sided $\mathcal{O}$-ideal in A such that $\mathcal{O} \subset \widetilde{\mathcal{O}}$. The orthogonal complement $V^\perp$ of a free A-submodule V of A^n is computed with respect to the form $b_n : A^n \times A^n \to A$, $(x, y) \mapsto \sum_{i=1}^n x_i y_i$, that is,

$$V^\perp = \{x \in A^n \mid b_n(x, y) = 0 \text{ for all } y \in V\}.$$

Theorem 1.4.2. *Every free A-submodule V of A^n of rank m satisfies the duality formula*

$$H_{\mathcal{O}}(V^\perp) = H_{\mathcal{O}}(V) \cdot \frac{[\widetilde{\mathcal{O}}^m : \mathcal{O}^m]}{[V \cap \widetilde{\mathcal{O}}^n : V \cap \mathcal{O}^n]}. \tag{1.19}$$

If $\mathcal{O}$ is a maximal *R-order in A, then*

$$H_{\mathcal{O}}(V^\perp) = H_{\mathcal{O}}(V).$$

Proof. Let V' denote the orthogonal complement of $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$ with respect to the $\mathbb{R}$-bilinear form $b := \operatorname{tr} \circ b_n$, and let $(\mathcal{O}^n)^* = \{x \in A_{\mathbb{R}}^n \mid b(x, \mathcal{O}^n) \subset \mathbb{Z}\}$ be the dual lattice of $\mathcal{O}^n$ with respect to b. By [B, Prop. 1(ii)], we have

$$\operatorname{vol}(V' \cap \mathcal{O}^n) = \operatorname{vol}(V_{\mathbb{R}} \cap \mathcal{O}^n) \operatorname{vol}(\mathcal{O}^n)/[V_{\mathbb{R}} \cap (\mathcal{O}^n)^* : V_{\mathbb{R}} \cap \mathcal{O}^n], \tag{1.20}$$

where vol denotes the volume induced by the metric b.

Firstly, we claim

$$V' \cap \mathcal{O}^n = V^\perp \cap \mathcal{O}^n. \tag{1.21}$$

Obviously, $V_{\mathbb{R}}^\perp$ is a subset of V', and since b_n restricted to $V \times V$ is non-degenerate, it follows $V_{\mathbb{R}}^\perp \oplus V_{\mathbb{R}} = A_{\mathbb{R}}^n$. But the same is true for V', i.e. $V' \oplus V_{\mathbb{R}} = A_{\mathbb{R}}^n$, whence $V_{\mathbb{R}}^\perp = V'$ and thus (1.21).

Secondly, we claim

$$[V_{\mathbb{R}} \cap (\mathcal{O}^n)^* : V_{\mathbb{R}} \cap \mathcal{O}^n] = [V \cap \widetilde{\mathcal{O}}^n : V \cap \mathcal{O}^n]. \tag{1.22}$$

It is clear that $\widetilde{\mathcal{O}}^n \subset (\mathcal{O}^n)^*$. Conversely, if $x = (x_1, \ldots, x_n) \in A_{\mathbb{R}}^n$ and $y = (y_1, \ldots, y_n) \in \mathcal{O}^n$ are such that $b(x,y) \in \mathbb{Z}$ but $\mathrm{tr}(x_i y_i) \notin \mathbb{Z}$ for some $i \in \{1, \ldots, n\}$, then $b(x, y') \notin \mathbb{Z}$, for $y' = (0, \ldots, 0, y_i, 0, \ldots, 0) \in \mathcal{O}^n$. Hence $\widetilde{\mathcal{O}}^n = (\mathcal{O}^n)^*$, which implies (1.22).

Thirdly, it follows from Theorem 1.3.4 that every free A-submodule $W \subset A^n$ of rank r satisfies

$$\begin{aligned} \log H_{\mathcal{O}}(W) &= -\widehat{\deg}_{\mathcal{O}}(W \cap \mathcal{O}^n, h_n) \\ &= -\widehat{\deg}_{\mathbb{Q}}(W \cap \mathcal{O}^n, \mathrm{tr}_{A_{\mathbb{R}}|\mathbb{R}} \circ h_n) + r\chi(\mathcal{O}). \end{aligned} \tag{1.23}$$

Moreover,

$$\begin{aligned} -\widehat{\deg}_{\mathbb{Q}}(W \cap \mathcal{O}^n, \mathrm{tr}_{A_{\mathbb{R}}|\mathbb{R}} \circ h_n) &= -\widehat{\deg}_{\mathbb{Q}}(W \cap \mathcal{O}^n, \mathrm{tr}_{A_{\mathbb{R}}|\mathbb{R}} \circ b_n) \\ &= \log \mathrm{vol}(W \cap \mathcal{O}^n), \end{aligned} \tag{1.24}$$

as already used several times. Furthermore it follows from the last equation of Section 1.(a) in [B] that $\log[\widetilde{\mathcal{O}} : \mathcal{O}] = -\chi(\mathcal{O})$. Together with (1.23) and (1.24), we see that the height of a free A-submodule $W \subset A^n$ of rank r may be computed as

$$H_{\mathcal{O}}(W) = \mathrm{vol}(W \cap \mathcal{O}^n)/[\widetilde{\mathcal{O}} : \mathcal{O}]^r. \tag{1.25}$$

Applying (1.25) to $V^\perp$ and V and combining the results with (1.20), (1.21) and (1.22) establishes (1.19).

To prove the second part of the theorem, we need

$$V \cap \widetilde{\mathcal{O}}^n = \widetilde{\mathcal{O}}(V \cap \mathcal{O}^n). \tag{1.26}$$

The inclusion $\supset$ is clear, and it suffices to prove the other inclusion locally. If $\mathcal{O}$ is a maximal $\mathbb{Z}$-order in a semisimple $\mathbb{Q}$-algebra A, it follows from [Re, (18.10)] that for every prime ideal p of $\mathbb{Z}$ there is a unit $u \in A^\times$ such that $\widetilde{\mathcal{O}}_p = u\mathcal{O}_p$. Hence every $v \in V \cap \widetilde{\mathcal{O}}_p^n$ can be written as $v = ux$ with

$x \in \mathcal{O}_p^n$. From this we get $x = u^{-1}v \in V$, therefore $x \in V \cap \mathcal{O}_p^n$, whence $v = ux \in \widetilde{\mathcal{O}}_p(V \cap \mathcal{O}_p^n)$, which establishes (1.26).

We thus have $[V \cap \widetilde{\mathcal{O}}^n : V \cap \mathcal{O}^n] = [\widetilde{\mathcal{O}}(V \cap \mathcal{O}^n) : V \cap \mathcal{O}^n]$. Since $V \cap \mathcal{O}^n$ is a full $\mathcal{O}$-lattice in V, there are full $\mathcal{O}$-ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_m$ in A such that $V \cap \mathcal{O}^n \cong \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_m$. Therefore

$$\begin{aligned}[\widetilde{\mathcal{O}}(V \cap \mathcal{O}^n) : V \cap \mathcal{O}^n] &= [\widetilde{\mathcal{O}}\mathfrak{a}_1 : \mathfrak{a}_1] \cdots [\widetilde{\mathcal{O}}\mathfrak{a}_m : \mathfrak{a}_m] \\ &= \mathfrak{N}(\mathfrak{a}_1)/\mathfrak{N}(\widetilde{\mathcal{O}}\mathfrak{a}_1) \cdots \mathfrak{N}(\mathfrak{a}_m)/\mathfrak{N}(\widetilde{\mathcal{O}}\mathfrak{a}_m).\end{aligned}$$

Finally, in the terminology of [Re, 24], $\widetilde{\mathcal{O}}$ is a two-sided $\mathcal{O}$-ideal in A, that is, its right and left order is $\mathcal{O}$. Hence for each $i \in \{1, \dots, m\}$, $\widetilde{\mathcal{O}}\mathfrak{a}_i$ is a proper product of normal ideals in A. Therefore [Re, (24.5)] implies $\mathfrak{N}(\widetilde{\mathcal{O}}\mathfrak{a}_i) = \mathfrak{N}(\widetilde{\mathcal{O}})\mathfrak{N}(\mathfrak{a}_i)$. Together with the formula $\mathfrak{N}(\widetilde{\mathcal{O}}) = [\widetilde{\mathcal{O}} : \mathcal{O}]^{-1}$, we conclude

$$[V \cap \widetilde{\mathcal{O}}^n : V \cap \mathcal{O}^n] = [\widetilde{\mathcal{O}}(V \cap \mathcal{O}^n) : V \cap \mathcal{O}^n] = [\widetilde{\mathcal{O}}^m : \mathcal{O}^m].$$

Now the second statement of the theorem follows from the first one. □

Corollary 1.4.3. *Every free A-submodule V of A^n satisfies*

$$H_{\mathcal{O}}(V^\perp) = H_{\widetilde{\mathcal{O}}}(V).$$

Proof. Applying (1.21), (1.22) and the index formula $[V \cap \widetilde{\mathcal{O}}^n : V \cap \mathcal{O}^n] = \operatorname{vol}(V \cap \mathcal{O}^n)/\operatorname{vol}(V \cap \widetilde{\mathcal{O}}^n)$, equation (1.20) can be rewritten in the form

$$\operatorname{vol}(V^\perp \cap \mathcal{O}^n) = \operatorname{vol}(\mathcal{O}^n)\operatorname{vol}(V \cap \widetilde{\mathcal{O}}^n).$$

Combining this with (1.25) and the formula $\operatorname{vol}(\widetilde{\mathcal{O}}) = \operatorname{vol}(\mathcal{O})^{-1}$ establishes the corollary. □

1.5 An application: Siegel's Lemma

An application of the height introduced in the last section is a version of Siegel's Lemma over division algebras of finite dimension over $\mathbb{Q}$. In analogy with the classical case, Minkowski's Second Theorem is the main ingredient in the proof of our version of Siegel's Lemma.

Minkowski's Second Theorem

We briefly recall Minkowski's Second Theorem. Let V be a real vector space of finite dimension n and let L be a full $\mathbb{Z}$-lattice in V. For each integer j, $1 \leq j \leq n$, the jth successive minimum $\lambda_j^{\mathbb{Q}}(L,S)$ of the lattice L with respect to a subset $S \subset V$ is defined to be the lower bound of the real numbers λ such that λS contains j $\mathbb{Q}$-linearly independent lattice points, that is

$$\lambda_j^{\mathbb{Q}}(L,S) = \inf\{\lambda > 0 \mid L \cap \lambda S \text{ contains } j \ \mathbb{Q}\text{-linearly independent points}\}.$$

In general it is not possible to estimate each single successive minimum. Instead Minkowski proved sharp estimates for the product of the successive minima. Given a Haar measure vol on V and a $\mathbb{Z}$-lattice L in V, the volume $\mathrm{vol}(L)$ of L is defined to be the volume of a fundamental domain $\Phi(L)$ of L. Minkowski proved

Theorem 1.5.1 (Minkowski's Second Theorem). *Let L be a full $\mathbb{Z}$-lattice in an n-dimensional real vector space V. Suppose S is a nonempty, open, convex, symmetric [-S=S], and bounded subset of V. Then*

$$\tfrac{2^n}{n!}\,\mathrm{vol}(L) \leq \lambda_1^{\mathbb{Q}}(L,S)\ldots\lambda_n^{\mathbb{Q}}(L,S)\,\mathrm{vol}(S) \leq 2^n\,\mathrm{vol}(L).$$

Proof. See for example [C, Chap. VIII, Thm. V]. □

It is important to note that the estimates in the theorem are independent of the chosen Haar measure, as the Haar measure on a locally compact topological group is uniquely determined up to a positive constant.

The same procedure can also be applied over division algebras. More precisely, let $\mathcal{O}$ be a $\mathbb{Z}$-order in a division algebra D of finite dimension over $\mathbb{Q}$, and let L be a left $\mathcal{O}$-lattice of rank n. For each $1 \leq j \leq n$, we define the jth successive minimum $\lambda_j^D(L,S)$ of L with respect to a subset S of $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$ to be the lower bound of the real numbers λ such that λS contains j D-linearly independent lattice points, i.e.

$$\inf\{\lambda > 0 \mid (1 \otimes M) \cap \lambda S \text{ contains } j \ D\text{-linearly independent points}\}.$$

As a straightforward generalisation of Minkowski's Second Theorem, we obtain

Corollary 1.5.2. *Let L be an $\mathcal{O}$-lattice of rank n and let S be a nonempty, open, convex, and bounded subset of $L_{\mathbb{R}}$. Then*

$$\left(\lambda_1^D(L,S)\cdots\lambda_n^D(L,S)\right)^d \operatorname{vol}(S) \leq 2^{nd}\operatorname{vol}(L), \tag{1.27}$$

where d is the dimension of D over $\mathbb{Q}$ and vol *is any Haar measure on the nd-dimensional real vector space $L_{\mathbb{R}}$.*

Proof. Since D has no zero divisors, for each $1 \leq i \leq n$, out of $(i-1)d+1$ $\mathbb{Q}$-linearly independent elements of L there are at least i D-linearly independent, hence

$$\left(\lambda_i^D(L,S)\right)^d \leq \lambda_{(i-1)d+1}^{\mathbb{Q}}(L,S)\lambda_{(i-1)d+2}^{\mathbb{Q}}(L,S)\ldots\lambda_{id}^{\mathbb{Q}}(L,S).$$

This leads to

$$\left(\lambda_1^D(L,S)\lambda_2^D(L,S)\ldots\lambda_n^D(L,S)\right)^d \leq \lambda_1^{\mathbb{Q}}(L,S)\lambda_2^{\mathbb{Q}}(L,S)\ldots\lambda_{nd}^{\mathbb{Q}}(L,S),$$

and applying Minkowski's Second Theorem yields (1.27). □

The same method of proof was also applied by Thunder [T] to obtain a generalisation of Minkowski's Second Theorem over number fields. Note that the proof of Corollary 1.5.2 cannot be completed in any obvious way to get a similar result over an arbitrary semisimple $\mathbb{Q}$-algebra. This is because our proof only works for rational algebras which have no zero divisors.

Siegel's Lemma

Let $\mathcal{O}$ be a $\mathbb{Z}$-order in a finite dimensional division algebra D over $\mathbb{Q}$. We define the height of a homomorphism $\phi : D^n \to D^m$ as the height of its kernel, i.e.

$$H_{\mathcal{O}}(\phi) = H_{\mathcal{O}}(\ker\phi).$$

Furthermore we define the height of an element $x \in D^n$ to be the height of the subspace generated by x, that is

$$H_{\mathcal{O}}(x) = H_{\mathcal{O}}(Dx).$$

Finally we let $\Delta_{\mathcal{O}}$ denote the discriminant of $\mathcal{O}$ with respect to the reduced trace. Note $\log\sqrt{|\Delta_{\mathcal{O}}|} = -\chi(\mathcal{O})$. Using these notations, we have the following version of Siegel's Lemma:

Theorem 1.5.3. *Let $\phi : D^n \to D^m$ be a surjective homomorphism. There exist $l = n-m$ vectors $x_1, \ldots, x_l \in \ker\phi \cap \mathcal{O}^n$ which are linearly independent over D and satisfy*

$$H_{\mathcal{O}}(x_1)\cdots H_{\mathcal{O}}(x_l) \leq \left(\frac{2\sqrt{|\Delta_{\mathcal{O}}|}}{\sqrt{d}V_{ld}^{1/ld}}\right)^{ld} H_{\mathcal{O}}(\phi)^d,$$

where d is the dimension of D over $\mathbb{Q}$ and V_{ld} the Lebesgue volume of the unit ball in $\mathbb{R}^{ld}$.

Proof. Fix a positive involution $*$ on $D_{\mathbb{R}} = D \otimes_{\mathbb{Q}} \mathbb{R}$ and let h_n be the canonical metric on $D^n_{\mathbb{R}}$ associated to the involution $*$. This gives rise to the scalar product $\mathrm{Tr}_{D_{\mathbb{R}}|\mathbb{R}} \circ h_n$ on the real vector space $D^n_{\mathbb{R}}$. We let $\|\ \|$ and vol denote the induced norm and volume on $D^n_{\mathbb{R}}$, respectively, and we consider the nonempty, open, convex, symmetric, and bounded subset

$$S = \{x \in (\ker\phi)_{\mathbb{R}} \mid \|x\| < V_{ld}^{-1/ld}\}.$$

This is a ball of radius $r = V_{ld}^{-1/ld}$ in the ld-dimensional real vector space $(\ker\phi)_{\mathbb{R}}$, thus its volume is $\mathrm{vol}(S) = r^{ld}V_{ld} = 1$. Applying Corollary 1.5.2 yields

$$\left(\lambda_1^D(\ker\phi\cap\mathcal{O}^n, S)\cdots\lambda_l^D(\ker\phi\cap\mathcal{O}^n, S)\right)^d \leq 2^{ld}\,\mathrm{vol}(\ker\phi\cap\mathcal{O}^n).$$

Hence we find l vectors $x_1, \ldots, x_l$ in $\ker\phi\cap\mathcal{O}^n$ which are linearly independent over D and satisfy

$$(\|x_1\|\ldots\|x_l\|)^d \leq (2r)^{ld}\,\mathrm{vol}(\ker\phi\cap\mathcal{O}^n). \tag{1.28}$$

Recall

$$\begin{aligned}\log\mathrm{vol}(\ker\phi\cap\mathcal{O}^n) &= -\widehat{\deg}_{\mathbb{Q}}\left(\ker\phi\cap\mathcal{O}^n, \mathrm{Tr}_{D_{\mathbb{R}}|\mathbb{R}}\circ h_n\right)\\ &= -d\cdot\widehat{\deg}_{\mathbb{Q}}\left(\ker\phi\cap\mathcal{O}^n, \mathrm{tr}_{D_{\mathbb{R}}|\mathbb{R}}\circ h_n\right)\\ &= -d\left(\widehat{\deg}_{\mathcal{O}}(\ker\phi\cap\mathcal{O}^n, h_n) - l\cdot\chi(\mathcal{O})\right)\\ &= d\cdot\log H_{\mathcal{O}}(\phi) - ld\cdot\chi(\mathcal{O}).\end{aligned} \tag{1.29}$$

The second equality follows from $\mathrm{Tr} = \mathrm{tr}^d$ and the third from Theorem 1.3.4. On the other hand, for every $x \in D^n$, we have $\mathcal{O}x \subset Dx\cap\mathcal{O}^n$, thus

$$\log H_{\mathcal{O}}(x) = -\widehat{\deg}_{\mathcal{O}}(Dx\cap\mathcal{O}^n, h_n) \leq -\widehat{\deg}_{\mathcal{O}}(\mathcal{O}x, h_n) = \tfrac{1}{2}\log\left|N_{D_{\mathbb{R}}|\mathbb{R}}(h_n(x,x))\right|.$$

Moreover, $\|x\|^2 = \mathrm{Tr}_{D_{\mathbb{R}}|\mathbb{R}}\left(h_n(x,x)\right)$, and since the automorphism given by left multiplication by $h_n(x,x)$ on $D_{\mathbb{R}}$ is symmetric and positive definite, we get

$$\left|N_{D_{\mathbb{R}}|\mathbb{R}}(h_n(x,x))\right| = N_{D_{\mathbb{R}}|\mathbb{R}}(h_n(x,x)) \leq \left(\tfrac{1}{d}\,\mathrm{Tr}_{D_{\mathbb{R}}|\mathbb{R}}\left(h_n(x,x)\right)\right)^d,$$

whence $H_{\mathcal{O}}(x) \leq \left(\frac{1}{\sqrt{d}}\,\|x\|\right)^d$. Combining this with (1.28) and (1.29) establishes the theorem. □

Using similar arguments, Liebendörfer and Rémond [LR2] have proven an almost identical version of Siegel's Lemma. But unlike the result presented here, they assume that the order $\mathcal{O}$ is maximal.

2 Noncommutative Arithmetic Surfaces

2.1 Noncommutative projective schemes

2.1.1 Definitions and notations

Graded rings and modules

Let us introduce the terminology and notation from graded ring and module theory, which we will use in this chapter. A good reference for this topic is the book of Nastasescu and Van Oystaeyen [NV].

In this chapter k will denote a noetherian commutative ring. A $\mathbb{Z}$-graded k-algebra is a k-algebra A together with a decomposition $A = \bigoplus_{i\in\mathbb{Z}} A_i$ of A into a direct sum of abelian groups A_i such that for all $i, j \in \mathbb{Z}$, $A_i \cdot A_j \subset A_{i+j}$. An element of A_i is called homogeneous element of degree i. Note that A_0 is a k-subalgebra of A and A_i is an A_0-bimodule for all i. We say that A is a locally finite k-algebra, if each component A_i is a finitely generated k-module. Finally, A is called $\mathbb{N}$-graded if $A_i = 0$ for all $i < 0$.

Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be a $\mathbb{Z}$-graded k-algebra. A $\mathbb{Z}$-graded right A-module is a right A-module M together with a decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$ of M into a direct sum of abelian groups M_i such that for all $i, j \in \mathbb{Z}$, $M_i A_j \subset M_{i+j}$. We will denote the category of $\mathbb{Z}$-graded right A-modules by $\operatorname{Gr} A$. In this category homomorphisms are of degree zero; thus if $M = \bigoplus_{i\in\mathbb{Z}} M_i$ and $N = \bigoplus_{i\in\mathbb{Z}} N_i$ are $\mathbb{Z}$-graded right A-modules and $f \in \operatorname{Hom}_{\operatorname{Gr} A}(M, N)$, then $f(M_i) \subset N_i$ for all integers i. Given a $\mathbb{Z}$-graded A-module M and an integer d, the A_0-module $\bigoplus_{i\geq d} M_i$ is denoted by $M_{\geq d}$. If A is $\mathbb{N}$-graded, then $M_{\geq d}$ is a $\mathbb{Z}$-graded A-submodule of M, and $A_{\geq d}$ is an $\mathbb{N}$-graded two-sided ideal of A. In the following, "graded" without any prefix will mean $\mathbb{Z}$-graded.

For a graded A-module $M = \bigoplus_{i\in\mathbb{Z}} M_i$ and any integer d, we let $M[d]$ be the graded A-module defined by $M[d]_i = M_{i+d}$ for all $i \in \mathbb{Z}$. It is clear that the rule $M \mapsto M[d]$ extends to an automorphism of $\operatorname{Gr} A$. We call $[1]$ the *degree-shift functor*. Note $[d] = [1]^d$.

Torsion and quotient category

Let A be a right noetherian $\mathbb{N}$-graded k-algebra. We say that an element x of a graded right A-module M is *torsion* if $xA_{\geq d} = 0$ for some d. The torsion elements in M form a graded A-submodule which we denote by $\tau(M)$ and call the *torsion submodule* of M. A module M is called *torsion-free* if $\tau(M) = 0$ and *torsion* if $M = \tau(M)$. The torsion modules form a dense subcategory of $\mathrm{Gr}\, A$ for which we will use the notation $\mathrm{Tors}\, A$. We set $\mathrm{QGr}\, A$ for the quotient category $\mathrm{Gr}\, A / \mathrm{Tors}\, A$; the formal definition of this category can be found in the appendix, but roughly speaking, $\mathrm{QGr}\, A$ has the same objects as $\mathrm{Gr}\, A$ but objects in $\mathrm{Tors}\, A$ become isomorphic to 0.

As for every quotient category there is a quotient functor π from $\mathrm{Gr}\, A$ to $\mathrm{QGr}\, A$. Since the category $\mathrm{Gr}\, A$ has enough injectives, there is a section functor σ from $\mathrm{QGr}\, A$ to $\mathrm{Gr}\, A$ which is right adjoint to π, cf. [Po, Sect. 4.4]. Hence $\mathrm{Tors}\, A$ is a localizing subcategory of $\mathrm{Gr}\, A$. The functor π is exact and the functor σ is left exact.

We will modify the notation introduced above by using lower case to indicate that we are working with finitely generated A-modules. Thus $\mathrm{gr}\, A$ denotes the category of finitely generated graded right A-modules, $\mathrm{tors}\, A$ denote the dense subcategory of $\mathrm{gr}\, A$ of torsion modules, and $\mathrm{qgr}\, A$ the quotient category $\mathrm{gr}\, A / \mathrm{tors}\, A$. The latter is the full subcategory of noetherian objects of $\mathrm{QGr}\, A$, cf. [AZ, Prop. 2.3].

Noncommutative projective schemes

We proceed with a review of the construction of noncommutative projective schemes, and we fix the notation we will use throughout. The standard reference for the theory of noncommutative projective schemes is the article [AZ] of Artin and Zhang.

In his fundamental paper [Se], Serre proved a theorem which describes the coherent sheaves on a projective scheme in terms of graded modules as follows. Let k be a commutative noetherian ring, let A be a finitely generated commutative $\mathbb{N}$-graded k-algebra, and let $X = \mathrm{Proj}\, A$ be the associated projective scheme. Let $\mathrm{coh}\, X$ denote the category of coherent sheaves on X, and let $\mathcal{O}_X(n)$ denote the nth power of the twisting sheaf of X. Define a functor $\Gamma_* : \mathrm{coh}\, X \to \mathrm{Gr}\, A$ by

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{F} \otimes \mathcal{O}_X(n))$$

and let $\pi : \operatorname{Gr} A \to \operatorname{QGr} A$ be the quotient functor. Then Serre [Se, Sect. 59, Prop. 7.8; H, Ex. II.5.9] proved

Theorem 2.1.1. *Suppose that A is generated over k by elements of degree 1. Then $\pi \circ \Gamma_*$ defines an equivalence of categories* $\operatorname{coh} X \to \operatorname{qgr} A$.

Since the categories $\operatorname{QGr} A$ and $\operatorname{qgr} A$ are also available when A is not commutative, this observation provides a way to make a definition in a more general setting. Let A be a right noetherian $\mathbb{N}$-graded k-algebra, let $\operatorname{QGr} A$ be the quotient category introduced above and let $\pi : \operatorname{Gr} A \to \operatorname{QGr} A$ be the quotient functor. The associated *noncommutative projective scheme* is the pair $\operatorname{Proj} A = (\operatorname{QGr} A, \mathcal{A})$ where $\mathcal{A} = \pi(A)$. The subcategory $\operatorname{Tors} A$ is stable under the degree-shift functor because M is a torsion module if and only if $M[1]$ is. Hence, by the universal property of quotient categories, cf. Theorem A.2 in the appendix, there is an induced automorphism s of $\operatorname{QGr} A$ defined by the equality $s \circ \pi = \pi \circ [1]$. We call s the *twisting functor* of $\operatorname{Proj} A$. Given an object $\mathcal{M}$ of $\operatorname{QGr} A$, we will often write $\mathcal{M}[i]$ instead of $s^i(\mathcal{M})$.

There is a representing functor $\Gamma : \operatorname{QGr} A \to \operatorname{Gr} A$ constructed as follows. For an object $\mathcal{M}$ in $\operatorname{QGr} A$, we define

$$\Gamma(\mathcal{M}) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\operatorname{QGr} A}(\mathcal{A}, \mathcal{M}[i]) .$$

If $a \in \Gamma(\mathcal{A})_i = \operatorname{Hom}_{\operatorname{QGr} A}(\mathcal{A}, \mathcal{A}[i])$, $b \in \Gamma(\mathcal{A})_j = \operatorname{Hom}_{\operatorname{QGr} A}(\mathcal{A}, \mathcal{A}[j])$ and $m \in \Gamma(\mathcal{M})_d = \operatorname{Hom}_{\operatorname{QGr} A}(\mathcal{A}, \mathcal{M}[d])$, we define multiplications by

$$ab = s^j(a) \circ b \quad \text{and} \quad ma = s^i(m) \circ a.$$

With this law of composition, $\Gamma(\mathcal{A})$ becomes a graded k-algebra and $\Gamma(\mathcal{M})$ a graded right $\Gamma(\mathcal{A})$-module. Moreover there is a homomorphism $\varphi : A \to \Gamma(\mathcal{A})$ of graded k-algebras sending $a \in A_i$ to $\pi(\lambda_a) \in \Gamma(\mathcal{A})_i$, where $\lambda_a : A \to A$ is left multiplication by a. So each $\Gamma(\mathcal{M})$ has a natural graded right A-module structure, and it is clear that Γ defines a functor from $\operatorname{QGr} A$ to $\operatorname{Gr} A$. The following lemma summarises some important properties of the representing functor Γ.

Lemma 2.1.2. *Let* $\operatorname{Proj} A$ *be a noncommutative projective scheme. Then the following statements hold:*

(i) *The representing functor* $\Gamma : \operatorname{QGr} A \to \operatorname{Gr} A$ *is isomorphic to the section functor* $\sigma : \operatorname{QGr} A \to \operatorname{Gr} A$;

(ii) Γ *is fully faithful;*

(iii) $\pi\Gamma \cong \mathrm{id}_{\mathrm{QGr}\,A}$.

Proof. (i) Given an object $\mathcal{M}$ of $\mathrm{QGr}\,A$, we let F denote the contravariant functor $\mathrm{Hom}_{\mathrm{QGr}\,A}(\pi(-),\mathcal{M})$ from $\mathrm{Gr}\,A$ to Mod-k. By [YZ, Prop. 1.1], there is a natural transformation $\eta : F \to \mathrm{Hom}_{\mathrm{Gr}\,A}(-, \underline{F}(A))$ such that $\eta_{A[i]}$ are isomorphisms for all $i \in \mathbb{Z}$, where

$$\underline{F}(A) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{QGr}\,A}\left(\pi\left(A[-i]\right), \mathcal{M}\right) \cong \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{QGr}\,A}\left(\pi\left(A\right), \mathcal{M}[i]\right) = \Gamma(\mathcal{M}).$$

Since the quotient functor π is exact and commutes with direct sums, F is contravariant left exact and converts direct sums into direct products. The same is true for the functor $\mathrm{Hom}_{\mathrm{Gr}\,A}\left(-, \underline{F}(A)\right)$. Since $\{A[i] \mid i \in \mathbb{Z}\}$ is a set of generators for the category $\mathrm{Gr}\,A$ and for all $i \in \mathbb{Z}$, $\eta_{A[i]}$ are isomorphisms, it follows with the Five Lemma that η is a natural isomorphism. In combination with the isomorphism $\underline{F}(A) \cong \Gamma(\mathcal{M})$, we obtain a natural isomorphism

$$\mathrm{Hom}_{\mathrm{QGr}\,A}\left(\pi(-), \mathcal{M}\right) \cong \mathrm{Hom}_{\mathrm{Gr}\,A}\left(-, \Gamma(\mathcal{M})\right).$$

This holds for every object $\mathcal{M}$ of $\mathrm{QGr}\,A$, thus Γ is right adjoint to the quotient functor π, whence $\Gamma \cong \sigma$ as claimed.

The statements (ii) and (iii) follow from the respective properties of the section functor σ, see Proposition A.3 in the appendix. $\square$

Serre duality

Being defined in terms of natural isomorphisms between Ext groups, also for noncommutative projective schemes there is a well-defined notion of Serre duality. Here we only want to give the relevant definitions; for more details we refer to [YZ], [J] and [RV].

The *cohomological dimension* of a noncommutative projective scheme $\mathrm{Proj}\,A$ is defined to be

$$\mathrm{cd}(\mathrm{Proj}\,A) = \max\{i \mid \mathrm{Ext}^i_{\mathrm{QGr}\,A}(\mathcal{A}, \mathcal{M}) \neq 0 \text{ for some } \mathcal{M} \in \mathrm{QGr}\,A\}.$$

Following Yekutieli and Zhang [YZ], we say that a noncommutative projective scheme $\mathrm{Proj}\,A$ of cohomological dimension $d < \infty$ *has a dualizing object,* if there is an object ω in $\mathrm{qgr}\,A$ and a natural isomorphism

$$\theta : \mathrm{Ext}^d_{\mathrm{qgr}\,A}(\mathcal{A}, -)^\vee \longrightarrow \mathrm{Hom}_{\mathrm{qgr}\,A}(-, \omega). \tag{2.1}$$

Here the dual is taken in the category of k-modules. Clearly, the dualizing object is unique, up to isomorphism, if it exists. Furthermore, if $\operatorname{Proj} A$ has a dualizing object ω, then for each $0 \leq i \leq d = \operatorname{cd}(\operatorname{Proj} A)$, there is a natural transformation

$$\theta^i : \operatorname{Ext}^i_{\operatorname{qgr} A}(-, \omega) \longrightarrow \operatorname{Ext}^{d-i}_{\operatorname{qgr} A}(\mathcal{A}, -)^\vee,$$

where θ^0 is the inverse of the natural isomorphism in (2.1). We say that $\operatorname{Proj} A$ satisfies *Serre duality* if θ^i are isomorphisms for all i.

2.1.2 Locally free objects

Let $\operatorname{Proj} A$ be a noncommutative projective scheme. To go on, we have to know which objects in $\operatorname{QGr} A$ play the role of locally free sheaves. Firstly we consider the commutative case, in particular we are looking for a characterisation of locally free sheaves on a commutative projective scheme, which can be translated to the noncommutative setting. The main step is

Lemma 2.1.3. *Let X be a projective scheme over a noetherian commutative ring k, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, and let $\mathcal{G}$ be any $\mathcal{O}_X$-module. Then the exact sequence*

$$\mathcal{G} \xrightarrow{g} \mathcal{F} \longrightarrow 0 \tag{2.2}$$

splits if and only if

$$\mathcal{G}_x \xrightarrow{g_x} \mathcal{F}_x \longrightarrow 0 \tag{2.3}$$

splits for all $x \in X$.

Proof. It is a general property of sheaf homomorphisms that (2.3) splits whenever (2.2) does. So we concentrate on the opposite implication. Let us collect the three results from Hartshorne's famous book [H] that we need to prove our lemma. We will use the same notation as Hartshorne. Firstly, if we apply Proposition 5.15 of [H, Chap. II] to the $\mathcal{H}om$ sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, we obtain a natural isomorphism

$$\Gamma_* \left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\right)^\sim \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}). \tag{2.4}$$

Secondly, if S denotes the homogeneous coordinate ring of X, then [H, (II.5.11.a)] asserts that for every graded S-module M and all $\mathfrak{p} \in \operatorname{Proj} S$,

$$(\widetilde{M})_{\mathfrak{p}} = M_{(\mathfrak{p})}. \tag{2.5}$$

Thirdly, by [H, (III.6.8)], for all $\mathfrak{p} \in \operatorname{Proj} S$ we have

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})_{\mathfrak{p}} \cong \operatorname{Hom}_{\mathcal{O}_{\mathfrak{p}}}(\mathcal{F}_{\mathfrak{p}}, \mathcal{G}_{\mathfrak{p}}). \tag{2.6}$$

Combining (2.4), (2.5) and (2.6) yields an isomorphism

$$\operatorname{Hom}_{\mathcal{O}_{\mathfrak{p}}}(\mathcal{F}_{\mathfrak{p}}, \mathcal{G}_{\mathfrak{p}}) \cong \Gamma_* \left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\right)_{(\mathfrak{p})},$$

which is clearly natural in $\mathcal{G}$. This shows that $(g_{\mathfrak{p}})_* = \operatorname{Hom}_{\mathcal{O}_{\mathfrak{p}}}(\mathcal{F}_{\mathfrak{p}}, g_{\mathfrak{p}})$ is surjective if and only if $(g_*)_{(\mathfrak{p})} = \Gamma_* \left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, g)\right)_{(\mathfrak{p})}$ is surjective. Now, if the exact sequence (2.3) splits, then $(g_{\mathfrak{p}})_*$ is surjective for all $\mathfrak{p} \in \operatorname{Proj} S$, hence $(g_*)_{(\mathfrak{p})}$ is surjective for all $\mathfrak{p} \in \operatorname{Proj} S$. In other words, $\operatorname{coker}(g_*)_{(\mathfrak{p})} = 0$ for all $\mathfrak{p} \in \operatorname{Proj} S$, which, by [HIO, (II.11.8)], is equivalent to $\operatorname{coker}(g_*) = 0$. This means that $g_* = \Gamma_* \left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, g)\right)$ is surjective. There thus exists some f in

$$\Gamma_* \left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\right) = \bigoplus_{n \in \mathbb{Z}} \Gamma\left(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})[n]\right) \cong \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}[n])$$

such that $\mathrm{id}_{\mathcal{F}} = g \circ f$. Since g and $\mathrm{id}_{\mathcal{F}}$ are both morphisms of degree zero, it follows that f is of degree zero as well, which shows that (2.2) splits. $\square$

It is well-known [H, (II.5.18)] that for every coherent sheaf $\mathcal{F}$ on a projective scheme X, there exist finitely many integers $n_1, \dots, n_I$ and a surjective homomorphism

$$g : \bigoplus_{i=1}^{I} \mathcal{O}_X(n_i) \to \mathcal{F}.$$

Now, if $\mathcal{F}$ is locally free then $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module and thus g_x splits for all $x \in X$. On the other hand, $\mathcal{O}_{X,x}$ are local rings and $\mathcal{O}_X(n_i)_x$ are free $\mathcal{O}_{X,x}$-modules, therefore, if g_x splits then $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module. Since this holds for all $x \in X$, $\mathcal{F}$ is locally free. So, applying the previous lemma, we see that $\mathcal{F}$ is locally free if and only if g splits. This motivates

Definition 2.1.4. *Let* $\operatorname{Proj} A$ *be a noncommutative projective scheme. An object* $\mathcal{M}$ *in* $\operatorname{QGr} A$ *is called* locally free *if there exists finitely many integers* $n_1, \dots, n_I$ *and a* split *epimorphism*

$$\bigoplus_{i=1}^{I} \mathcal{A}[n_i] \longrightarrow \mathcal{M} \longrightarrow 0.$$

Note that in general the objects $\mathcal{A}[n]$ are *not* projective in $\mathrm{QGr}\, A$, therefore a locally free object in $\mathrm{QGr}\, A$ must not be projective. Nevertheless we have the following characterisation of locally free objects.

Proposition 2.1.5. *For an object $\mathcal{M}$ in $\mathrm{QGr}\, A$ the following assertions are equivalent:*

(i) $\mathcal{M}$ is locally free;

(ii) $\Gamma(\mathcal{M})$ is finitely generated and projective in $\mathrm{Gr}\, \Gamma(\mathcal{A})$;

(iii) $\Gamma(\mathcal{M})$ is a finitely generated projective $\Gamma(\mathcal{A})$-module.

Conversely, if a graded right A-module M is finitely generated and projective over A or over $\Gamma(\mathcal{A})$, then $\pi(M)$ is locally free in $\mathrm{QGr}\, A$.

Proof. If M is a finitely generated projective graded A-module, then there exist finitely many integers $n_1, \dots, n_I$ and a graded split epimorphism $f : \bigoplus_{i=1}^{I} A[n_i] \to M$. Let g be a section for f. Applying the quotient functor π yields

$$\pi(f) \circ \pi(g) = \pi(f \circ g) = \pi(\mathrm{id}_M) = \mathrm{id}_{\pi(M)} .$$

Hence $\pi(f) : \bigoplus_{i=1}^{I} \mathcal{A}[n_i] \to \pi(M)$ is a split epimorphism as well, which shows that $\pi(M)$ is locally free in $\mathrm{QGr}\, A$.

It follows from Lemma 2.1.2(iii) that $\pi\left(\Gamma(\mathcal{A})\right) \cong \mathcal{A}$, hence the argument in the last paragraph remains true if we replace A by $\Gamma(\mathcal{A})$. This also establishes the implication (ii) $\Longrightarrow$ (i).

(i) $\Longrightarrow$ (ii) : If $\mathcal{M}$ is locally free in $\mathrm{QGr}\, A$, there exist finitely many integers $n_1, \dots, n_I$ and a split epimorphism $f : \bigoplus_{i=1}^{I} \mathcal{A}[n_i] \to \mathcal{M}$. Let g be a section for f. As we have seen above, every functor maps retractions to retractions, hence $\Gamma(f) : \bigoplus_{i=1}^{I} \Gamma(\mathcal{A})[n_i] \to \Gamma(\mathcal{M})$ is a split epimorphism as well. But $\Gamma(\mathcal{A})[i]$ is projective in $\mathrm{Gr}\, \Gamma(\mathcal{A})$ for every $i \in \mathbb{Z}$, which yields the projectivity of $\Gamma(\mathcal{M})$.

(ii) $\Longleftrightarrow$ (iii) : [NV, Cor. I.2.2]. □

Another nice property of locally free objects is summarised in

Proposition 2.1.6. *Let D be an abelian category, let $F, G : \mathrm{QGr}\, A \to \mathsf{D}$ be two additive functors and let $\eta : F \to G$ be a natural transformation. If $\eta_{\mathcal{A}[i]}$ is an isomorphism for all $i \in \mathbb{Z}$, then $\eta_{\mathcal{M}}$ is an isomorphism for every locally free $\mathcal{M}$ in $\mathrm{QGr}\, A$.*

Proof. If $\mathcal{M}$ is locally free, then there are finitely many integers $n_1, \ldots, n_I$ and a split epimorphism $f : \mathcal{G} := \bigoplus_{i=1}^{I} \mathcal{A}[n_i] \to \mathcal{M}$. Let g be a section for f and consider the commutative diagram

$$\begin{array}{ccccc} F(\mathcal{M}) & \xrightarrow{F(g)} & F(\mathcal{G}) & \xrightarrow{F(f)} & F(\mathcal{M}) \\ {\scriptstyle \eta_{\mathcal{M}}}\downarrow & & {\scriptstyle \eta_{\mathcal{G}}}\downarrow & & {\scriptstyle \eta_{\mathcal{M}}}\downarrow \\ G(\mathcal{M}) & \xrightarrow{G(g)} & G(\mathcal{G}) & \xrightarrow{G(f)} & G(\mathcal{M}). \end{array}$$

Since F and G are additive functors, $\eta_{\mathcal{G}}$ is an isomorphism because $F(\mathcal{G}) \cong \bigoplus_i F(\mathcal{A}[n_i])$ and $G(\mathcal{G}) \cong \bigoplus_i G(\mathcal{A}[n_i])$. As we have seen in the proof of the previous lemma, every functor maps retractions to retractions and sections to sections, thus $F(f)$ and $G(f)$ are epimorphisms and $F(g)$ and $G(g)$ are monomorphisms. Hence the left square implies that $\eta_{\mathcal{M}}$ is a monomorphism and the right square implies that $\eta_{\mathcal{M}}$ is an epimorphism. Since D is an abelian category, this is equivalent to say that $\eta_{\mathcal{M}}$ is an isomorphism. $\square$

2.1.3 Invertible objects

Let X be a projective scheme. Recall that a coherent $\mathcal{O}_X$-module $\mathcal{L}$ is called invertible if $\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X$. This means that the functor $\mathcal{L} \otimes -$ is an autoequivalence of the category of coherent sheaves on X. Moreover

$$\mathcal{L}[1] = \mathcal{L} \otimes \mathcal{O}_X[1] \cong \mathcal{O}_X[1] \otimes \mathcal{L}.$$

Hence the autoequivalence $\mathcal{L} \otimes -$ commutes, up to natural isomorphism, with the twisting functor $\mathcal{O}_X[1] \otimes -$. This motivates

Definition 2.1.7. *Let* $\operatorname{Proj} A$ *be a noncommutative projective scheme. An object* $\mathcal{L}$ *in* $\operatorname{qgr} A$ *is called* invertible *if there exists an autoequivalence* t *of* $\operatorname{qgr} A$ *such that* $t(\mathcal{A}) \cong \mathcal{L}$ *and which commutes with the twisting functor* s*, i.e.* $s \circ t \cong t \circ s$.

Given an graded right A-module L, we let $L^\vee$ denote the graded left A-module $\underline{\operatorname{Hom}}_{\operatorname{Gr} A}(L, A) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\operatorname{Gr} A}(L, A[i])$. Invertible objects are characterised in

Theorem 2.1.8. *Let* $\operatorname{Proj} A$ *be a noncommutative projective scheme and let* $\mathcal{L}$ *be an object of* $\operatorname{qgr} A$*. Then,* $\mathcal{L}$ *is invertible if and only if* $\Gamma(\mathcal{L})$ *is a graded* A*-bimodule and* $t' = \pi\left(\Gamma(-) \otimes_A \Gamma(\mathcal{L})\right)$ *is an autoequivalence of* $\operatorname{qgr} A$ *such that* $t'(\mathcal{A}) \cong \mathcal{L}$*. Moreover,* $\pi\left(\Gamma(-) \otimes_A \Gamma(\mathcal{L})^\vee\right)$ *is a quasi-inverse of* t'*.*

Proof. Suppose that $L = \Gamma(\mathcal{L})$ is a graded A-bimodule and that $t' = \pi\left(\Gamma(-) \otimes_A \Gamma(\mathcal{L})\right)$ is an autoequivalence of $\operatorname{qgr} A$ such that $t'(\mathcal{A}) \cong \mathcal{L}$. In order to prove that $\mathcal{L}$ is invertible, it remains to show that t commutes with the twisting functor s. We have

$$\Gamma(s(-)) \otimes_A L = \Gamma(-)[1] \otimes_A L \cong \left(\Gamma(-) \otimes_A L\right)[1],$$

where the last isomorphism is a basic property of the graded tensor product [NV, I.2.17]. This finally leads to

$$t' \circ s = \pi\left(\Gamma\left(s(-)\right) \otimes_A L\right) \cong \pi\left(\left(\Gamma(-) \otimes_A L\right)[1]\right) = \left(\pi\left(\Gamma(-) \otimes_A L\right)\right)[1] = s \circ t'.$$

To prove the other implication, we suppose that $\mathcal{L}$ is invertible and that t is an autoequivalence of $\operatorname{qgr} A$ such that $t(\mathcal{A}) \cong \mathcal{L}$. Firstly, we show that $L = \Gamma(\mathcal{L})$ is a graded A-bimodule. As we do not want to overload the notation, we assume that $t \circ s = s \circ t$; if the natural isomorphism $\epsilon : t \circ s \to s \circ t$ is not the identity, the forthcoming arguments work as well, unless one always has to take ϵ into account.

If $a \in \Gamma(\mathcal{A})_i = \operatorname{Hom}_{\operatorname{QGr} A}(\mathcal{A}, \mathcal{A}[i])$, $b \in \Gamma(\mathcal{A})_j = \operatorname{Hom}_{\operatorname{QGr} A}(\mathcal{A}, \mathcal{A}[j])$ and $x \in L_d = \operatorname{Hom}_{\operatorname{QGr} A}(\mathcal{A}, t(\mathcal{A})[d])$, we define multiplications by

$$ax = t(s^d(a)) \circ x \text{ and } xb = s^j(x) \circ b.$$

Since $t \circ s = s \circ t$, these operations induce a graded $\Gamma(\mathcal{A})$-bimodule structure on L, hence L is in fact a graded A-bimodule.

Secondly we are going to prove that the functors $t \circ \pi$ and $\pi(- \otimes_A L)$ from $\operatorname{gr} A$ to $\operatorname{qgr} A$ are isomorphic. Let $\mathcal{E}$ be any object of $\operatorname{QGr} A$. Since t is an autoequivalence, it is exact. We also know that π is an exact functor. Hence both,

$$F = \operatorname{Hom}_{\operatorname{QGr} A}(t \circ \pi(-), \mathcal{E}) \text{ and } G = \operatorname{Hom}_{\operatorname{QGr} A}(\pi(- \otimes_A L), \mathcal{E})$$

are contravariant left exact functors from $\operatorname{gr} A$ to $\operatorname{Mod-}k$. This allows us to apply Watts' Theorem for $\operatorname{gr} A$ [YZ, Thm. 1.3] to conclude

$$F \cong \operatorname{Hom}_{\operatorname{Gr} A}(-, \underline{F}(A)) \text{ and } G \cong \operatorname{Hom}_{\operatorname{Gr} A}(-, \underline{G}(A)).$$

Note

$$\underline{F}(A) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\operatorname{QGr} A}\left(t \circ \pi(A[-i]), \mathcal{E}\right) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\operatorname{QGr} A}\left(t(\mathcal{A})[-i], \mathcal{E}\right)$$

and

$$\underline{G}(A) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{QGr}\,A}\left(\pi\left(A[-i]\otimes_A L\right), \mathcal{E}\right) \cong \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{QGr}\,A}\left(\pi(L)[-i], \mathcal{E}\right).$$

By Lemma 2.1.2(iii), we have $\pi(L) \cong \pi\Gamma t(\mathcal{A}) \cong t(\mathcal{A})$, which shows $\underline{F}(A) \cong \underline{G}(A)$ and thus $F \cong G$. As this is true for every object $\mathcal{E}$ of QGr A, the functors $t \circ \pi$ and $\pi(- \otimes_A L)$ are isomorphic. Therefore $t\pi\Gamma \cong \pi\left(\Gamma(-)\otimes_A L\right)$, and again we use $\pi\Gamma \cong \mathrm{id}_{\mathrm{qgr}\,A}$ to conclude $t \cong \pi\left(\Gamma(-)\otimes_A L\right)$. But t is an autoequivalence of qgr A and hence so is $\pi\left(\Gamma(-)\otimes_A L\right)$.

To finish the proof of the theorem, it remains to show that $\pi\left(\Gamma(-)\otimes_A L^\vee\right)$ is a quasi-inverse of $\pi\left(\Gamma(-)\otimes_A L\right)$. Let t^{-1} be a quasi-inverse of t and let $\eta : tt^{-1} \to \mathrm{id}_{\mathrm{qgr}\,A}$ be the associated natural isomorphism. Since we have just proven that $\pi\left(\Gamma(-)\otimes_A L\right) \cong t$ with $L = \Gamma(\mathcal{L}) \cong \Gamma t(\mathcal{A})$, it suffices to show $L^\vee \cong \Gamma t^{-1}(\mathcal{A})$. For this, we consider the map

$$\phi : \underline{\mathrm{Hom}}_{\mathrm{QGr}\,A}\left(\mathcal{A}, t^{-1}(\mathcal{A})\right) \to \underline{\mathrm{Hom}}_{\mathrm{Gr}\,A}\left(\Gamma t(\mathcal{A}), A\right) = L^\vee$$

given by $f \mapsto \Gamma\left(s^i(\eta_{\mathcal{A}}) \circ t(f)\right)$ whenever $f \in \mathrm{Hom}_{\mathrm{QGr}\,A}\left(\mathcal{A}, t^{-1}(\mathcal{A})[i]\right)$. We claim that ϕ is an isomorphism of graded right A-modules. Lemma 2.1.2(ii) asserts that Γ is fully faithful, so ϕ is clearly bijective. Hence it remains to show that ϕ is A-linear. For $f \in \mathrm{Hom}_{\mathrm{QGr}\,A}\left(\mathcal{A}, t^{-1}(\mathcal{A})[i]\right)$ and $b \in \Gamma(\mathcal{A})_j$, we have

$$\begin{aligned}
\phi(fb) &= \phi(s^j(f) \circ b)\\
&= \Gamma\left(s^{i+j}(\eta_{\mathcal{A}}) \circ t(s^j(f)\circ b)\right)\\
&= \Gamma\left(s^{i+j}(\eta_{\mathcal{A}}) \circ t(s^j(f))\right) \circ \Gamma(t(b))\\
&\overset{(1)}{=} \Gamma s^j\left(s^i(\eta_{\mathcal{A}}) \circ t(f)\right) \circ \Gamma(t(b))\\
&\overset{(2)}{=} \Gamma\left(s^i(\eta_{\mathcal{A}}) \circ t(f)\right) \circ \Gamma(t(b))\\
&= \phi(f) \circ \Gamma\left(t(b)\right).
\end{aligned}$$

Equality (1) holds because t and s commute, (2) holds because $\Gamma\left(s^j(\varphi)\right) = \Gamma(\varphi)[j] = \Gamma(\varphi)$ for all homomorphisms φ in QGr A. On the other hand, if $b \in \Gamma(\mathcal{A})_i$ and $x \in L_d$ then

$$\left[\Gamma \circ t(b)\right](x) = s^d(t(b)) \circ x = t(s^d(b)) \circ x = bx,$$

therefore

$$\phi(fb)(x) = \phi(f)\left(\Gamma(t(b))(x)\right) = \phi(f)(bx).$$

This shows that ϕ is $\Gamma(\mathcal{A})$-linear, which implies that $\Gamma\left(t^{-1}(\mathcal{A})\right) \cong L^\vee$ as graded right A-modules and thus completes the proof of the theorem. $\square$

2.1.4 Base change

Let K be a k-algebra, let A be an $\mathbb{N}$-graded k-algebra, and let A_K denote the $\mathbb{N}$-graded k-algebra $A \otimes_k K$. If K is a flat k-module, then the functor $- \otimes_k K : \operatorname{Gr} A \to \operatorname{Gr} A_K$ is exact. The quotient functor $\pi_K : \operatorname{Gr} A_K \to \operatorname{QGr} A_K$ is also exact, so is their composition. Clearly, if a graded right A-module M is torsion, the graded right A_K-module $M_K = M \otimes_k K$ is also torsion. Therefore $\pi_K(M_K) = 0$ whenever M is torsion. Hence we may apply the universal property of quotient categories, cf. Theorem A.2 in the appendix, to conclude that there exists a unique functor $F : \operatorname{QGr} A \to \operatorname{QGr} A_K$ such that $\pi_K \circ (- \otimes_k K) = F \circ \pi$, where as usual π denotes the quotient functor from $\operatorname{Gr} A$ to $\operatorname{QGr} A$. Given an object $\mathcal{M}$ of $\operatorname{QGr} A$, we will use the notation $\mathcal{M}_K = F(\mathcal{M})$. In this situation the following base change lemma holds:

Lemma 2.1.9. *Suppose that A and A_K are right noetherian and that K is a flat k-module. Let $\mathcal{M}$ and $\mathcal{N}$ be two objects of* $\operatorname{QGr} A$. *If $\mathcal{N}$ is noetherian, then for all $i \geq 0$,*

$$\operatorname{Ext}^i_{\operatorname{QGr} A_K}(\mathcal{N}_K, \mathcal{M}_K) \cong \operatorname{Ext}^i_{\operatorname{QGr} A}(\mathcal{N}, \mathcal{M}) \otimes_k K.$$

Proof. Let M and N be graded right A-modules such that $\mathcal{M} = \pi(M)$ and $\mathcal{N} = \pi(N)$ respectively. Then [AZ, Prop. 7.2(1)] asserts that for all $i \geq 0$,

$$\underline{\operatorname{Ext}}^i_{\operatorname{QGr} A}(\mathcal{N}, \mathcal{M}) \cong \varinjlim_n \underline{\operatorname{Ext}}^i_{\operatorname{Gr} A}(N_{\geq n}, M). \tag{2.7}$$

Recall $\mathcal{N}_K = \pi_K(N_K)$ and $\mathcal{M}_K = \pi_K(M_K)$. Since $\mathcal{N}$ is noetherian, this is also true for N and thus for N_K, whence $\mathcal{N}_K$ is noetherian. By assumption, A_K is right noetherian, so we may apply [AZ, Prop 7.2(1)] once again to conclude that for all $i \geq 0$,

$$\underline{\operatorname{Ext}}^i_{\operatorname{QGr} A_K}(\mathcal{N}_K, \mathcal{M}_K) \cong \varinjlim_n \underline{\operatorname{Ext}}^i_{\operatorname{Gr} A_K}\left((N_K)_{\geq n}, M_K\right).$$

Clearly, $(N_K)_{\geq n} = (N_{\geq n})_K$, and [NV, I.2.12] tells us $\underline{\operatorname{Ext}}^i_{\operatorname{Gr} A_K}\left((N_{\geq n})_K, M_K\right) = \operatorname{Ext}^i_{A_K}\left((N_{\geq n})_K, M_K\right)$. Since K is flat over k, it follows from a change of ring theorem, cf. [Re, (2.39)], that for all i and n,

$$\operatorname{Ext}^i_{A_K}\left((N_{\geq n})_K, M_K\right) \cong \operatorname{Ext}^i_A(N_{\geq n}, M) \otimes_k K.$$

But the tensor product commutes with direct limits, so we finally get

$$\underline{\operatorname{Ext}}^i_{\operatorname{QGr} A_K}(\mathcal{N}_K, \mathcal{M}_K) \cong \underline{\operatorname{Ext}}^i_{\operatorname{QGr} A}(\mathcal{N}, \mathcal{M}) \otimes_k K.$$

Taking the degree zero parts yields the claim. □

The base change lemma completes our study of arbitrary noncommutative projective schemes, and we now specalise to noncommutative arithmetic surfaces.

2.2 Noncommutative arithmetic surfaces

2.2.1 Definition

Following Soulé [So], an arithmetic surface is a regular scheme X, projective and flat over $\operatorname{Spec}\mathbb{Z}$ of Krull dimension two. We now isolate the conditions that we want to be satisfied by our noncommutative analogues of the homogeneous coordinate ring S of X and of $\operatorname{coh}(X)$, the category of coherent sheaves on X. They clearly satisfy the following conditions:

- S is noetherian and locally finite, and the associated graded real algebra $S_{\mathbb{R}}$ is still noetherian.
- $\operatorname{coh}(X)$ is H-finite. In other words, for every coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and all integers $i \geq 0$, the cohomology groups $H^i(X, \mathcal{F})$ are finitely generated;
- $\operatorname{coh}(X)$ has cohomological dimension 1 in the sense that $H^i(X, \mathcal{F}) = 0$ for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ and all $i > 1$;
- $\operatorname{coh}(X)$ has a dualizing sheaf ω_X.

This motivates

Definition 2.2.1. *A* noncommutative arithmetic surface *is a noncommutative projective scheme* $\operatorname{Proj} A$, *which satisfies the following conditions:*

- *A is an $\mathbb{N}$-graded right noetherian locally finite $\mathbb{Z}$-algebra such that the associated real algebra $A_{\mathbb{R}} = A \otimes_{\mathbb{Z}} \mathbb{R}$ is also right noetherian;*
- $\operatorname{qgr} A$ *is H-finite and has cohomological dimension 1, i.e. for every object $\mathcal{M}$ in* $\operatorname{qgr} A$ *and all $i \geq 0$, the Ext groups* $\operatorname{Ext}^i_{\operatorname{qgr} A}(\mathcal{A}, \mathcal{M})$ *are finitely generated and* $\operatorname{Ext}^i_{\operatorname{qgr} A}(\mathcal{A}, \mathcal{M}) = 0$ *whenever $i > 1$;*
- $\operatorname{qgr} A$ *has a dualizing object ω.*

The next subsection provides some examples of noncommutative arithmetic surfaces.

2.2.2 Examples

Example 1: Noncommutative arithmetic surfaces derived from commutative ones

Let X be a commutative arithmetic surface and let $\mathcal{O}$ be a coherent sheaf of $\mathcal{O}_X$-algebras. We let $\mathrm{coh}(\mathcal{O})$ denote the category of coherent sheaves with a structure of right $\mathcal{O}$-module, and we claim that $\mathrm{coh}(\mathcal{O})$ is a noncommutative arithmetic surface. To establish this claim, we have to check the several conditions that are imposed in the definition of a noncommutative arithmetic surface.

Firstly, for every $n \in \mathbb{Z}$, we set $\mathcal{O}[n] = \mathcal{O} \otimes_{\mathcal{O}_X} \mathcal{O}_X[n]$. Then the homogeneous coordinate ring of $\mathrm{coh}(\mathcal{O})$ is defined to be

$$B = \bigoplus_{n \in \mathbb{N}} H^0(X, \mathcal{O}[n]).$$

For every object $\mathcal{M}$ of $\mathrm{coh}(\mathcal{O})$, we have $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{M}) = H^0(X, \mathcal{M})$, therefore the truth of the statement that the triple $(\mathrm{coh}(\mathcal{O}), \mathcal{O}, - \otimes_{\mathcal{O}} \mathcal{O}[n])$ satisfies the conditions (H1), (H2)' and (H3) of Corollary 4.6 of [AZ] follows directly from the fact that X fulfils them. Hence [AZ, Corollary 4.6] tells us that B is a right noetherian locally finite $\mathbb{Z}$-algebra and the pair $(\mathrm{coh}(\mathcal{O}), \mathcal{O})$ is isomorphic to the noncommutative projective scheme $\mathrm{Proj}\, B$. In particular, the categories $\mathrm{coh}(\mathcal{O})$ and $\mathrm{qgr}\, B$ are equivalent. Since the homogeneous coordinate ring S of X is noetherian and $\mathcal{O}$ is a coherent $\mathcal{O}_X$-module, it follows that B is a finitely generated $\mathbb{N}$-graded S-module. After tensoring with $\mathbb{R}$, $S_{\mathbb{R}}$ is still noetherian and $B_{\mathbb{R}}$ finitely generated over $S_{\mathbb{R}}$, so also $B_{\mathbb{R}}$ is right noetherian.

Secondly, let M be a finitely generated graded right B-module and denote by $\mathcal{M}$ and $\mathcal{M}_{\mathcal{S}}$ the corresponding objects of $\mathrm{qgr}\, B$ and $\mathrm{qgr}\, S$. Combining Proposition 3.11(3) and Theorem 8.3(2),(3) of [AZ] shows that for all $i \geq 0$,

$$\mathrm{Ext}^i_{\mathrm{qgr}\, B}(\mathcal{B}, \mathcal{M}) \cong \mathrm{Ext}^i_{\mathrm{qgr}\, S}(\mathcal{S}, \mathcal{M}_{\mathcal{S}}). \tag{2.8}$$

On the other hand it follows from Serre's Theorem (cf. Theorem 2.1.1) that the categories $\mathrm{qgr}\, S$ and $\mathrm{coh}(X)$ are equivalent, therefore $\mathrm{qgr}\, S$ is H-finite and has cohomological dimension one. The isomorphism in (2.8) shows that $\mathrm{Proj}\, B$ inherits these properties from $\mathrm{qgr}\, S$.

It remains to prove that $\mathrm{coh}(\mathcal{O})$ has a dualizing object. If ω_X denotes the dualizing sheaf of X, then there are natural isomorphism

$$\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}, -)^{\vee} = H^1(X, -)^{\vee} \cong \mathrm{Hom}_X(-, \omega_X)$$

of functors from $\mathrm{coh}(\mathcal{O})$ to the category of abelian groups. This implies that the $\mathcal{H}om$ sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}, \omega_X)$ is a representing object of the functor $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}, -)^\vee$, and so it is a dualizing object of $\mathrm{coh}(\mathcal{O})$.

All together this shows that $\mathrm{coh}(\mathcal{O})$ is indeed a noncommutative arithmetic surface. The noncommutative projective scheme $\mathrm{coh}(\mathcal{O})$ is an example of what Artin and Zhang call classical projective schemes. They use, in analogy with [EGA, 1], the notation $\underline{\mathrm{Spec}}\ \mathcal{O}$. Classical projective schemes are studied in [AZ, 6].

Example 2: Maximal orders

We now specialise the last example a little bit. Again we let X be a commutative arithmetic surface. Let K be the function field of X and let $\mathcal{O}$ be a sheaf of maximal $\mathcal{O}_X$-orders in a finite dimensional semisimple K-algebra A. In this situation, the dualizing sheaf $\omega = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}, \omega_X)$ is an invertible $\mathcal{O}$-module. Indeed, since $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}, \omega_X) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \omega_X$ and ω_X is an invertible sheaf, it suffices to show that $\widetilde{\mathcal{O}} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}, \mathcal{O}_X)$ is an invertible $\mathcal{O}$-module. We prove this locally. So let $x \in X$. Then $\widetilde{\mathcal{O}}_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_x, \mathcal{O}_{X,x})$. Since $\mathcal{O}_x$ is an $\mathcal{O}_{X,x}$-order in the semisimple K-algebra A and the trace form $(a, b) \mapsto tr(ab)$ is a nondegenerate K-bilinear form on A, it follows that $\widetilde{\mathcal{O}}_x$ is isomorphic to the $\mathcal{O}_x$-module $\{a \in A \mid tr\,(a\mathcal{O}_x) \subset \mathcal{O}_x\}$, which is in fact an invertible $\mathcal{O}_x$-module because $\mathcal{O}_x$ is a maximal order in A, cf. [Re, 25].

Note that we have just shown that for every $x \in X$, $\widetilde{\mathcal{O}}_x$ is the inverse different. Hence we may view $\widetilde{\mathcal{O}}$ as the canonical bundle of the extension $\mathrm{coh}(\mathcal{O}) \to X$.

Example 3: Noncommutative projective line

Let R be a ring and let $R[T_0, T_1]$ be the polynomial ring in two indeterminates over R. We call $\mathrm{Proj}\,(R[T_0, T_1])$ the noncommutative projective line over R and denote it by $\mathbb{P}^1_R$. If the ring R is finitely generated as $\mathbb{Z}$-module, then $\mathbb{P}^1_R$ is a noncommutative arithmetic surface. We will show that it is a special case of Example 1.

For this, we consider the projective line $X = \mathbb{P}^1_{\mathbb{Z}}$ over the integers. Then $\mathcal{O} = R \otimes_{\mathbb{Z}} \mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_X$-algebras. We claim that the homogeneous coordinate ring $B = \bigoplus_{n \in \mathbb{N}} H^0(X, \mathcal{O}[n])$ is isomorphic to $R[T_0, T_1]$. Indeed, since X is noetherian, cohomology commutes with arbitrary direct sums [H, (III.2.9.1)], thus

$$B \cong H^0(X, \bigoplus_{n\in\mathbb{N}} \mathcal{O}[n]) \cong H^0(X, R\otimes_{\mathbb{Z}} \bigoplus_{n\in\mathbb{N}} \mathcal{O}_X[n]) \cong R\otimes_{\mathbb{Z}} \Gamma_+(\mathcal{O}_X),$$

where $\Gamma_+(\mathcal{O}_X) = \bigoplus_{n\in\mathbb{N}} H^0(X, \mathcal{O}_X[n]) \cong \mathbb{Z}[T_0, T_1]$. This shows $B \cong R\otimes_{\mathbb{Z}} \mathbb{Z}[T_0, T_1] \cong R[T_0, T_1]$. So it follows from Example 1 that the categories $\mathrm{coh}(\mathcal{O})$ and $\mathrm{qgr}\,(R[T_0, T_1])$ are equivalent and that $\mathrm{Proj}\,(R[T_0, T_1])$ is a noncommutative arithmetic surface.

In the polynomial ring $R[T_0, T_1]$ the indeterminates T_0 and T_1 lie in the center. Of course this is quite a strong restriction in a noncommutative setting. Fortunately we can omit the assumption that T_1 and T_2 belong to the center by considering twisted polynomial rings. They are obtained as follows. Let σ be a graded automorphism of the polynomial ring $R[T_0, T_1]$. We define a new multiplication on the underlying graded R-module $R[T_0, T_1]$ by

$$p * q = p\sigma^d(q),$$

where p and q are homogeneous polynomials and $\deg(p) = d$. The graded ring thus obtained is called twisted polynomial ring and is denoted by $R[T_0, T_1]^\sigma$. By [Z], the categories $\mathrm{Gr}\, R[T_0, T_1]$ and $\mathrm{Gr}\, R[T_0, T_1]^\sigma$ are equivalent and hence $\mathrm{Proj}\,(R[T_0, T_1]) \cong \mathrm{Proj}\,(R[T_0, T_1]^\sigma)$. So, twisted polynomial rings in two indeterminates over a noncommutative ring R all define the same noncommutative projective line over R. Twisted algebras are studied in [Z].

For example, we may consider the graded ring automorphism σ of $R[T_0, T_1]$ given by

$$rT_0 \mapsto \tilde{\sigma}(r)uT_0 \quad \text{and} \quad rT_1 \mapsto \tilde{\sigma}(r)T_1$$

for all $r \in R$, where u is a unit which lies in the center of R, and where $\tilde{\sigma}$ is an automorphism of R. In the twisted polynomial ring $R[T_0, T_1]^\sigma$, one has

$$T_0 * T_1 = T_0T_1,\ T_1 * T_0 = uT_0T_1,\ T_0 * T_0 = uT_0^2,\ T_1 * T_1 = T_1^2.$$

Example 4: Noncommutative plane projective curves

A noncommutative plane projective curve is obtained as follows. Let R be a ring and let $p \in R[T_0, T_1, T_2]$ be a homogeneous normal polynomial of positive degree. Recall that an element of a ring is called normal if the principal ideal generated by this element is two-sided. This allows to form the factor ring $A = R[T_0, T_1, T_2]/(p)$. We call $\operatorname{Proj} A$ a noncommutative plane projective curve.

Again we specialise to the case where the ring R is finitely generated as $\mathbb{Z}$-module and where $p \in R[T_0, T_1, T_2]$ is a homogeneous normal polynomial of positive degree such that $(p') = (p) \cap \mathbb{Z}[T_0, T_1, T_2]$ is a prime ideal of $\mathbb{Z}[T_0, T_1, T_2]$. We set $S = \mathbb{Z}[T_0, T_1, T_2]/(p')$. Then $X = \operatorname{Proj} S$ is an arithmetic surface. Moreover we have a natural homomorphism $\phi : S \to A$ of noetherian $\mathbb{N}$-graded $\mathbb{Z}$-algebras, and A is finitely generated as S-module. Hence the sheaf $\widetilde{A}$ associated to A is a coherent sheaf of $\mathcal{O}_X$-algebras, and by Example 1, $\operatorname{coh}(\widetilde{A})$ is a noncommutative arithmetic surface. If B denotes the homogeneous coordinate ring of $\operatorname{coh}(\widetilde{A})$, the categories $\operatorname{qgr} B$ and $\operatorname{coh}(\widetilde{A})$ are equivalent. But the category $\operatorname{qgr} B$ is also equivalent to the category $\operatorname{qgr} A$, which shows that $\operatorname{Proj} A$ is a noncommutative arithmetic surface.

As in Example 3, we may consider twisted algebras in order to omit the restriction that the indeterminates lie in the center.

2.2.3 Arithmetic vector bundles

Hermitian vector bundles are very important objects in Arakelov geometry. A hermitian vector bundle on an arithmetic variety X is a pair $\overline{E} = (E, h)$ consisting of a locally free sheaf E on X and a *smooth* hermitian metric on the holomorphic vector bundle $E_{\mathbb{C}}$ induced by E on the associated complex algebraic variety $X_{\mathbb{C}}$.

Unfortunately we are not able to adapt this definition of a hermitian vector bundle to our noncommutative setting. The main problem is that we do not know what should be the differential structure on a noncommutative complex algebraic variety. This problem was also mentioned by other authors, for example Polishchuk [P] writes: "However, it is rather disappointing that at present there is almost no connection between noncommutative algebraic varieties over $\mathbb{C}$ and noncommutative topological spaces, which according to Connes are described by C^*-algebras."

The lack of smooth hermitian metrics on vector bundles on noncommutative algebraic varieties forces us to substitute the infinite part of a hermitian vector bundle on an arithmetic variety. To get a feeling what this substitute could be, we first study the commutative case.

Another infinite part in Arakelov theory

Firstly we consider arithmetic vector bundles on the arithmetic curve $\operatorname{Spec}\mathbb{Z}$. An arithmetic vector bundle on $\operatorname{Spec}\mathbb{Z}$ is a pair $\overline{E} = (E, h)$, where E is a free $\mathbb{Z}$-module of finite rank and h is a scalar product on the real vector space $E_{\mathbb{R}} = E \otimes_{\mathbb{Z}} \mathbb{R}$. Now the crucial idea is to replace the scalar product h on $E_{\mathbb{R}}$ by an automorphism β of the real vector space $E_{\mathbb{R}}$. In this way, we get the object $\widetilde{E} = (E, \beta)$. Of course we want to understand how $\overline{E}$ and $\widetilde{E}$ are related.

Recall that for any finite dimensional real vector space V there is a bijection between the nondegenerate bilinear forms and the automorphisms of V. Unfortunately this bijection is not canonical but depends on the choice of a basis of V. More precisely, given a basis $Y = \{v_1, \ldots, v_n\}$, $n = \dim V$, of V, we let $h_Y : V \times V \to \mathbb{R}$ denote the scalar product defined by $h_Y(x, y) = x_1 y_1 + \cdots + x_n y_n$, where $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are the coordinates of x and y in the basis Y, respectively. The bijection mentioned above maps an automorphism α of V to the nondegenerate bilinear form $h_{Y,\alpha}$, which is defined by $h_{Y,\alpha}(x, y) = h_Y(x, \alpha(y))$ for all $x, y \in V$.

Now, if $\overline{E} = (E, h)$ is an arithmetic vector bundle on $\operatorname{Spec}\mathbb{Z}$, then we may choose a basis Y of $E_{\mathbb{R}}$ which already lies in E, i.e. Y is also a basis of the free $\mathbb{Z}$-module E. In this way we get the object $\widetilde{E}_Y = (E, \beta_Y)$, where β_Y is the automorphism of $E_{\mathbb{R}}$ which satisfies $h_{Y,\beta_Y} = h$. The fact that the volume of E does not depend on the choice of a basis of E implies

Remark 2.2.2. *Let $\overline{E} = (E, h)$ be an arithmetic vector bundle on* $\operatorname{Spec}\mathbb{Z}$ *and let Y be a basis of the free $\mathbb{Z}$-module E. Then*

$$-\log|\det \beta_Y| = \widehat{\deg}\,\overline{E}.$$

In particular, $\det \beta_Y$ *does not depend on the choice of the basis Y of E.*

At least for arithmetic line bundles the above construction can be reversed. Namely, if L is an invertible $\mathbb{Z}$-module then $L_{\mathbb{R}}$ is a one-dimensional real vector space. Therefore every automorphism α of $L_{\mathbb{R}}$ is given by multiplication by some nonzero $r \in \mathbb{R}$. Let x be a basis of L over $\mathbb{Z}$ and let h_α denote the scalar product on $L_{\mathbb{R}}$ which is defined by the equation $h_\alpha(x, x) = r^2$. Note that h_α does not depend on the choice of the basis x of L because the only other basis of L is $-x$. This allows us to associate in a well-defined manner the arithmetic line bundle $\overline{L} = (L, h_\alpha)$ to the pair $\widetilde{L} = (L, \alpha)$. We have the following:

Lemma 2.2.3. *Let $\widetilde{L} = (L, \alpha)$ and $\widetilde{M} = (M, \beta)$ be two pairs consisting of an invertible $\mathbb{Z}$-module and an automorphism of the induced real vector space. The associated arithmetic line bundles $\overline{L} = (L, h_\alpha)$ and $\overline{M} = (M, h_\beta)$ are isomorphic if and only if $|\det \alpha| = |\det \beta|$. Moreover, $\widehat{\deg}\, \overline{L} = -\log |\det \alpha|$.*

Proof. The automorphisms α and β are given by multiplication by some nonzero real numbers r and s, respectively. Then, $|\det \alpha| = |r|$ and $|\det \beta| = |s|$. Let x be a basis of L over $\mathbb{Z}$ and let $\varphi : L \to M$ be an isomorphism. Since $\varphi(x)$ is a basis of M, we have $h_\beta(\varphi(x), \varphi(x)) = s^2$. Hence $\varphi_{\mathbb{R}}$ is an isometry if and only if $r^2 = s^2$ which is equivalent to $|r| = |s|$. Moreover, we have $\widehat{\deg}\, \overline{L} = -\frac{1}{2} \log h_\alpha(x, x) = -\log |r|$. □

So at least for arithmetic curves, we obtain essentially the same theory whether the infinite part is given by a scalar product or by an automorphism. Let us now consider hermitian vector bundles on arithmetic surfaces. If X is an arithmetic surface, the associated complex variety $X_{\mathbb{C}}$ is a Riemann surface. It is possible to endow every line bundle L with a distinguished hermitian metric g. This metric is defined via the Green's function for the Riemann surface $X_{\mathbb{C}}$ and is therefore called the Green's metric. Already Arakelov [A1] observed that the Green's metric is admissible and that every other admissible metric h on L is a scalar multiple of the Green's metric, i.e. there is some positive real number α such that $h = \alpha g$.

On the other hand for every line bundle L on the arithmetic surface X, we have

$$\operatorname{Hom}_{X_{\mathbb{R}}}(L_{\mathbb{R}}, L_{\mathbb{R}}) \cong \operatorname{Hom}_{X_{\mathbb{R}}}(\mathcal{O}_{X_{\mathbb{R}}}, \mathcal{O}_{X_{\mathbb{R}}}) \cong H^0(X_{\mathbb{R}}, \mathcal{O}_{X_{\mathbb{R}}}) \cong \mathbb{R}.$$

This shows that the set of admissible hermitian line bundles on X embeds into the set of pairs (L, α) consisting of a line bundle and an automorphism of the real line bundle $L_{\mathbb{R}}$.

All these considerations motivate

Definition 2.2.4. *An* arithmetic vector bundle *on a noncommutative arithmetic surface* $\operatorname{Proj} A$ *is a pair $\overline{\mathcal{E}} = (\mathcal{E}, \beta)$ consisting of an object $\mathcal{E}$ of* $\operatorname{qgr} A$ *and an automorphism $\beta : \mathcal{E}_{\mathbb{R}} \to \mathcal{E}_{\mathbb{R}}$. If $\mathcal{E}$ is an invertible object, then $\overline{\mathcal{E}}$ is called an* arithmetic line bundle *on* $\operatorname{Proj} A$.

2.3 Arithmetic intersection and Riemann-Roch theorem

2.3.1 Intersection on commutative arithmetic surfaces

Following Soulé [So], there are two ways to compute the intersection number of two hermitian line bundles $\overline{L}$ and $\overline{M}$ on an arithmetic surface X. The first possibility works as follows. Pick two non-zero sections l and m of L and M, respectively, such that the divisors $\mathrm{div}(l)$ and $\mathrm{div}(m)$ of l and m have no component in common. Then the intersection number of $\overline{L}$ and $\overline{M}$ is computed as

$$(\overline{L}, \overline{M}) = (l, m)_{fin} + (l, m)_{\infty},$$

where the finite part $(l, m)_{fin}$ is simply the usual intersection number of the divisors $\mathrm{div}(l)$ and $\mathrm{div}(m)$, and the infinite part is some expression that depends on the metrics; see [So, 1.1] for the precise formula.

Since in noncommutative geometry only the category of (quasi-)coherent $\mathcal{O}_X$-modules is available and particularly there are no points, the concept of divisors is not defined in this framework; hence this method of computing the intersection number fails in a noncommutative setting. Therefore we concentrate on the other approach which involves the determinant of the cohomology.

Determinant of the cohomology

In order to define the determinant of the cohomology, we need the determinant of any finitely generated $\mathbb{Z}$-module and not just of the free ones. If M is a finitely generated $\mathbb{Z}$-module, then M is the internal direct sum of its free part M_f and its torsion part M_{tor}. We let $n = |M_{tor}|$ and define

$$\det M = \det M_f \otimes_{\mathbb{Z}} \tfrac{1}{n}\mathbb{Z}.$$

The determinant of the cohomology of a vector bundle E on an arithmetic surface X is defined to be the invertible $\mathbb{Z}$-module

$$\lambda(E) = \det H^0(X, E) \otimes_{\mathbb{Z}} \left(\det H^1(X, E)\right)^{-1}. \tag{2.9}$$

Given two line bundles L and M on X, one puts

$$\langle L, M\rangle = \lambda(L \otimes M) \otimes_{\mathbb{Z}} \lambda(L)^{-1} \otimes_{\mathbb{Z}} \lambda(M)^{-1} \otimes_{\mathbb{Z}} \lambda(\mathcal{O}_X). \tag{2.10}$$

This again is an invertible $\mathbb{Z}$-module and its norm equals the intersection number of L and M.

Now, if $\overline{L}$ and $\overline{M}$ are two hermitian line bundle on X, one tries to construct out of the given metrics on $L_{\mathbb{C}}$ and $M_{\mathbb{C}}$ a hermitian scalar product on the complex vector space $\langle L, M\rangle_{\mathbb{C}}$. In this way one obtains a hermitian line bundle $\langle\overline{L}, \overline{M}\rangle$ on $\operatorname{Spec}\mathbb{Z}$ which, in view of (2.10), is a natural generalisation of the intersection of L and M. The problem of endowing $\langle L, M\rangle_{\mathbb{C}}$ with a suitable hermitian metric was solved by constructing for every hermitian line bundle $\overline{N}$ on X, a hermitian scalar product on the determinant of the cohomology $\lambda(N)_{\mathbb{C}}$. One solution is due to Faltings [F2], the other goes back to Quillen [Q]. The two scalar products on $\lambda(N)_{\mathbb{C}}$ differ by a constant, which only depends on the Riemann surface $X_{\mathbb{C}}$ but is independent on the hermitian line bundle $\overline{N}$ in question, cf. [So, 4.4].

2.3.2 Intersection on noncommutative arithmetic surfaces

Determinant of the cohomology

Let $\operatorname{Proj} A$ be a noncommutative arithmetic surface and fix some automorphism α of the dualizing object $\omega_{\mathbb{R}}$. Let $\overline{\mathcal{L}} = (\mathcal{L}, \beta)$ be an arithmetic line bundle and $\overline{\mathcal{E}} = (\mathcal{E}, \gamma)$ be an arithmetic vector bundle on $\operatorname{Proj} A$. Recall that β induces homomorphisms

$$\beta_n^* : \operatorname{Ext}^n_{\operatorname{qgr} A_{\mathbb{R}}}(\mathcal{L}_{\mathbb{R}}, \mathcal{E}_{\mathbb{R}}) \to \operatorname{Ext}^n_{\operatorname{qgr} A_{\mathbb{R}}}(\mathcal{L}_{\mathbb{R}}, \mathcal{E}_{\mathbb{R}}), \quad n \geq 0.$$

Likewise γ induces homomorphisms

$$(\gamma_*)_n : \operatorname{Ext}^n_{\operatorname{qgr} A_{\mathbb{R}}}(\mathcal{L}_{\mathbb{R}}, \mathcal{E}_{\mathbb{R}}) \to \operatorname{Ext}^n_{\operatorname{qgr} A_{\mathbb{R}}}(\mathcal{L}_{\mathbb{R}}, \mathcal{E}_{\mathbb{R}}), \quad n \geq 0.$$

We set $\beta^* = \beta_0^*$ and $\gamma_* = (\gamma_*)_0$. To avoid ambiguities, we sometimes write $\operatorname{Ext}^n_{\operatorname{qgr} A_{\mathbb{R}}}(\beta, \mathcal{E}_{\mathbb{R}})$ instead of β_n^* and similarly for $(\gamma_*)_n$.

Inspired by the above considerations in the commutative case, we define two arithmetic line bundles on $\operatorname{Spec}\mathbb{Z}$, namely

$$\det\operatorname{Hom}\left(\overline{\mathcal{L}}, \overline{\mathcal{E}}\right) = \left(\det\operatorname{Hom}_{\operatorname{qgr} A}(\mathcal{L}, \mathcal{E}), \det\left((\beta^{-1})^* \circ \gamma_*\right)\right)$$

and

$$\det\operatorname{Ext}\left(\overline{\mathcal{L}}, \overline{\mathcal{E}}\right) = \left(\det\operatorname{Ext}^1_{\operatorname{qgr} A}(\mathcal{L}, \mathcal{E}), \det\left((\beta^{-1})_1^* \circ (\gamma_*)_1\right)\det(\alpha_*)^{-1}\right),$$

where $\alpha_* = \operatorname{Hom}_{\operatorname{qgr} A_{\mathbb{R}}}\left(\mathcal{E}_{\mathbb{R}} \otimes \mathcal{L}_{\mathbb{R}}^{-1}, \alpha\right)$ and $\mathcal{E}_{\mathbb{R}} \otimes \mathcal{L}_{\mathbb{R}}^{-1} = \pi\left(\Gamma(\mathcal{E}) \otimes_A \Gamma(\mathcal{L})^{\vee}\right)_{\mathbb{R}}$. By definition of a noncommutative arithmetic surface, all the Ext groups occurring above are finitely generated. Moreover the base change lemma, Lemma 2.1.9, ensures that this remains true after tensoring with $\mathbb{R}$, i.e. the Ext groups $\operatorname{Ext}^n_{\operatorname{qgr} A_{\mathbb{R}}}(\mathcal{L}_{\mathbb{R}}, \mathcal{E}_{\mathbb{R}})$ are finite dimensional real vector spaces, hence the determinants are reasonable and everything makes sense.

The expression $\det(\alpha_*)^{-1}$ occurs for the following reason. If $\overline{\mathcal{A}} = (\mathcal{A}, \mathrm{id})$ is the trivial arithmetic bundle on $\operatorname{Proj} A$, then

$$\begin{aligned}\left(\det \operatorname{Ext}(\overline{\mathcal{A}}, \overline{\mathcal{A}})\right)^{-1} &= \left(\left(\det \operatorname{Ext}^1_{\operatorname{qgr} A}(\mathcal{A}, \mathcal{A})\right)^{-1}, \det(\alpha_*)\right) \\ &\cong (\det \operatorname{Hom}_{\operatorname{qgr} A}(\mathcal{A}, \omega), \det(\alpha_*)) \\ &= \det \operatorname{Hom}\left(\overline{\mathcal{A}}, \overline{\omega}\right),\end{aligned}$$

which is a natural condition to be imposed in the definition of a metric on the determinant of the cohomology, cf. [La, Thm. V.3.2]. In other words, the expression $\det(\alpha_*)$ is necessary in order that the natural isomorphism between Ext groups given by the dualizing sheaf induces an isomorphism of the corresponding arithmetic line bundles. For more details we refer to Theorem 2.3.3 below.

However, we set

$$\lambda(\overline{\mathcal{L}}, \overline{\mathcal{E}}) = \det \operatorname{Hom}\left(\overline{\mathcal{L}}, \overline{\mathcal{E}}\right) \otimes_{\mathbb{Z}} \left(\det \operatorname{Ext}\left(\overline{\mathcal{L}}, \overline{\mathcal{E}}\right)\right)^{-1}.$$

If $\overline{\mathcal{L}} = \overline{\mathcal{A}}$ is the trivial arithmetic bundle, then we write $\lambda(\overline{\mathcal{E}})$ instead of $\lambda(\overline{\mathcal{A}}, \overline{\mathcal{E}})$ and call it the *determinant of the cohomology* of $\overline{\mathcal{E}}$.

Recall that for every $\mathcal{O}_X$-module $\mathcal{E}$ on a commutative scheme X and every $i \geq 0$, the Ext group $\operatorname{Ext}^i_X(\mathcal{O}_X, \mathcal{E})$ and the cohomology group $H^i(X, \mathcal{E})$ are naturally isomorphic. Hence in view of (2.9), our definition of the determinant of the cohomology is a natural generalisation of the one in the commutative case.

Compatibility with Serre duality

In order to prove that the determinant of the cohomology is compatible with Serre duality, we need the following two lemmata.

Lemma 2.3.1. *Let k be a field, let $F, G : \mathsf{C} \to$ mod-k be two functors from any category C to the category of finite dimensional k-vector spaces, let $\eta : F \to G$ be a natural transformation, and let A be an object of C. If η_A is an isomorphism, then $\det F(f) = \det G(f)$ for every endomorphism $f : A \to A$.*

Proof. Given an endomorphism $f : A \to A$, there is a commutative diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(f)} \\ F(A) & \xrightarrow{\eta_A} & G(A). \end{array}$$

If η_A is an isomorphism, we thus obtain $F(f) = \eta_A^{-1} \circ G(f) \circ \eta_A$, whence $\det F(f) = \det G(f)$. □

Lemma 2.3.2. *Let A be a finite dimensional simple algebra over a field k, and let M be a finitely generated A-bimodule. Given $a \in A$, let λ_a and ρ_a denote left and right multiplication by a on M, respectively. Then for every $a \in A$,*

$$\det_k(\lambda_a) = \det_k(\rho_a).$$

Proof. Let K denote the center of the simple algebra A, and let r be the dimension of A over K. It follows from [Re, (7.13)] that the enveloping algebra $A^e = A \otimes_K A^\circ$ is isomorphic to the algebra $M_r(K)$ of $r \times r$-matrices over the field K. We may view every A-bimodule N as a left A^e-module, by means of the formula

$$(a \otimes b^\circ)n = anb, \quad \text{for all } a \in A, b^\circ \in A^\circ, n \in N.$$

Since $A^e \cong M_r(K)$, every minimal left ideal V of A^e is isomorphic to the left A-module of r-component column vectors with entries in K, and hence has dimension r over K. Moreover, every finitely generated left A^e-module is isomorphic to a finite direct sum of copies of V. In particular, this applies to the left A^e-module A, whence $A \cong V$ for dimensional reasons. This implies that there is a natural number m such that $M \cong A^m$ as A-bimodules. Hence for every $a \in A$, $\det_k(\lambda_a) = m \det_k(f_a)$ and $\det(\rho_a) = m \det_k(g_a)$, where $f_a : A \to A$ is left multiplication and $g_a : A \to A$ is right multiplication by a. Finally, it follows from [Re, (9.32)] that $\det_k(f_a) = N_{A|k}(a) = \det_k(g_a)$ which completes the proof. □

Now we are ready to prove

Theorem 2.3.3. *Let $\overline{\mathcal{L}}$ be an arithmetic line bundle on a noncommutative arithmetic surface* Proj A. *Suppose that* Proj $A_{\mathbb{R}}$ *satisfies Serre duality. If* $\mathrm{End}_{\mathrm{qgr}\, A_{\mathbb{R}}}(\mathcal{A}_{\mathbb{R}})$ *is a simple ring, then the determinant of the cohomology is compatible with Serre duality, that is,*

$$\lambda(\overline{\mathcal{L}}, \overline{\omega}) \cong \lambda(\overline{\mathcal{L}}).$$

Proof. Recall

$$\lambda(\overline{\mathcal{L}},\overline{\omega}) = \det \mathrm{Hom}(\overline{\mathcal{L}},\overline{\omega}) \otimes \left(\det \mathrm{Ext}(\overline{\mathcal{L}},\overline{\omega})\right)^{-1}$$

and

$$\lambda(\overline{\mathcal{L}}) = \det \mathrm{Hom}(\overline{\mathcal{A}},\overline{\mathcal{L}}) \otimes \left(\det \mathrm{Ext}(\overline{\mathcal{A}},\overline{\mathcal{L}})\right)^{-1}.$$

If $\overline{\mathcal{L}} = (\mathcal{L},\beta)$ and $\overline{\omega} = (\omega,\alpha)$ then

$$\det \mathrm{Hom}\left(\overline{\mathcal{L}},\overline{\omega}\right) = \left(\det \mathrm{Hom}_{\mathrm{qgr}\,A}(\mathcal{L},\omega), \det\left((\beta^{-1})^*\right)\det(\alpha_*)\right) \tag{2.11}$$

and

$$\left(\det \mathrm{Ext}(\overline{\mathcal{A}},\overline{\mathcal{L}})\right)^{-1} = \left(\left(\det \mathrm{Ext}^1_{\mathrm{qgr}\,A}(\mathcal{A},\mathcal{L})\right)^{-1}, \det(\beta_*)_1^{-1}\det(\alpha_*)\right), \tag{2.12}$$

where $(\beta^{-1})^* = \mathrm{Hom}_{\mathrm{qgr}\,A_{\mathbb{R}}}\left(\beta^{-1},\omega_{\mathbb{R}}\right)$, $(\beta_*)_1 = \mathrm{Ext}^1_{\mathrm{qgr}\,A_{\mathbb{R}}}(\mathcal{A}_{\mathbb{R}},\beta)$ and $\alpha_* = \mathrm{Hom}_{\mathrm{qgr}\,A_{\mathbb{R}}}(\mathcal{L}_{\mathbb{R}},\alpha)$. By definition of the dualizing object, the two functors $\mathrm{Ext}^1_{\mathrm{qgr}\,A_{\mathbb{R}}}(\mathcal{A}_{\mathbb{R}},-)^\vee$ and $\mathrm{Hom}_{\mathrm{qgr}\,A_{\mathbb{R}}}(-,\omega_{\mathbb{R}})$ are isomorphic, so Lemma 2.3.1 ensures

$$\det\left((\beta^{-1})^*\right) = \det\left((\beta_*^{-1})_1^\vee\right). \tag{2.13}$$

On the other hand we have

$$\det\left((\beta_*^{-1})_1^\vee\right) = \det\left((\beta_*^{-1})_1\right) = \det(\beta_*)_1^{-1}. \tag{2.14}$$

Combining (2.11), (2.12), (2.13) and (2.14) yields

$$\det \mathrm{Hom}\left(\overline{\mathcal{L}},\overline{\omega}\right) \cong \left(\det \mathrm{Ext}\left(\overline{\mathcal{A}},\overline{\mathcal{L}}\right)\right)^{-1}. \tag{2.15}$$

It remains to prove

$$\det \mathrm{Hom}\left(\overline{\mathcal{A}},\overline{\mathcal{L}}\right) \cong \left(\det \mathrm{Ext}\left(\overline{\mathcal{L}},\overline{\omega}\right)\right)^{-1}. \tag{2.16}$$

By definition,

$$\det \mathrm{Hom}\left(\overline{\mathcal{A}},\overline{\mathcal{L}}\right) = \left(\det \mathrm{Hom}_{\mathrm{qgr}\,A}(\mathcal{A},\mathcal{L}), \det(\beta_*)\right) \tag{2.17}$$

and

$$\left(\det \mathrm{Ext}\left(\overline{\mathcal{L}},\overline{\omega}\right)\right)^{-1} = \left(\left(\det \mathrm{Ext}^1_{\mathrm{qgr}\,A}(\mathcal{L},\omega)\right)^{-1}, \det\left((\beta^{-1})_1^* \circ (\alpha_*)_1\right)^{-1}\det(\alpha_*)\right), \tag{2.18}$$

where $\alpha_* = \mathrm{Hom}_{\mathrm{qgr}\, A_{\mathbb{R}}}\left(\omega_{\mathbb{R}} \otimes \mathcal{L}_{\mathbb{R}}^{-1}, \alpha\right)$. Since $\mathrm{Proj}\, A_{\mathbb{R}}$ satisfies Serre duality, the functors $\mathrm{Ext}^1_{\mathrm{qgr}\, A_{\mathbb{R}}}(-, \omega_{\mathbb{R}})$ and $\mathrm{Hom}_{\mathrm{qgr}\, A_{\mathbb{R}}}(\mathcal{A}_{\mathbb{R}}, -)^{\vee}$ are isomorphic, and applying Lemma 2.3.1 yields

$$\det\left(\beta_1^*\right) = \det\left((\beta_*)^{\vee}\right). \tag{2.19}$$

Since $\det\left((\beta^{-1})_1^*\right)^{-1} = \det\left(\beta_1^*\right)$ and $\det\left((\beta_*)^{\vee}\right) = \det\left(\beta_*\right)$, we may combine (2.17), (2.18) and (2.19) to see that

$$\det \mathrm{Hom}(\overline{\mathcal{A}}, \overline{\mathcal{L}}) \cong \left(\left(\det \mathrm{Ext}^1_{\mathrm{qgr}\, A}(\mathcal{L}, \omega)\right)^{-1}, \det\left((\beta^{-1})_1^*\right)^{-1}\right).$$

Hence in order to prove (2.16), it suffices to show

$$\det\left((\alpha_*)_1\right)^{-1} \det(\alpha_*) = 1. \tag{2.20}$$

Since $\mathcal{L}_{\mathbb{R}}$ is invertible, it follows from Theorem 2.1.8 that the functor $t_{\mathbb{R}} = \pi_{\mathbb{R}}\left(\Gamma_{\mathbb{R}}(-) \otimes_{A_{\mathbb{R}}} L_{\mathbb{R}}\right)$ is an autoequivalence of $\mathrm{qgr}\, A_{\mathbb{R}}$ and $t_{\mathbb{R}}^{-1} = \pi_{\mathbb{R}}\left(\Gamma_{\mathbb{R}}(-) \otimes_{A_{\mathbb{R}}} L_{\mathbb{R}}^{\vee}\right)$ is a quasi-inverse of it. We thus have natural isomorphisms

$$\mathrm{Ext}^1_{\mathrm{qgr}\, A_{\mathbb{R}}}\left(\mathcal{L}_{\mathbb{R}}, -\right) \cong \mathrm{Ext}^1_{\mathrm{qgr}\, A_{\mathbb{R}}}\left(t_{\mathbb{R}}(\mathcal{A}_{\mathbb{R}}), -\right) \cong \mathrm{Ext}^1_{\mathrm{qgr}\, A_{\mathbb{R}}}\left(\mathcal{A}_{\mathbb{R}}, t_{\mathbb{R}}^{-1}(-)\right).$$

In combination with the natural isomorphism

$$\mathrm{Ext}^1_{\mathrm{qgr}\, A_{\mathbb{R}}}\left(\mathcal{A}_{\mathbb{R}}, t_{\mathbb{R}}^{-1}(-)\right)^{\vee} \cong \mathrm{Hom}_{\mathrm{qgr}\, A_{\mathbb{R}}}\left(t_{\mathbb{R}}^{-1}(-), \omega_{\mathbb{R}}\right),$$

which is provided by definition of the dualizing object, we obtain a natural isomorphism

$$\mathrm{Ext}^1_{\mathrm{qgr}\, A_{\mathbb{R}}}\left(\mathcal{L}_{\mathbb{R}}, -\right)^{\vee} \cong \mathrm{Hom}_{\mathrm{qgr}\, A_{\mathbb{R}}}\left(t_{\mathbb{R}}^{-1}(-), \omega_{\mathbb{R}}\right).$$

Applying Lemma 2.3.1 yields

$$\det\left((\alpha_*)_1\right) = \det\left((\alpha_*)_1^{\vee}\right) = \det\left(t_{\mathbb{R}}^{-1}(\alpha)^*\right). \tag{2.21}$$

Let E denote the endomorphism ring $\mathrm{End}_{\mathrm{qgr}\, A_{\mathbb{R}}}(\omega_{\mathbb{R}})$ of the dualizing object $\omega_{\mathbb{R}}$. There is an E-bimodule structure on the abelian group $G = \mathrm{Hom}_{\mathrm{qgr}\, A_{\mathbb{R}}}\left(t_{\mathbb{R}}^{-1}(\omega_{\mathbb{R}}), \omega_{\mathbb{R}}\right)$ defined by

$$e\varphi f = e \circ \varphi \circ t_{\mathbb{R}}^{-1}(f), \quad \text{for all } e, f \in E, \varphi \in G.$$

In terms of this E-bimodule structure, applying the homomorphism $\alpha_* = \mathrm{Hom}_{\mathrm{qgr}\, A_{\mathbb{R}}}\big(t_{\mathbb{R}}^{-1}(\omega_{\mathbb{R}}), \alpha\big)$ is simply left multiplication by α. Analogously, applying $t_{\mathbb{R}}^{-1}(\alpha)^*$ is right multiplication by α. Therefore, if E is a finite dimensional simple $\mathbb{R}$-algebra, then it follows from Lemma 2.3.2 that $\det(\alpha_*) = \det(t_{\mathbb{R}}^{-1}(\alpha)^*)$, which together with (2.21) establishes (2.20). It thus remains to show that E is simple and finite dimensional over $\mathbb{R}$.

Since $\mathrm{Proj}\, A_{\mathbb{R}}$ satisfies Serre duality, there exist two natural isomorphisms

$$\theta^i : \mathrm{Ext}^i_{\mathrm{qgr}\, A_{\mathbb{R}}}(-, \omega_{\mathbb{R}}) \to \mathrm{Ext}^{1-i}_{\mathrm{qgr}\, A_{\mathbb{R}}}(\mathcal{A}_{\mathbb{R}}, -)^{\vee}, \quad i = 0, 1.$$

Therefore

$$E = \mathrm{Hom}(\omega_{\mathbb{R}}, \omega_{\mathbb{R}}) \cong \mathrm{Ext}^1(\mathcal{A}_{\mathbb{R}}, \omega_{\mathbb{R}})^{\vee} \cong \mathrm{Hom}(\mathcal{A}_{\mathbb{R}}, \mathcal{A}_{\mathbb{R}}). \tag{2.22}$$

By assumption, $B = \Gamma_{\mathbb{R}}(\mathcal{A}_{\mathbb{R}})_0 = \mathrm{Hom}_{\mathrm{qgr}\, A_{\mathbb{R}}}(\mathcal{A}_{\mathbb{R}}, \mathcal{A}_{\mathbb{R}})$ is a simple $\mathbb{R}$-algebra, whose dimension over $\mathbb{R}$ is finite because $\mathrm{Proj}\, A_{\mathbb{R}}$ is H-finite. The isomorphism in (2.22) is an isomorphism of real vector spaces, so it only implies that E has the same finite dimension as B, but it does not ensure that $E \cong B$ as rings. To verify this, we still have to work a little bit.

Left multiplication λ_b on $\Gamma_{\mathbb{R}}(\omega_{\mathbb{R}})$ induces the ring homomorphism

$$\lambda : B \to E' = \mathrm{End}_{\mathrm{Gr}\, A_{\mathbb{R}}}\left(\Gamma_{\mathbb{R}}(\omega_{\mathbb{R}})\right)^{\circ}, \ b \mapsto \lambda_b.$$

Since $E' \neq 0$, the two-sided ideal $\ker \lambda$ of the simple ring B is proper. This implies that λ is injective. On the other hand by Lemma 2.1.2(ii), the representing functor $\Gamma_{\mathbb{R}} : \mathrm{QGr}\, A_{\mathbb{R}} \to \mathrm{Gr}\, A_{\mathbb{R}}$ is fully faithful, thus the rings E and E' are isomorphic. But λ is also a homomorphism of real vector spaces, so it follows from (2.22) that λ is in fact an isomorphism, which shows that the rings E and B are isomorphic. Hence E is indeed a finite dimensional simple $\mathbb{R}$-algebra, and the theorem is established. □

We proceed now with the definition of the intersection of an arithmetic line bundle $\overline{\mathcal{L}}$ with an arithmetic vector bundle $\overline{\mathcal{E}}$ on a noncommutative arithmetic surface $\mathrm{Proj}\, A$. We define the intersection of $\overline{\mathcal{L}}$ with $\overline{\mathcal{E}}$ to be

$$\langle \overline{\mathcal{L}}, \overline{\mathcal{E}} \rangle = \lambda(\overline{\mathcal{L}}, \overline{\mathcal{E}}) \otimes_{\mathbb{Z}} \lambda(\overline{\mathcal{L}}, \overline{\mathcal{A}})^{-1} \otimes_{\mathbb{Z}} \lambda(\overline{\mathcal{E}})^{-1} \otimes_{\mathbb{Z}} \lambda(\overline{\mathcal{A}}). \tag{2.23}$$

This is an arithmetic line bundle on $\mathrm{Spec}\, \mathbb{Z}$. The *intersection number* $(\overline{\mathcal{L}}, \overline{\mathcal{E}})$ is now simply the negative of the arithmetic (or Arakelov) degree of $\langle \overline{\mathcal{L}}, \overline{\mathcal{E}} \rangle$, i.e.

$$(\overline{\mathcal{L}}, \overline{\mathcal{E}}) = -\widehat{\deg}\left(\langle \overline{\mathcal{L}}, \overline{\mathcal{E}} \rangle\right). \tag{2.24}$$

The minus sign occurs for the following reason. If $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ are two arithmetic line bundles on a commutative arithmetic surface X, then

$$\lambda(\overline{\mathcal{L}}, \overline{\mathcal{M}}) \cong \lambda(\overline{\mathcal{O}}_X, \overline{\mathcal{L}}^{-1} \otimes \overline{\mathcal{M}}) = \lambda(\overline{\mathcal{L}}^{-1} \otimes \overline{\mathcal{M}}).$$

Putting this into (2.23) yields

$$\langle \overline{\mathcal{L}}, \overline{\mathcal{M}} \rangle \cong \lambda(\overline{\mathcal{L}}^{-1} \otimes \overline{\mathcal{M}}) \otimes \lambda(\overline{\mathcal{L}}^{-1})^{-1} \otimes \lambda(\overline{\mathcal{M}})^{-1} \otimes \lambda(\overline{\mathcal{O}}_X).$$

Comparing the right hand side with (2.10), we see that it computes the intersection of $\overline{\mathcal{L}}^{-1}$ and $\overline{\mathcal{M}}$. Since $\widehat{\deg}\left(\langle \overline{\mathcal{L}}^{-1}, \overline{\mathcal{M}} \rangle\right) = -\widehat{\deg}\left(\langle \overline{\mathcal{L}}, \overline{\mathcal{M}} \rangle\right)$, this explains the minus sign in (2.24).

2.3.3 Riemann-Roch theorem and formula

We have the following Riemann-Roch theorem:

Theorem 2.3.4. *Let $\overline{\mathcal{L}}$ be an arithmetic line bundle on a noncommutative arithmetic surface* $\operatorname{Proj} A$*, and suppose that* $\operatorname{Proj} A_{\mathbb{R}}$ *satisfies Serre duality. If* $\operatorname{End}_{\operatorname{qgr} A_{\mathbb{R}}}(\mathcal{A}_{\mathbb{R}})$ *is a simple ring, there is an isomorphism*

$$\lambda(\overline{\mathcal{L}})^{\otimes -2} \otimes \lambda(\overline{\mathcal{A}})^{\otimes 2} \cong \langle \overline{\mathcal{L}}, \overline{\mathcal{L}} \rangle \otimes \langle \overline{\mathcal{L}}, \overline{\omega} \rangle^{-1}. \tag{2.25}$$

Proof. Once

$$\lambda(\overline{\mathcal{L}}, \overline{\mathcal{L}}) \cong \lambda(\overline{\mathcal{A}}) \tag{2.26}$$

is established, the theorem follows immediately. Indeed, if (2.26) holds then

$$\begin{aligned} \langle \overline{\mathcal{L}}, \overline{\mathcal{L}} \rangle &= \lambda(\overline{\mathcal{L}}, \overline{\mathcal{L}}) \otimes \lambda(\overline{\mathcal{L}}, \overline{\mathcal{A}})^{-1} \otimes \lambda(\overline{\mathcal{L}})^{-1} \otimes \lambda(\overline{\mathcal{A}}) \\ &\cong \lambda(\overline{\mathcal{A}})^{\otimes 2} \otimes \lambda(\overline{\mathcal{L}}, \overline{\mathcal{A}})^{-1} \otimes \lambda(\overline{\mathcal{L}})^{-1}. \end{aligned} \tag{2.27}$$

On the other hand by Theorem 2.3.3, we have $\lambda(\overline{\mathcal{L}}, \overline{\omega}) \cong \lambda(\overline{\mathcal{L}})$ and $\lambda(\overline{\omega}) \cong \lambda(\overline{\mathcal{A}})$, whence

$$\begin{aligned} \langle \overline{\mathcal{L}}, \overline{\omega} \rangle^{-1} &= \lambda(\overline{\mathcal{L}}, \overline{\omega})^{-1} \otimes \lambda(\overline{\mathcal{L}}, \overline{\mathcal{A}}) \otimes \lambda(\overline{\omega}) \otimes \lambda(\overline{\mathcal{A}})^{-1} \\ &\cong \lambda(\overline{\mathcal{L}})^{-1} \otimes \lambda(\overline{\mathcal{L}}, \overline{\mathcal{A}}). \end{aligned} \tag{2.28}$$

Combining (2.27) and (2.28) yields (2.25).

So let us prove (2.26). Since $\mathcal{L}_{\mathbb{R}}$ is invertible, there is an autoequivalence $t_{\mathbb{R}}$ of $\operatorname{qgr} A_{\mathbb{R}}$ such that $t_{\mathbb{R}}(\mathcal{A}_{\mathbb{R}}) \cong \mathcal{L}_{\mathbb{R}}$. But every equivalence is a fully faithful functor, therefore the rings $E = \operatorname{End}_{\operatorname{qgr} A_{\mathbb{R}}}(\mathcal{L}_{\mathbb{R}})$ and $\operatorname{End}_{\operatorname{qgr} A_{\mathbb{R}}}(\mathcal{A}_{\mathbb{R}})$ are isomorphic, which shows that E is a finite dimensional simple $\mathbb{R}$-algebra. If $\overline{\mathcal{L}} = (\mathcal{L}, \beta)$ then, in terms of the ring structure of E, $\beta_* : \mathrm{End}_{\mathrm{qgr}\, A_{\mathbb{R}}}(\mathcal{L}_{\mathbb{R}}) \to \mathrm{End}_{\mathrm{qgr}\, A_{\mathbb{R}}}(\mathcal{L}_{\mathbb{R}})$ is left multiplication by β and likewise, β^* is right multiplication by β. Hence it follows from Lemma 2.3.2 that $\det(\beta_*) = \det(\beta^*)$, whence $\det\big((\beta^{-1})^*\beta_*\big) = 1$, which shows

$$\det \mathrm{Hom}(\overline{\mathcal{L}}, \overline{\mathcal{L}}) \cong \det \mathrm{Hom}(\overline{\mathcal{A}}, \overline{\mathcal{A}}). \tag{2.29}$$

The same argument also applies to the bundle $\mathrm{Ext}(\overline{\mathcal{L}}, \overline{\mathcal{L}})$. More precisely, the abelian group $\mathrm{Ext}^1_{\mathrm{qgr}\, A_{\mathbb{R}}}(\mathcal{L}, \mathcal{L})$ has a natural E-bimodule structure for which $(\beta_*)_1$ is left multiplication by β, and β_1^* is right multiplication by β. Since E is a finite dimensional simple $\mathbb{R}$-algebra, it thus follows from Lemma 2.3.2 that $\det\left((\beta_*)_1\right) = \det\left((\beta)_1^*\right)$, whence

$$\det\big((\beta^{-1})_1^*(\beta_*)_1\big) = 1. \tag{2.30}$$

On the other hand since $\mathcal{L}_{\mathbb{R}}$ is invertible, there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{qgr}\, A_{\mathbb{R}}}\left(\mathcal{L}_{\mathbb{R}} \otimes \mathcal{L}_{\mathbb{R}}^{-1}, -\right) \cong \mathrm{Hom}_{\mathrm{qgr}\, A_{\mathbb{R}}}\left(\mathcal{A}_{\mathbb{R}}, -\right),$$

and applying Lemma 2.3.1 to the automorphism $\alpha : \omega_{\mathbb{R}} \to \omega_{\mathbb{R}}$ yields

$$\det \mathrm{Hom}_{\mathrm{qgr}\, A_{\mathbb{R}}}\left(\mathcal{L}_{\mathbb{R}} \otimes \mathcal{L}_{\mathbb{R}}^{-1}, \alpha\right) = \det \mathrm{Hom}_{\mathrm{qgr}\, A_{\mathbb{R}}}\left(\mathcal{A}_{\mathbb{R}}, \alpha\right). \tag{2.31}$$

Combining (2.30) and (2.31) shows

$$\det \mathrm{Ext}\left(\overline{\mathcal{L}}, \overline{\mathcal{L}}\right) \cong \det \mathrm{Ext}\left(\overline{\mathcal{A}}, \overline{\mathcal{A}}\right),$$

which together with (2.29) establishes (2.26) and thus completes the proof. □

To get a Riemann-Roch formula, we still have to introduce the Euler characteristic of an arithmetic vector bundle on a noncommutative arithmetic surface. Let M be a finitely generated $\mathbb{Z}$-module and suppose that M has a volume. Recall the Euler characteristic

$$\chi(M) = -\log \mathrm{vol}(M_{\mathbb{R}}/M) + \log |M_{tor}|.$$

Hence, if $\overline{E}$ is a hermitian vector bundle on $\mathrm{Spec}\, \mathbb{Z}$, then $\chi(\overline{E}) = \widehat{\deg}\, \overline{E}$. Now, in analogy with [La, p. 112], we define the *Euler characteristic* of an arithmetic vector bundle $\overline{\mathcal{E}}$ on a noncommutative arithmetic surface $\mathrm{Proj}\, A$ to be

$$\chi(\overline{\mathcal{E}}) = \chi\left(\mathrm{Hom}(\overline{\mathcal{A}}, \overline{\mathcal{E}})\right) - \chi\left(\mathrm{Ext}(\overline{\mathcal{A}}, \overline{\mathcal{E}})\right) = \widehat{\deg}\, \lambda(\overline{\mathcal{E}}).$$

Using this notion, taking degrees in (2.25) yields the Riemann-Roch formula

$$\chi(\overline{\mathcal{L}}) = \frac{1}{2}\big((\overline{\mathcal{L}}, \overline{\mathcal{L}}) - (\overline{\mathcal{L}}, \overline{\omega})\big) + \chi(\overline{\mathcal{A}}).$$

Note that this formula looks exactly like the Riemann-Roch formula for hermitian line bundles on commutative arithmetic surfaces, cf. [Mo, Thm. 6.13].

A Quotient Categories

In this appendix, we briefly resume the construction of quotient categories and list some important results. Two good references for quotient categories are Gabriel's article [G] and Popescus's book [Po].

A non-empty full subcategory T of an abelian category A is called *dense* if for all short exact sequences $0 \to A' \to A \to A'' \to 0$ in A, A belongs to T if and only if both A' and A'' do. In particular, the zero object is in T. An object A of A is called *torsion* if it lies in T and *torsion-free* if its only subobject belonging to T is the zero object.

Definition A.1. *Let* T *be a dense subcategory of an abelian category* A. *The quotient category* A/T *is defined as follows:*

- *its objects are the objects of* A;
- *if* A *and* B *are objects of* A *then*

$$\mathrm{Hom}_{\mathsf{A}/\mathsf{T}}(A, B) := \varinjlim \mathrm{Hom}_{\mathsf{A}}(A', B/B'),$$

  where the direct limit is taken over all subobjects A' *of* A *and all subobjects* B' *of* B *with the property that* A/A' *and* B' *belong to* T;

- *the composition of morphisms in* A/T *is induced by that in* A.

Of course one has to show that this definition makes sense, cf. [Po, Thm. 4.3.3]. Every quotient category comes with a quotient functor

$$\pi : \mathsf{A} \longrightarrow \mathsf{A}/\mathsf{T}$$

defined by $\pi(A) = A$ on objects and $\pi(f)$ as the image of f in the direct limit on morphisms. The quotient functor satisfies $\pi(A) \cong 0$ if and only if A belongs to T. This is a direct consequence of Lemme 2 in [G, Chap. III]. Furthermore, quotient categories together with their quotient functors satisfy the following universality:

Theorem A.2. *Let* T *be a dense subcategory of an abelian category* A. *Then* *(i)* *the quotient category* A/T *is abelian and the quotient functor* $\pi : \mathsf{A} \to \mathsf{A}/\mathsf{T}$ *is exact;*

(ii) *and if* $F : \mathsf{A} \to \mathsf{D}$ *is an exact functor to another abelian category* D *such that* $F(A) \cong 0$ *whenever* A *is in* T, *then there is a unique functor* $G : \mathsf{A}/\mathsf{T} \to \mathsf{D}$ *such that* $F = G \circ \pi$, *that is such that the diagram*

$$\begin{tikzcd} \mathsf{A} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathsf{D} \\ \mathsf{A}/\mathsf{T} \arrow[ur, dashed, "G"'] & \end{tikzcd}$$

commutes.

Proof. [G, Prop. III.1, Cor. III.2; Po, Thm. 4.3.8, Cor. 4.3.11]. □

Following [Po], a dense subcategory T of an abelian category A is called a *localizing* subcategory if the quotient functor $\pi : \mathsf{A} \to \mathsf{A}/\mathsf{T}$ has a right adjoint. We write σ for the right adjoint and call it the *section functor.* The key result is that when A has injective envelopes, T is a localizing subcategory if and only if it is closed under direct sums, cf. [Po, 4.4]. The following proposition summarises some important properties of the section functor.

Proposition A.3. *Suppose that* T *is a localizing subcategory of the abelian category* A. *Let* $\pi : \mathsf{A} \to \mathsf{A}/\mathsf{T}$ *and* $\sigma : \mathsf{A}/\mathsf{T} \to \mathsf{A}$ *denote the quotient and the section functors. Let* $\mathcal{F}$ *be an object of* A/T. *Then*

(i) $\sigma\mathcal{F}$ *is torsion-free;*

(ii) *if* $f \in \mathrm{Hom}_{\mathsf{A}}(M, N)$ *and* $\pi(f)$ *is an isomorphism, then the induced map* $f^* : \mathrm{Hom}_{\mathsf{A}}(N, \sigma\mathcal{F}) \to \mathrm{Hom}_{\mathsf{A}}(M, \sigma\mathcal{F})$ *is an isomorphism as well;*

(iii) *the map* $\pi : \mathrm{Hom}_{\mathsf{A}}(M, \sigma\mathcal{F}) \to \mathrm{Hom}_{\mathsf{A}/\mathsf{T}}(\pi M, \pi\sigma\mathcal{F})$ *is an isomorphism for every object* M *of* A;

(iv) $\pi\sigma \cong \mathrm{id}_{\mathsf{A}/\mathsf{T}}$;

(v) σ *is fully faithful.*

Proof. [G, Lemme III.2.1, Lemme III.2.2, Prop. III. 2.3.a]. □

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Curriculum Vitae

19.12.1975	Born in Frauenfeld, Switzerland
1983-1987	Freie Volksschule Lenzburg
1987-1988	Primarschule Neuenhof
1988-1992	Bezirksschule Wettingen
1992-1996	Kantonsschule Baden, attaining Matura Typus C
1996-2001	Studies in mathematics, political economics and international law at the University of Zürich, attaining the diploma in mathematics, diploma thesis: "Sukzessive Minima und Arakelov-Steigungen" supervised by Prof. Dr. Gisbert Wüstholz
1999-2001	Research assistant at the Socioeconomic Institute of the University of Zürich
2002-2006	Ph.D. studies in mathematics at the ETH Zürich under the supervision of Prof. Dr. Gisbert Wüstholz, teaching assistant at the Department of Mathematics of the ETH Zürich