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A Proof of Luttinger's Theorem

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presented by
ANTOINE PRAZ
Dip. Phys. ETHZ
born June 22, 1976
citizen of Switzerland

accepted on the recommendation of
Prof. Dr. Eugene Trubowitz, examiner
Prof. Dr. Horst Knörrer, co-examiner

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A Alexandre et Marina

Abstract

"Luttinger's theorem" states that the volume enclosed by the Fermi surface of a system of interacting fermions at zero temperature is independent of the strength of the interaction. We give a rigorous proof of Luttinger's theorem to all orders of perturbation theory, based on an argument due to E. Trubowitz.

The first step consists in the analysis of a system of weakly interacting fermions in a finite volume. The dual lattice provides a natural infrared cutoff which allows the use of nonperturbative methods, developed by J. Feldman, H. Knörrer and E. Trubowitz for the insulator. The second step implements renormalization group ideas to control the thermodynamic limit of the Green functions, order by order in the interaction strength.

Résumé

Dans un système de fermions n'interagissant pas entre eux, l'état fondamental consiste en un produit de tous les états simples, dont l'énergie est inferieure a l'énergie de Fermi. Cette configuration est une conséquence du principe d'exclusion de Pauli, qui interdit à deux fermions d'occuper un meme état.

La surface délimitant les états occupés des états non-occupés dans l'espace des moments est appelée la surface de Fermi. Par un calcul simple, on vérifie aisément que la densité de fermions est, à un facteur 2 près, le volume contenu à l'intérieur de la surface de Fermi.

Lorsque les fermions interagissent entre eux, cette définition de la surface de Fermi perd son sens. L'état fondamental du système ne s'exprime plus comme le produit d'états simples. Si le nombre moyen de particules par état est une fonction discontinue du moment à temperature zero, alors le système est un liquide de Fermi. Dans ce cas, la surface de Fermi est la surface où, dans l'espace des moments, le nombre moyen de particules par état est discontinu.

Selon le théorème de Luttinger, le volume contenu à l'interieur de la surface de Fermi d'un système de fermions interagissant entre eux ne depend pas de l'interaction. Une preuve perturbative rigoureuse de ce théorème, basée sur un argument de E. Trubowitz, est donnée dans ce travail.

Pour ce faire, un système de fermions interagissant entre eux est d'abord analysé dans un volume fini. La structure discrète de l'espace dual fournit une coupure infrarouge naturelle, ce qui permet d'utiliser la méthode non-perturbative développée par J. Feldman, H. Knörrer et E. Trubowitz pour les isolants. Le groupe de renormalisation permet finallement de contrôler la limite thermodynamique des fonctions de Green à chaques ordres.

Contents

Chapter 1

Introduction and Overview

In a system of noninteracting fermions at zero temperature, the ground state is given by the product of all single-particle momentum states, with one-particle energy less than the Fermi energy. This configuration is a consequence of Pauli's exclusion principle, which forbids the one-particle states to be occupied with more than one fermion.

The surface in momentum space representing the limit of occupation of the one-particle momentum states is called the Fermi surface. A simple computation shows that the density of fermions in the system is given, up to a factor two due to spin multiplicity, by the volume enclosed in the Fermi surface.

In a system of interacting fermions, this definition of the Fermi surface becomes meaningless, since the ground state is no more given by a product of single-particle states. If the mean occupation number in momentum space exhibits a sharp discontinuity at zero temperature, we refer to the system as a Fermi liquid. In that case, the Fermi surface is the surface on which the discontinuity in the mean occupation number occurs.

Luttinger's theorem, first formulated in 1960 in [1], states that keeping the density fixed, the volume enclosed in the Fermi surface is independent of the interaction strength.

The aim of the present work is to give a rigorous proof of Luttinger's theorem to all orders in perturbation theory. Luttinger's theorem follows directly from the conservation of the particle number under changes in the interaction strength.

1.1 Field Theory for Many-Fermion Systems

Consider a system of many spin $\frac{1}{2}$ fermions on a discrete torus $\mathbf{\Lambda}_L = \mathbb{Z}^d/L\mathbb{Z}^d$, $d = 2$ or 3, containing L^d points. Let $\sigma \in \{\uparrow, \downarrow\} \simeq \{-1, 1\}$ be the projection of the spin on the vertical axis, measured in units of $\frac{\hbar}{2}$. For $\mathbf{x} \in \mathbf{\Lambda}_L$ and $\sigma \in \{\uparrow, \downarrow\}$, let $c_\sigma^+(\mathbf{x})$ and $c_\sigma(\mathbf{x})$ be fermionic creation and annihilation operators obeying the anticommutation relations $\{c_\sigma^+(\mathbf{x}), c_{\sigma'}(\mathbf{x}')\} = \delta_{\sigma\sigma'}\delta_{\mathbf{x}\mathbf{x}'}$ and let $\mathcal{F}$ be the fermionic Fock space generated by

this algebra. The free Hamiltonian of the system is defined as

$$H_0 = \sum_{\substack{\mathbf{x},\mathbf{x}'\in\mathbf{\Lambda}_L \\ \sigma\in\{\uparrow,\downarrow\}}} c_\sigma^+(\mathbf{x})T(\mathbf{x}-\mathbf{x}')c_\sigma(\mathbf{x}')$$

where T is the hopping amplitude between sites of the lattice. Let A be an operator on $\mathcal{F}$, i.e a polynomial in the creation and annihilation operators. In the free-fermion approximation, the thermal expectation value of the operator A at zero temperature in the thermodynamic limit is

$$\langle A\rangle = \lim_{\mathbf{\Lambda}_L\to\mathbb{Z}^d}\lim_{\beta\to\infty}\frac{1}{Z_{\beta,\mathbf{\Lambda}_L}}\operatorname{tr}\left(e^{-\beta(H_0-\mu N)}A\right),$$

where N is the number operator on $\mathcal{F}$, $\beta = 1/T$ the inverse temperature and $Z_{\beta,\mathbf{\Lambda}_L} = \operatorname{tr} e^{-\beta(H_0-\mu N)}$ is the grand canonical partition function; μ is the chemical potential. The trace formula has the functional integral representation[1]

$$\langle A\rangle = \int \mathcal{A}(\bar\psi,\psi)d\mu_{C_{bare}}(\bar\psi,\psi),$$

where $\bar\psi_\sigma(x)$ and $\psi_\sigma(x)$ are Grassman fields with $x = (x^0,\mathbf{x}) \in \mathbb{R}\times\mathbb{Z}^d$, x^0 being the imaginary time, and $\mathcal{A}$ is the polynomial in $\bar\psi_\sigma(x)$ and $\psi_\sigma(x)$ corresponding to the normal ordered form of the operator A. The Grassman Gaussian measure is formally given by

$$d\mu_{C_{bare}}(\bar\psi,\psi) = \frac{1}{\mathcal{N}}e^{-\frac{1}{2}(\bar\psi,C_{bare}^{-1}\psi)}\prod_{\substack{x\in\mathbb{R}\times\mathbb{Z}^d \\ \sigma\in\{\uparrow,\downarrow\}}} d\bar\psi_\sigma(x)d\psi_\sigma(x),$$

where $\mathcal{N}$ is a normalization factor, and

$$(\bar\psi, C_{bare}^{-1}\psi) = \sum_{\sigma\in\{\uparrow,\downarrow\}}\int dx^0dy^0\sum_{\mathbf{x},\mathbf{y}\in\mathbb{Z}^d}\bar\psi_\sigma(x)C_{bare}^{-1}(x,y)\psi_\sigma(y)$$

with the inverse propagator

$$C_{bare}^{-1}(x,y) = \delta_{\sigma\sigma'}\delta(x^0-y^0)(\delta_{\mathbf{xy}}(\partial_{x^0}+\mu) - T(\mathbf{x}-\mathbf{y})).$$

In the thermodynamic limit, the momentum space is the first Brillouin zone $\mathbb{T}$, i.e. the torus $\mathbb{T} = \mathbb{R}^d/2\pi\mathbb{Z}^d$. Under the Fourier transform

$$\begin{aligned}\bar\psi_\sigma(x) &= \int_{\mathbb{R}\times\mathbb{T}}\frac{dp^{d+1}}{(2\pi)^{d+1}}e^{-ip_0x^0+i\mathbf{px}}\hat{\bar\psi}_\sigma(p)\\ \psi_\sigma(x) &= \int_{\mathbb{R}\times\mathbb{T}}\frac{dp^{d+1}}{(2\pi)^{d+1}}e^{ip_0x^0-i\mathbf{px}}\hat\psi_\sigma(p)\end{aligned}$$

[1]For a complete introduction into this subject, see [5].

the bilinear forms in the measure $d\mu_{C_{bare}}(\bar{\psi},\psi)$ becomes, dropping the hats on the Fourier transforms,

$$(\bar{\psi}, C_{bare}^{-1}\psi) = \sum_{\sigma\in\{\uparrow,\downarrow\}} \int \frac{dp^{d+1}}{(2\pi)^{d+1}} \bar{\psi}_\sigma(p)(ip_0 - \varepsilon(\mathbf{p}) + \mu)\psi_\sigma(p),$$

where the band function

$$\varepsilon(\mathbf{p}) = \sum_{\mathbf{x}\in\mathbb{Z}^d} e^{-i\mathbf{p}\mathbf{x}} T(\mathbf{x})$$

has been introduced. The propagator

$$C_{bare}(p) = \frac{e^{ip_0 0_+}}{ip_0 - \varepsilon(\mathbf{p}) + \mu}$$

is singular for $p_0 = 0$ and $\mathbf{p} \in S^{(0)}$, where

$$S_\mu^{(0)} = \{\mathbf{p} \in \mathbb{T} : \varepsilon(\mathbf{p}) = \mu\}$$

is the free Fermi surface.

We turn to a system of interacting fermions on the finite lattice $\mathbf{\Lambda}_L$. Let the Hamiltonian of the system be $H = H_0 + \lambda V$, where the interaction is given by a two-particle translation invariant potential v:

$$V = \frac{1}{2} \sum_{\substack{\mathbf{x},\mathbf{x}'\in\mathbf{\Lambda} \\ \sigma,\sigma'\in\{\uparrow,\downarrow\}}} v_{\sigma\sigma'}(\mathbf{x}-\mathbf{x}') c_\sigma^+(\mathbf{x}) c_{\sigma'}^+(\mathbf{x}') c_{\sigma'}(\mathbf{x}') c_\sigma(\mathbf{x}).$$

The thermal expectation value at zero temperature of an operator A on $\mathcal{F}$ becomes

$$\langle A\rangle_L = \lim_{\beta\to\infty} \frac{1}{Z_{\beta,\mathbf{\Lambda}_L}} \operatorname{tr}\left(e^{-\beta(H-\mu N)} A\right),$$

where now $Z_{\beta,\mathbf{\Lambda}_L} = \operatorname{tr} e^{-\beta(H-\mu N)}$. The connected Green functions of the system on the finite lattice $\mathbf{\Lambda}_L$ are defined as

$$G_{2m}^{(L)}(\mathbf{x}_1,\sigma_1;\cdots;\mathbf{x}_{2m},\sigma_{2m}) =< c_{\sigma_1}^+(\mathbf{x}_1)\cdots c_{\sigma_m}^+(\mathbf{x}_m) c_{\sigma_{m+1}}(\mathbf{x}_{m+1})\cdots c_{\sigma_{2m}}(\mathbf{x}_{2m}) >_{L,connected} .$$

The functional representation

$$G_{2m}^{(L)}(x_1,\sigma_1;\cdots;x_{2m},\sigma_{2m}) = \frac{1}{Z_{\mathbf{\Lambda}_L}} \int e^{\lambda\mathcal{V}(\bar{\psi},\psi)} \bar{\psi}_{\sigma_1}(x_1)\cdots\psi_{\sigma_{2m}}(x_{2m}) d\mu_{C_{bare}}(\bar{\psi},\psi)\Big|_{connected}$$

does not exist in thermodynamic limit, i.e. in the limit $\mathbf{\Lambda}_L \to \mathbb{Z}^d$, reflecting the fact that the Fermi surface gets distorted by the interaction. The propagator in the Grassman Gaussian measure has therefore the wrong surface of singularities, which leads to unphysical divergences in the thermodynamic limit.

The renormalization procedure allows to cure the divergences on the level of perturbation theory. Formally, the bare dispersion relation is split in the interacting dispersion relation $e(\mathbf{p};\lambda)$ and a counterterm $K(\mathbf{p};\lambda,e)$, which is removed from the propagator and put in the interaction. Thus

$$\varepsilon(\mathbf{p})-\mu = e(\mathbf{p};\lambda)+K(\mathbf{p};\lambda,e).$$

In the mathematical analysis of the problem, we begin directly with the interacting dispersion relation $e(\mathbf{p})$ rather than the band function. The counterterm is then chosen such that the Fermi surface defined by the zero set of $e(\mathbf{p})$ remains fixed. The existence of the functional integral representation in perturbation theory in the infinite volume is then given by the following theorem:

Theorem 1.1: Let $e(\mathbf{p},\mu)$ be a dispersion relation and v be an interaction satisfying the following assumptions for $r\geq 3$:

A1 The interaction $\hat{v}_{\sigma\tau}\in C^r(\mathbb{T},\mathbb{C})$. The supremum norm over $\mathbb{T}$ of the first r derivatives of $\hat{v}_{\sigma\tau}$ is finite and $\hat{v}_{\sigma\tau}(\mathbf{p})=\hat{v}_{\tau\sigma}(-\mathbf{p})$.

A2 There is an interval $\mathcal{M}$ of positive numbers and a compact set $U\subset\mathbb{T}$, such that $\forall\mu\in\mathcal{M}$, $e_\mu\in C^r(U,\mathbb{R})$ and e_μ is at least once differentiable in μ. Further, the Fermi surface

$$S_\mu := \{\mathbf{p}\in\mathbb{T}\,|\,e_\mu(\mathbf{p})=0\}$$

is entirely in U, $S_\mu\subset U$ and $\nabla e_\mu(\mathbf{p})\neq 0$ for all $\mathbf{p}\in S_\mu$.

A3 For all $\mu\in\mathcal{M}$, the Fermi surface S_μ is strictly convex.

Then there is a counterterm $K(\mathbf{p},\lambda;e)$ defined as a formal power series in λ such that

(i) The connected Green functions generated by the generating functional

$$\mathcal{G}(\bar{\phi},\phi;\lambda,e)=\log\frac{1}{Z}\int e^{\lambda\mathcal{V}(\bar{\psi},\psi)+\mathcal{E}(\bar{\psi},\psi;\lambda,e)+(\bar{\phi},\psi)+(\bar{\psi},\phi)}d\mu_{C_e}(\bar{\psi},\psi)$$

are well-defined power series in λ in the thermodynamic limit.[2] Here $d\mu_{C_e}(\bar{\psi},\psi)$ is the Grassman Gaussian measure with covariance[3]

$$C_e(p)=\frac{e^{ip_0 0_+}}{ip_0-e(\mathbf{p})},$$

[2]The generating functional $\mathcal{G}$ is probably not analytic in the dispersion relation e, i.e. there is no formal power series of $\mathcal{G}$ in e. If $\mathcal{G}$ would be analytic in e, it would also be analytic in the band function $e+K$, which is in contradiction with the presence of infrared divergences.

[3]The lattice structure in position space furnishes an ultraviolet cutoff, since the momentum $\mathbf{p}$ is on the torus $\mathbb{T}$.

the interaction is

$$\mathcal{V}(\bar{\psi},\psi) = \sum_{\substack{\mathbf{x},\mathbf{y}\in\mathbb{Z}^d \\ \sigma,\tau\in\{\uparrow,\downarrow\}}} \int dx^0\, v_{\sigma\tau}(\mathbf{x}-\mathbf{y})\bar{\psi}_\sigma(x^0,\mathbf{x})\bar{\psi}_\tau(x^0,\mathbf{y})\psi_\tau(x^0,\mathbf{y})\psi_\sigma(x^0,\mathbf{x}),$$

and the counterterm

$$\mathcal{E}(\bar{\psi},\psi;\lambda,e) = \sum_{\substack{\mathbf{x},\mathbf{y}\in\mathbb{Z}^d \\ \sigma\in\{\uparrow,\downarrow\}}} \int dx^0\, \check{K}(\mathbf{x}-\mathbf{y};\lambda,e)\bar{\psi}_\sigma(x^0,\mathbf{x})\psi_\sigma(x^0,\mathbf{y}),$$

where $\check{K}$ is the inverse Fourier transform of K.

(ii) The self-energy $\Sigma(p,\lambda)$, defined as the formal power series in λ satisfying the equation

$$\hat{S}_2(p,\lambda) = \frac{1}{ip_0 - e(\mathbf{p}) - \Sigma(p;\lambda)},$$

where $\hat{S}_2$ is the two-point Green function, is $2+\epsilon$-times differentiable with respect to p for an $\epsilon > 0$. The counterterm is fixed by the renormalization condition

$$\Sigma(0,\mathbf{p};\lambda) = 0 \text{ for } \mathbf{p}\in S.$$

(iii) Further, there is a small ball $\mathcal{B}$ in the Banach space of the dispersion relations, such that the renormalization map

$$\begin{array}{rccc} R_\lambda: & \mathcal{B} & \to & \mathcal{B} \\ & e(\mathbf{p}) & \mapsto & e(\mathbf{p}) + K(\mathbf{p};\lambda,e) \end{array}$$

is invertible in the sense of formal power series.

Corollary 1.2: The occupation number

$$\begin{aligned} n_\sigma(\mathbf{p};\lambda) &= \lim_{\mathbf{\Lambda}_L\to\mathbb{Z}^d} \langle c_\sigma^+(\mathbf{p})c_\sigma(\mathbf{p})\rangle_L \\ &= \lim_{\tau\to 0_+} \int \frac{dp_0}{2\pi}\hat{S}_2(p,\lambda)e^{ip_0\tau} \end{aligned}$$

is a well-defined formal power series in λ, such that for all $R>0$, there is a $\lambda_R>0$ such that for all λ with $|\lambda|<\lambda_R$, the occupation number

$$n_\sigma^R(\mathbf{p};\lambda) = \sum_{r=0}^{R} \lambda^r n_{r,\sigma}(\mathbf{p})$$

has a jump on $S = \{\mathbf{p}\in\mathbb{T} : e_\mu(\mathbf{p}) = 0\}$.

We refer to the literature for a precise formulation of theorem 1.1 and its corollary 1.2. The proof of the existence of the Green functions is given in [7]. The invertibility of

the renormalization map is proved in [8],[9] and [10]. The proof of the corollary can be found in [11].

Theorem 1.3: The density of fermions, defined as the formal power series

$$\rho(\lambda) := \sum_{r\geq 0} \lambda^r \rho_r$$

where

$$\rho_r := \lim_{\tau\to 0_+} \sum_{\sigma\in\{\uparrow,\downarrow\}} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \hat{S}_{2,r}(k) e^{ik_0\tau}$$

exists, and is independent of the coupling constant λ:

$$\rho(\lambda) = \rho(0) = 2\,\mathrm{Vol}(S).$$

The proof of this theorem, which is the angle stone of the proof of Luttinger's theorem, is the main result of the present work.

1.2 Luttinger's Theorem

Definition 1.4: The physical (or interacting) Fermi surface S_μ of the system is the surface of discontinuity of the occupation number $n_\sigma(\mathbf{p}, \lambda)$.

Luttinger's Theorem: Let $\varepsilon(\mathbf{k}) - \mu$ be a dispersion relation satisfying the assumptions **A2** and **A3**. Assume further that the interaction $v_{\sigma\tau}$ between the fermions satisfies **A1**, and let $\bar{\rho}$ be the (given) density of fermions in the system.

Then the volume enclosed by the Fermi surface is independent of the interaction strength.

Proof of Luttinger's theorem: In the free-fermion approximation, the Fermi surface is

$$S_\mu^{(0)} = \{\mathbf{k} \in \mathbb{T} \,:\, \varepsilon(\mathbf{k}) = \mu\}.$$

Since by assumption $\varepsilon(\mathbf{k}) - \mu$ has convex level sets, there is a chemical potential μ_0 such that

$$\bar{\rho} = 2\,\mathrm{Vol}(S_{\mu_0}^{(0)}).$$

We turn now to the interacting system. For each μ' in a neighborhood of μ_0, there is an interacting dispersion relation

$$e_{int}(\mathbf{k}, \mu'; \lambda) = R_\lambda^{-1}(\varepsilon(\cdot) - \mu')(\mathbf{k})$$

defined as a formal power series $e_{int}(\mathbf{k}, \mu'; \lambda) = \sum_{r \geq 0} \lambda^r e_r(\mathbf{k}, \mu')$. For each $R > 0$, there is a $\lambda_R > 0$ such that for λ with $|\lambda| \leq \lambda_R$, the dispersion relation

$$e^R(\mathbf{k}, \mu'; \lambda) = \sum_{r=0}^{R} \lambda^r e_r(\mathbf{k}, \mu')$$

satisfies the assumptions **A2** and **A3**. For λ' with $|\lambda'| \leq |\lambda|$, consider the model defined by the generating functional

$$\mathcal{G}(\bar{\phi}, \phi; \lambda', e^R(\cdot; \lambda)) = \int e^{\lambda' \mathcal{V}(\bar{\psi}, \psi) + \mathcal{E}(\bar{\psi}, \psi; \lambda', e^R(\cdot; \lambda)) + (\bar{\phi}, \psi) + (\bar{\psi}, \phi)} d\mu_{C_{e^R(\cdot;\lambda)}}(\bar{\psi}, \psi),$$

which is well-defined by theorem 1.1 with e replaced by $e^R(\cdot; \lambda)$ and λ replaced by λ'. The Green functions of the physical model are obtained setting $\lambda' = \lambda$ in the generating functional $\mathcal{G}(\bar{\phi}, \phi; \lambda', e^R(\cdot; \lambda))$.[4]

By the corollary 1.2, we know that the occupation number corresponding to the model described by $\mathcal{G}(\bar{\phi}, \phi; \lambda', e^R(\cdot; \lambda))$ has a jump on the surface

$$S_{\mu'}^{(R)} = \{\mathbf{k} \in \mathbb{T} \,:\, e^R(\mathbf{k}, \mu'; \lambda) = 0\},$$

which is therefore the interacting Fermi surface up to the order R in perturbation theory. The theorem 1.3 implies that the density of the model is

$$\rho(\lambda'; e^R(\cdot; \lambda)) = \rho(0; e^R(\cdot; \lambda)) = 2\,\mathrm{Vol}(S_{\mu'}^{(R)}).$$

In order to achieve the right physical model, the chemical potential μ' has to be chosen such that the density of the system is just $\bar{\rho}$. Observe that $e^R(\mathbf{k}, \mu'; \lambda) = \varepsilon(\mathbf{k}) - \mu' + O(\lambda)$, such that for λ_R small enough, $\mathrm{Vol}(S_{\mu'}^{(R)})$ is a strictly increasing function of μ'. Thus, there is a μ near μ_0 (depending on λ) such that

$$\bar{\rho} = 2\,\mathrm{Vol}(S_{\mu}^{(R)}).$$

Hence,

$$\mathrm{Vol}(S_{\mu}^{(R)}) = \mathrm{Vol}(S_{\mu_0}^{(0)}),$$

and to each order R in perturbation theory, the volume enclosed in the Fermi surface is independent of the interaction strength. □

1.3 Sketch of the Proof of $\rho(\lambda) = \rho(0)$

In order to prove that the density is independent of the coupling constant λ, we consider for each $L \in \mathbb{N}$ the model on the finite lattice $\mathbf{\Lambda}_L = \mathbb{Z}_L^d$. The finite volume induces a

[4] Observe that the Green functions are probably not analytic in λ, since the Green functions are not analytic in the dispersion relation e^R.

natural infrared cutoff on the propagator, such that the model is analytic in the coupling constant λ. Further, the number operator has eigenvalues in the set of the natural numbers. Since the number operator commutes with the Hamiltonian, the ground state of the system is an eigenvector of the number operator, and if the ground state is non-degenerate, the expectation value of the number operator at zero temperature is an integer. On the other hand, the expectation value of the number operator is analytic in the coupling constant.

The expectation value of the number operator is therefore an analytic function of λ, taking values in a discrete set. It follows that the expectation value for the number operator is constant[5]. Thus, for each finite volume, the density is independent of the coupling constant λ.

In order to obtain the claim of theorem 1.3, the thermodynamic limit $L \to \infty$ has to be controlled. In this work, we construct for each L a counterterm, such that the Green functions on the finite volume tends for $L \to \infty$ to the Green functions defined in the theorem 1.1. Observe that the proof of convergence in the thermodynamic limit differs from the proof of theorem 1.1 given in [7], where a system in an infinite volume is considered from the beginning, and the Green functions are defined with an infrared cutoff. The limit in which the cutoff vanishes has to be controlled. This approach allows to work on a continuous Brillouin zone rather than on a discrete one.

In chapter two, we present a rigorous proof of 1.3, assuming the analyticity of the Green functions on the finite lattice, and their convergence to the Green functions of the physical model defined in theorem 1.1. The proof of the analyticity is given in chapter three, and the convergence is proved in chapters five and six.

1.3.1 The Thermodynamic Limit

In the thermodynamic limit, the natural infrared cutoff due to the lattice structure of the dual space is removed. The radius of analyticity of the Green functions shrinks to zero (before divergences appears in the computations), and naive (i.e. one-scale) non-perturbative analysis breaks down.

Still, one can try to compute the thermodynamic limit of the Green functions at fixed order in the coupling constant λ, i.e. the coefficients of the Taylor expansion of the Green functions in λ. As it is well known, the Green functions at fixed order in λ are obtained by the techniques of Feynman diagrams, the propagator

$$C(p) = \frac{e^{ik_0 0_+}}{ik_0 - e(\mathbf{k})}$$

being associated to the lines of the graphs (See [4] or [3]). Here $e(\mathbf{k})$ is the dispersion relation of the model, and $k_0 \in \mathbb{R}$ is the Euclidean frequency at $T = 0$. Singularities

[5]This argument is due to H. Knörrer and E. Trubowitz.

appears on the Fermi surface $S = \{\mathbf{k} \in \mathbb{T} : e(\mathbf{k}) = 0\}$ in the propagator for $k_0 = 0$. The accumulation of propagators in the two-legged subdiagrams leads to infrared divergences in the Green functions at fixed order in λ.

1.3.2 The Renormalization

The divergences appearing in the Green functions at fixed order in λ are not physical, but reflect the fact that the Fermi surface of a system of interacting fermions gets distorted. The diagrammatic expansion is then performed in the vicinity of the wrong surface of singularities.

The renormalization procedure allows to take the change in the dispersion relation due to interaction into account, and cures the infrared divergences. The renormalization consists first in a resummation of the two-legged contributions, in order to get expressions of the type

$$\frac{\Sigma_r(k_0, \mathbf{k})}{ik_0 - e(\mathbf{k})},$$

rather than bare propagators. Σ_r turns out to be a contribution to the self-energy. In a second step, an appropriated counterterm is introduced in order to remove the divergences. The renormalization condition

$$\Sigma_r^{Ren}(0, \mathbf{k}) = 0 \text{ for } \mathbf{k} \in S$$

allows to fix the counterterm, which is constructed projecting the self-energy Σ_r onto the Fermi surface S. The renormalization procedure can then formally be seen as the replacement

$$\Sigma_r(k_0, \mathbf{k}) \to \Sigma_r(k_0, \mathbf{k}) - \Sigma_r(0, \mathbf{k})|_{\mathbf{k} \in S} = \Sigma_r^{Ren}(k_0, \mathbf{k}).$$

Thus, the divergences in the propagator are compensated by the vanishing denominator obtained by the renormalization substraction.

Although, the projection onto the Fermi surface is not adequate in the finite volume. In order to preserve the periodic boundary conditions, the operation has to be performed on the dual lattice, rather than in the full Brillouin zone. Thus, we define a projection from points of the dual lattice $\mathbf{\Lambda}_L^\sharp$ onto points of the dual lattice that are close to the Fermi surface. The projection on the dual lattice is defined such that in the limit $L \to \infty$, the renormalization procedure described above is recovered.

Unfortunately, the counterterm defined with this projection fails in the suppression of the divergences. Precisely, the renormalized self-energy will not vanish on the Fermi surface, but rather be of order

$$\Sigma_r^{Ren}(0, \mathbf{k}) \sim \frac{2\pi}{L},$$

where $2\pi/L$ is the dual lattice spacing. Let $c_L > 0$ be the natural infrared cutoff provided by the lattice, i.e. $|e(\mathbf{k})| > c_L$ for $\mathbf{k} \in \mathbf{\Lambda}_L^\sharp$. Geometric considerations show that c_L is

typically bounded by $\sim L^{-d+1}$ in d dimension[6]. Thus

$$\frac{\Sigma_r^{Ren}(k_0, \mathbf{k})}{ik_0 - e(\mathbf{k})} \sim \frac{L^{-1}}{c_L} \sim L^{d-2},$$

and the infrared divergences remain present even after the renormalization. (For $d = 2$, only special choices of the dispersion relation achieve the infrared cutoff $c_L \sim L^{-1}$.) This is the reason why a regularization of the dispersion relation has to be implemented, in order to introduce an effective cutoff at scales $\sim 1/L$.

1.3.3 Scale Decomposition

Scale decomposition is the tool used to control the thermodynamic limit of the Green functions to all orders in perturbation theory. The basic idea of scale decomposition is to decompose the vicinity of the Fermi surface, where the divergences are located, into shells fitting into each other, the width of the shells getting smaller near the Fermi surface. The contributions of Feynman graphs at fixed scales are bounded by power counting, and careful summation over the scales shows the convergence of the Green functions.

Note that when the width of the shell is of order $1/L$, the scale decomposition gets sensitive to the lattice structure of the dual space. The power counting becomes worst, and the scale decomposition proves to be useless in that case. The infrared cutoff implemented at energy $\sim 1/L$ stops the scale decomposition in the $1/L$ neighborhood of the Fermi surface, avoiding this problem.

1.4 Spin Magnetization

As in the original work of Luttinger [1], the argument proving the theorem 1.3 can be extend to the spin magnetization. It follows that the spin magnetization is a function depending only on the (physical) dispersion relation.

Consider a system of fermions in a weak, constant magnetic field h, oriented along the vertical axis, and assume that the effect of the magnetic field on the orbital motion of the fermions is encoded in the band function $\varepsilon(\mathbf{k})$. The free Hamiltonian takes the form

$$H_0 = L^{-d} \sum_{\substack{\mathbf{k} \in \mathbf{\Lambda}_L^\sharp \\ \sigma \in \{\uparrow, \downarrow\}}} (\varepsilon(\mathbf{k}) - (-1)^\sigma \mu_B h)\, c_\sigma^+(\mathbf{k}) c_\sigma(\mathbf{k}),$$

[6]This can be proved as follows. Consider the lattice $\mathbb{Z}^d$, and a sphere of radius $k_F L$. Suppose that there is a shell of width $2c_L$ around the sphere, that contains no point of the lattice in its interior. The number of points in the shell is approximatively given by $const\, c_L L^{d-1}$. Since there are no point in the shell, we expect $const\, c_L L^{d-1} < 1$, and therefore the cutoff c_L has to be smaller than $\sim L^{1-d}$

where μ_B is the magnetic moment of the fermions and $(-1)^{\uparrow} = -(-1)^{\downarrow} = -1$. For the chemical potential μ, the band function ε defines two Fermi surfaces in the free fermion approximations,

$$S_\sigma^{(0)} = \{\mathbf{k} \in \mathbb{T} \,:\, \varepsilon(\mathbf{k}) - (-1)^\sigma \mu_B h = \mu\}$$

The density of the system in the free fermion approximation is then given by

$$\rho(0) = \mathrm{Vol}(S_\uparrow^{(0)}) + \mathrm{Vol}(S_\downarrow^{(0)}),$$

and the spin magnetization by

$$m(0) = \mu_B \left(\mathrm{Vol}(S_\uparrow^{(0)}) - \mathrm{Vol}(S_\downarrow^{(0)})\right).$$

We turn now to the system of interacting fermions. Theorem 1.1 can be extended to spin dependent dispersion relation, with a spin dependent counterterm. Theorem 1.3 now reads

Theorem 1.3bis: The density of fermions, defined as the formal power series

$$\rho(\lambda) = \sum_{r \geq 0} \rho_r \lambda^r$$

where

$$\rho_r = \lim_{\tau \to 0_+} \sum_{\sigma \in \{\uparrow,\downarrow\}} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \hat{S}_{\sigma,r}(k) e^{ik_0\tau}$$

exists, and is independent of the coupling constant λ. Moreover, the spin magnetization, defined by

$$m(\lambda) = \sum_{r \geq 0} m_r \lambda^r$$

with

$$m_r = \mu_B \lim_{\tau \to 0_+} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \left(\hat{S}_{\uparrow,r}(k) - \hat{S}_{\downarrow,r}(k)\right) e^{ik_0\tau}$$

in the sense of formal power series, is independent of the coupling constant λ.

Definition 1.4bis: The physical Fermi surfaces S_σ are the surface on which the occupation numbers $n_\sigma(\mathbf{p})$ have a jump.

Luttinger's Theorem: Let $\varepsilon(\mathbf{k}) - \mu$ be a dispersion relation that satisfies the assumptions **A2** and **A3**, and $v_{\sigma\tau}$ a potential satisfying the assumption **A1**. Assume further that the magnetic field h is small enough, such that $\varepsilon(\mathbf{k}) \pm \mu_B h - \mu$ also satisfy the assumptions **A2** and **A3**, and let $\bar{\rho}$ be the (given) density of the system.

Then the sum of the volumes enclosed in the Fermi surfaces is independent of the interaction strength,

$$\mathrm{Vol}(S_{\uparrow}^{(\lambda)}) + \mathrm{Vol}(S_{\downarrow}^{(\lambda)}) = \mathrm{Vol}(S_{\uparrow}^{(0)}) + \mathrm{Vol}(S_{\downarrow}^{(0)}).$$

Further, the spin magnetization is given by the difference between the volumes enclosed in the Fermi surfaces, times the magnetic moment of the fermions,

$$m = \mu_B \left(\mathrm{Vol}(S_{\uparrow}^{(\lambda)}) - \mathrm{Vol}(S_{\downarrow}^{(\lambda)}) \right).$$

Observe that the spin magnetization depends only of the interacting dispersion relation.

The proof of this version of Luttinger's theorem is identical to the proof given in the absence of magnetic field, and can be found in the appendix.

Chapter 2

The Results

In this chapter we present the main results that lead to the proof of theorem 1.3. We first construct a sequence of approximations for the model on the finite lattice, that converges in the thermodynamic limit to the physical model of theorem 1.1. We then present the main results concerning the analyticity and convergence of these approximations. The proof of these results follows in the next chapters.

2.1 The Model

Definition 2.1:

(i) For $L \in \mathbb{N}$, let $\mathbf{\Lambda}_L$ be the finite lattice defined by

$$\mathbf{\Lambda}_L = \mathbb{Z}_L^d,$$

where $\mathbb{Z}_L = \mathbb{Z}/L\mathbb{Z}$. The dual lattice of $\mathbf{\Lambda}_L$ is

$$\mathbf{\Lambda}_L^\sharp = \frac{2\pi}{L}\mathbb{Z}_L^d.$$

The lattice $\mathbf{\Lambda}_L$ and its dual $\mathbf{\Lambda}_L^\sharp$ contain L^d points.

(ii) On the finite lattice $\mathbf{\Lambda}_L^\sharp$, let $c_\sigma^+(\mathbf{k})$ and $c_\sigma(\mathbf{k})$ be the creation and annihilation operators satisfying the fermionic anticommutation relations

$$\begin{aligned}
\{c_\sigma^+(\mathbf{k}), c_{\sigma'}(\mathbf{k}')\} &= (2\pi L)^d \delta_{\sigma\sigma'}\delta_{\mathbf{k},\mathbf{k}'} \\
\{c_\sigma(\mathbf{k}), c_{\sigma'}(\mathbf{k}')\} &= 0 \\
\{c_\sigma^+(\mathbf{k}), c_{\sigma'}^+(\mathbf{k}')\} &= 0.
\end{aligned}$$

Remark 2.2: Let $\mathbb{T}$ be the first Brillouin zone, $\mathbb{T} = \mathbb{R}^d/2\pi\mathbb{Z}^d$ being the $d-$dimensional torus. There is an embedding of $\mathbf{\Lambda}_L^\sharp$ into $\mathbb{T}$ which maps the class of $\mathbf{k} = \frac{2\pi}{L}(k_1, \ldots, k_d)$ to $\frac{2\pi}{L}(k_1, \ldots, k_d) \in \mathbb{T}$, for $k_i \in \{0, \ldots, L-1\}$, $i = 1, \ldots, d$. In the limit $L \to \infty$, the embedding of the set $\mathbf{\Lambda}_L^\sharp$ in $\mathbb{T}$ tends to a dense subset of $\mathbb{T}$.

Definition 2.3: Let $\mathcal{M}$ be an interval of positive numbers. For $r \geq 2$ and $\mu \in \mathcal{M}$, let $e_\mu : \mathbb{T} \to \mathbb{R}$ be a piecewise C^r-function on the torus $\mathbb{T}$, called the dispersion relation, and v be a C^r-function, such that for all $\mathbf{k} \in \mathbf{\Lambda}_L^\sharp$, $v(\mathbf{k}) = v(-\mathbf{k})$.[1] Define the Hamiltonian of the system of fermions by

$$H^{(L)} - \mu N^{(L)} = L^{-d} \sum_{\substack{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp \\ \sigma\in\{\uparrow,\downarrow\}}} e_\mu(\mathbf{k}) c_\sigma^+(\mathbf{k}) c_\sigma(\mathbf{k}) + \lambda V^{(L)} + K^{(L)}(\lambda),$$

where the interaction is given by

$$V^{(L)} = \frac{1}{2L^{3d}} \sum_{\substack{\mathbf{k}_1,\cdots,\mathbf{k}_4\in\mathbf{\Lambda}_L^\sharp \\ \sigma,\tau\in\{\uparrow,\downarrow\}}} \delta_{\mathbf{k}_1+\mathbf{k}_3,\mathbf{k}_2+\mathbf{k}_4} v(\mathbf{k}_1 - \mathbf{k}_2) c_\sigma^+(\mathbf{k}_1) c_\tau^+(\mathbf{k}_3) c_\sigma(\mathbf{k}_2) c_\tau(\mathbf{k}_4).$$

and the counterterm

$$K^{(L)}(\lambda) = L^{-d} \sum_{\substack{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp \\ \sigma\in\{\uparrow,\downarrow\}}} K^{(L)}(\mathbf{k}, \lambda) c_\sigma^+(\mathbf{k}) c_\sigma(\mathbf{k})$$

has to be determined by the renormalization procedure. The number operator on the fermionic Fock space is defined by

$$N^{(L)} = L^{-d} \sum_{\substack{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp \\ \sigma\in\{\uparrow,\downarrow\}}} c_\sigma^+(\mathbf{k}) c_\sigma(\mathbf{k}).$$

The free Hamiltonian is given by

$$H_0^{(L)} - \mu N^{(L)} = L^{-d} \sum_{\substack{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp \\ \sigma\in\{\uparrow,\downarrow\}}} e_\mu(\mathbf{k}) c_\sigma^+(\mathbf{k}) c_\sigma(\mathbf{k}) + K^{(L)}(\lambda),$$

since e_μ is the interacting band function.

Remark 2.4: In position space, the creation and annihilation operators are obtained by Fourier transform:

$$c_\sigma^+(\mathbf{x}) = L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} c_\sigma^+(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}} \quad \text{and} \quad c_\sigma(\mathbf{x}) = L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} c_\sigma(\mathbf{k}) e^{-i\mathbf{k}\mathbf{x}},$$

[1] For simplicity, we consider a spin independent interaction. Further, we will often use $e(\mathbf{k})$ instead of $e_\mu(\mathbf{k})$.

satisfying the anticommutation relation

$$\{c_\sigma^+(\mathbf{x}), c_\tau(\mathbf{y})\} = (2\pi)^d \delta_{\sigma\tau}\delta_{\mathbf{xy}}.$$

For vanishing interaction, the free Hamiltonian can be expressed as

$$H_0^{(L)} = \sum_{\substack{\mathbf{x},\mathbf{y}\in\mathbf{\Lambda}_L \\ \sigma\in\{\uparrow,\downarrow\}}} T^{(L)}(\mathbf{x}-\mathbf{y}) c_\sigma^+(\mathbf{x}) c_\sigma(\mathbf{y}),$$

where

$$T^{(L)}(\mathbf{x}) = L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \varepsilon(\mathbf{k}) e^{-i\mathbf{kx}},$$

for the band function $\varepsilon(\mathbf{k})$. The dispersion relation is then $e_\mu(\mathbf{k}) = \varepsilon(\mathbf{k}) - \mu$. $T^{(L)}$ is a function on the lattice $\mathbf{\Lambda}_L$ that satisfies periodic boundary conditions. The interaction in position space is

$$V^{(L)} = \frac{1}{2} \sum_{\substack{\mathbf{x},\mathbf{y}\in\mathbf{\Lambda}_L \\ \sigma,\tau\in\{\uparrow,\downarrow\}}} V^{(L)}(\mathbf{x}-\mathbf{y}) c_\sigma^+(\mathbf{x}) c_\tau^+(\mathbf{y}) c_\sigma(\mathbf{x}) c_\tau(\mathbf{y}),$$

with the translation invariant potential

$$V^{(L)}(\mathbf{x}-\mathbf{y}) = L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} v(\mathbf{k}) e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}.$$

Finally, the position space representation of the number operator is

$$N^{(L)} = \sum_{\substack{\mathbf{x}\in\mathbf{\Lambda}_L \\ \sigma\in\{\uparrow,\downarrow\}}} c_\sigma^+(\mathbf{x}) c_\sigma(\mathbf{x}).$$

2.2 The Assumptions

Definition 2.5: The Fermi surface is the zero set of $e_\mu(\mathbf{k})$

$$S_\mu := \{\mathbf{k} \in \mathbb{T} \,|\, e_\mu(\mathbf{k}) = 0\}.$$

Assume the following assumptions:

A1 The interaction $v \in C^r(\mathbb{T}, \mathbb{C})$. The supremum norm over $\mathbb{T}$ of the first r derivatives of v is finite and $v(\mathbf{k}) = v(-\mathbf{k})$.

A2 There is an interval $\mathcal{M}$ of positive numbers, and a compact set $U \subset \mathbb{T}$, such that $\forall \mu \in \mathcal{M}$, $S_\mu \subset U$ and $e_\mu \in C^r(U, \mathbb{R})$. The dispersion relation is at least once differentiable in μ with $\partial_\mu e_\mu(\mathbf{k}) < 0$. Further, $\nabla e_\mu(\mathbf{p}) \neq 0$ for all $\mathbf{p} \in S_\mu$.

A3 For all $\mu \in \mathcal{M}$, the Fermi surface S_μ is strictly convex.

Remark 2.6:

(i) The assumption **A1** assures the interaction to be positive definite and short-range in position space.

(ii) The second assumption excludes singular points on the Fermi surface. The fact that the derivative of e_μ is negative reflect the fact that for $\lambda = 0$, $e_\mu(\mathbf{k}) = \varepsilon(\mathbf{k}) - \mu$.

(iii) Assumptions **A2** and **A3** imply the following bound:

Volume improvement estimate: For $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, let

$$I_2(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \sup_{\substack{\mathbf{q}\in\mathbb{T} \\ v_1,v_2\in\{1,-1\}}} \int_{\mathbb{T}\times\mathbb{T}} d^d\mathbf{p}_1 d^d\mathbf{p}_2 \, 1_{|e(\mathbf{p}_1)|<\varepsilon_1} 1_{|e(\mathbf{p}_2)|<\varepsilon_2} 1_{|e(v_1\mathbf{p}_1+v_2\mathbf{p}_2+\mathbf{q})|<\varepsilon_3}.$$

Then there is a constant $C_{vol} > 0$ and

$$\epsilon \geq \frac{d-1}{d-1/2} > \frac{1}{2}$$

such that

$$I_2(\varepsilon_1, \varepsilon_2, \varepsilon_3) \leq C_{vol}\varepsilon_1\varepsilon_2\varepsilon_3^\epsilon.$$

We refer to [7] for a proof of the volume improvement estimate.

Remark 2.7:

(i) The set $M_L = \{\mu \in \mathcal{M} \,|\, \exists\, \mathbf{k} \in \mathbf{\Lambda}_L^\sharp \text{ with } e_\mu(\mathbf{k}) = 0\}$ is finite.

(ii) For each $\mu \in \mathcal{M} \setminus M_L$, there is a constant c_L that depends on μ and L such that

$$\forall\, \mathbf{k} \in \mathbf{\Lambda}_L^\sharp,\ |e_\mu(\mathbf{k})| > c_L.$$

(iii) The set $M = \bigcup_{L\in\mathbb{N}} M_L$ is countable.

If we choose dispersion relations e_μ with $\mu \in \mathcal{M}/M$, the model on the finite lattice has an infrared cutoff denoted by c_L. This natural cutoff provided by the dual lattice structure is to small in order to prove the convergence of the model in thermodynamic limit. We will thus introduce a regularized dispersion relation which implements an effective infrared cutoff at scale $\sim 1/L$.

2.3 The Thermodynamics

Definition 2.8: We define the following norms on $\mathbb{T}$, $\mathbb{R}\times\mathbb{T}$, $\mathbf{\Lambda}_L^\sharp$ and $\mathbb{R}\times\mathbf{\Lambda}_L^\sharp$:

(i) For a function $f:(\mathbb{R}\times\mathbf{\Lambda}_L^\sharp\times\{\uparrow,\downarrow\})^n\to\mathbb{C}$, the supremum norm in momentum space is defined by

$$|f|_0=\sup_{\sigma_1,\ldots,\sigma_n\in\{\uparrow,\downarrow\}}\ \sup_{k_1,\ldots,k_n\in\mathbb{R}\times\mathbf{\Lambda}_L^\sharp}|f(k_1,\sigma_1,\ldots,k_n,\sigma_n)|.$$

(ii) For a function $f:(\mathbb{R}\times\mathbb{T}\times\{\uparrow,\downarrow\})^n\to\mathbb{C}$,

$$||f||_0=\sup_{\sigma_1,\ldots,\sigma_n\in\{\uparrow,\downarrow\}}\ \sup_{k_1,\ldots,k_n\in\mathbb{R}\times\mathbb{T}}|f(k_1,\sigma_1,\ldots,k_n,\sigma_n)|.$$

(iii) If the function $f:(\mathbb{R}\times\mathbb{T}\times\{\uparrow,\downarrow\})^n\to\mathbb{C}$ is differentiable, then the derivative norm is defined by

$$||f||_1=||f||_0+\max_{\substack{i=1,\ldots,n\\ \alpha=0,\ldots,d}}\ \sup_{\substack{k_1,\ldots,k_n\in\mathbb{R}\times\mathbb{T}\\ \sigma_1,\ldots,\sigma_n\in\{\uparrow,\downarrow\}}}|\partial_{i\alpha}f(k_1,\sigma_1,\ldots,k_n,\sigma_n)|,$$

where $\partial_{i\alpha}$ is the partial derivative with respect to the $\alpha-$component of k_i.

(iv) For a function $f:(\mathbb{R}\times\mathbb{T}\times\{\uparrow,\downarrow\})^n\to\mathbb{C}$, the integral norm (or L_1-norm) in momentum space is defined as

$$||f||'=\sum_{\sigma_1,\ldots,\sigma_n\in\{\uparrow,\downarrow\}}\int\frac{d^{d+1}p_1}{(2\pi)^{d+1}}\cdots\frac{d^{d+1}p_n}{(2\pi)^{d+1}}|f(p_1,\sigma_1,\ldots,p_n,\sigma_n)|.$$

For a function $f:(\mathbb{R}\times\mathbf{\Lambda}_L^\sharp\times\{\uparrow,\downarrow\})^n\to\mathbb{C}$, the L_1-norm is defined by

$$|f|'=L^{-dn}\sum_{\substack{\mathbf{k}_1,\ldots,\mathbf{k}_n\in\mathbf{\Lambda}_L^\sharp\\ \sigma_1,\ldots,\sigma_n\in\{\uparrow,\downarrow\}}}\int\frac{dk_{01}}{2\pi}\cdots\frac{dk_{0n}}{2\pi}|f(k_1,\sigma_1,\ldots,k_n,\sigma_n)|.$$

Definition 2.9: Let $A>4\pi\max\{1,||e||_1\}$. The set of possible counterterms is defined by

$$\mathcal{K}=\left\{u:\mathbf{\Lambda}_L^\sharp\times\mathbb{C}\to\mathbb{C}\,|\,u\text{ is analytic in }\lambda\in\mathbb{C}\text{ with }u(\mathbf{k},0)=0,\text{and }\sup_\lambda|u|_0\le\frac{A}{2L}\right\}.$$

By analytic in λ, we understand that there is a $\lambda_0>0$ such that u is analytic in λ in a ball of radius λ_0 around 0. The supremum is taken over all λ with $|\lambda|<\lambda_0$.

Definition 2.10:

(i) Let e be a dispersion relation defined in 2.3 and $A > 4\pi \sup\{1, ||e||_1\}$. Set

$$e^{(L)}(\mathbf{k}) = \begin{cases} e(\mathbf{k}), & \text{if } |e(\mathbf{k})| \geq \frac{A}{L} \\ \operatorname{sgn}(e(\mathbf{k}))\frac{A}{L}, & \text{if } |e(\mathbf{k})| < \frac{A}{L}. \end{cases}$$

(ii) For each $u \in \mathcal{K}$, define the regularized Hamiltonian

$$\overline{H}_u^{(L)} = \overline{H}_0^{(L)} + \lambda V^{(L)} + L^{-d} \sum_{\substack{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp \\ \sigma\in\{\uparrow,\downarrow\}}} u(\mathbf{k},\lambda) c_\sigma^+(\mathbf{k}) c_\sigma(\mathbf{k}),$$

where[2]

$$\overline{H}_0^{(L)} = L^{-d} \sum_{\substack{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp \\ \sigma\in\{\uparrow,\downarrow\}}} e^{(L)}(\mathbf{k}) c_\sigma^+(\mathbf{k}) c_\sigma(\mathbf{k}).$$

Remark 2.11:

(i) The dispersion relation $e^{(L)}$ is motivated by the scale decomposition that will be used in the section 5. At energy scales of orders of the lattice spacing, i.e. with $e(\mathbf{k}) \sim \frac{1}{L}$, the power counting gets worst. The dispersion relation $e^{(L)}$ introduces an effective cut-off at that scale, without modifying the ground state of the system.

(ii) For L big enough,

$$\frac{A}{L} \leq |e^{(L)}(\mathbf{k})| \leq E := ||e||_0.$$

Remark 2.12: For each $u \in \mathcal{K}$, the number operator commutes with the Hamiltonian $\overline{H}_u^{(L)}$:

$$[\overline{H}_u^{(L)}, N^{(L)}] = 0.$$

<u>Proof:</u> One easily verify that

$$[N, c_\sigma^+(\mathbf{k})] = (2\pi L)^d c_\sigma^+(\mathbf{k})$$

and

$$[N, c_\sigma(\mathbf{k})] = -(2\pi L)^d c_\sigma(\mathbf{k}).$$

[2]Observe that $\overline{H}_0^{(L)}$ does not correspond to the free Hamiltonian, since $e^{(L)}$ is the interacting dispersion relation.

Using $[A, BC] = [A, B]C + B[A, C]$, we get

$$[N, c_\sigma^+(\mathbf{k})c_\sigma(\mathbf{k})] = [N, c_\sigma^+(\mathbf{k})]c_\sigma(\mathbf{k}) + c_\sigma^+(\mathbf{k})[N, c_\sigma(\mathbf{k})] = 0.$$

Thus, the number operator commutes with the free Hamiltonian. Further, let $c_i^+ = c_{\sigma_i}^+(\mathbf{p}_i)$ and $c_i = c_{\sigma_i}(\mathbf{p}_i)$. Then

$$[c_\tau^+(\mathbf{k})c_\tau(\mathbf{k}), c_1^+c_2^+c_3c_4] = (2\pi L)^d(\delta_{\tau\sigma_1}\delta_{\mathbf{k}\mathbf{p}_1} + \delta_{\tau\sigma_2}\delta_{\mathbf{k}\mathbf{p}_2} - \delta_{\tau\sigma_3}\delta_{\mathbf{k}\mathbf{p}_3} - \delta_{\tau\sigma_4}\delta_{\mathbf{k}\mathbf{p}_4})c_1^+c_2^+c_3c_4.$$

Hence, summing over $\mathbf{k}$,

$$[N_\tau^{(L)}, V] = L^{-d} \sum_{\mathbf{p}_1,\ldots,\mathbf{p}_4} v_{\sigma_1\cdots\sigma_4}(\mathbf{p}_1, \ldots, \mathbf{p}_4)c_1^+c_2^+c_3c_4(\delta_{\tau\sigma_1} + \delta_{\tau\sigma_2} - \delta_{\tau\sigma_3} - \delta_{\tau\sigma_4})$$

where $N_\tau^{(L)} = L^{-d} \sum_{\mathbf{k}\in\Lambda_L^\sharp} c_\tau^+(\mathbf{k})c_\tau(\mathbf{k})$. Multiplying with the spin structure $\delta_{\sigma_1\sigma_2}\delta_{\sigma_3\sigma_4}$ contained in the interaction, we see that the right hand side vanishes, such that the interaction commutes with the number operator. □

Definition 2.13:

(i) The grand canonical partition function at zero temperature is

$$Z^{(L)} = \lim_{\beta\to\infty} \operatorname{tr} e^{-\beta\overline{H}_u^{(L)}},$$

where the trace is taken over the fermionic Fock space.

(ii) For an observable A, i.e. a polynomial in the fermionic operators, the thermal expectation value at zero temperature is given by

$$\begin{aligned}\langle A\rangle_L &= \lim_{\beta\to\infty} \frac{1}{Z_\beta^{(L)}} \operatorname{tr}(e^{-\beta\overline{H}_u^{(L)}} A) \\ &= \frac{1}{tr P_0} \operatorname{tr}(P_0 A),\end{aligned}$$

where P_0 is the spectral projection onto the zero energy states.

Lemma 2.14: For all $\mu \in \mathcal{M}$ and each $L \in \mathbb{N}$, there is a $\lambda_\mu > 0$ such that for all λ with $|\lambda| < \lambda_\mu$, the operator $\overline{H}_u^{(L)}$ has a non-degenerate ground state $\Omega^{(L)}$. In particular,

$$\langle A\rangle_L = \frac{(\Omega^{(L)}, A\Omega^{(L)})}{(\Omega^{(L)}, \Omega^{(L)})}$$

where $(\cdot, \cdot)$ denotes the scalar product on the Fock space $\mathcal{F}$.

Proof: First consider the free operator $\overline{H}_0^{(L)}$, whose state with minimal energy is

$$\Omega^{(L)} = \prod_{\substack{\mathbf{k} \text{ with } e^{(L)}(\mathbf{k}) \leq 0 \\ \sigma \in \{\uparrow,\downarrow\}}} c_\sigma^+(\mathbf{k})|0\rangle,$$

and its energy

$$E_0^{(L)} = 2L^{-d} \sum_{\substack{\mathbf{k} \in \mathbf{\Lambda}_L^\sharp \\ e^{(L)}(\mathbf{k}) \leq 0}} e^{(L)}(\mathbf{k}).$$

Observe that

$$\overline{H}_0^{(L)} c_\sigma^+(\mathbf{k}) = c_\sigma^+(\mathbf{k}) \overline{H}_0^{(L)} + e^{(L)}(\mathbf{k}) c_\sigma^+(\mathbf{k}),$$

and

$$\overline{H}_0^{(L)} c_\sigma(\mathbf{k}) = c_\sigma(\mathbf{k}) \overline{H}_0^{(L)} - e^{(L)}(\mathbf{k}) c_\sigma(\mathbf{k}).$$

Recalling that for $\mathbf{k}$ with $e^{(L)}(\mathbf{k}) > 0$, a quasi-particle is created applying $c_\sigma^+(\mathbf{k})$ to the ground state $\Omega^{(L)}$, and for $\mathbf{k}$ with $e^{(L)}(\mathbf{k}) \leq 0$, a quasi-particle is created applying $c_\sigma(\mathbf{k})$ to $\Omega^{(L)}$, we see that the energy of the one-particle state $c_\sigma^{(+)}(\mathbf{k})\Omega^{(L)}$ is given by

$$E_1 = E_0 + |e^{(L)}(\mathbf{k})| > E_0,$$

since $|e^{(L)}(\mathbf{k})| > A/L$. Proceeding inductively, one sees that the energy of the $n-$particles state $c_{\sigma_1}^{(+)}(\mathbf{k}_1) \cdots c_{\sigma_n}^{(+)}(\mathbf{k}_n)\Omega^{(L)}$ is given by

$$E_n = E_0 + \sum_{i=1}^{n} |e(\mathbf{k}_i)| > E_0.$$

By the orthogonality of the $n-$particle states, and the diagonality of $\overline{H}_0^{(L)}$, we deduce that the ground states is non-degenerate.

The full Hamiltonian is analytic in λ, hence it exists a $\lambda_\mu > 0$ such that for λ with $|\lambda| < \lambda_\mu$, the ground state of $\overline{H}_u^{(L)}$ is non-degenerated as well. □

Definition 2.15: The density of fermions in the system with Hamiltonian $\overline{H}_u^{(L)}$ is defined by

$$\rho^{(L)}(\lambda, u) = L^{-d} \sum_{\substack{\mathbf{k} \in \mathbf{\Lambda}_L^\sharp \\ \sigma \in \{\uparrow,\downarrow\}}} \langle c_\sigma^+(\mathbf{k}) c_\sigma(\mathbf{k}) \rangle_L.$$

2.4 Green Functions

The thermal expectation value of an observable can be expressed in terms of Gaussian Grassman integrals (See [5] for a rigorous proof). In particular, the generating functional

for the connected amputated temperature Green functions is formally given by

$$\mathcal{G}(\bar{\phi},\phi) = \log \frac{1}{Z} \int e^{\lambda \mathcal{V}(\bar{\psi}+\bar{\phi},\psi+\phi)} d\mu_{C_u}(\bar{\psi},\psi),$$

where $d\mu_C(\bar{\psi},\psi)$ is the Grassman Gaussian measure with covariance

$$C_u(x,\sigma,y,\tau) = \delta_{\sigma\tau} L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi} \frac{e^{-i<k,x-y>}}{ik_0 - e^{(L)}(\mathbf{k}) - u(\mathbf{k})}$$

on the Grassman algebra generated by the fermionic fields $\bar{\psi}_\sigma(x), \psi_\sigma(x)$. Here $< k, x >= -k_0 x^0 + \mathbf{kx}$ and $u \in \mathcal{K}$. The interaction is

$$\mathcal{V}(\bar{\psi},\psi) = \sum_{\substack{\mathbf{x},\mathbf{y}\in\mathbf{\Lambda}_L \\ \sigma,\tau\in\{\uparrow,\downarrow\}}} \int dx^0\, V^{(L)}(\mathbf{x}-\mathbf{y}) \bar{\psi}_\sigma(x^0,\mathbf{x}) \bar{\psi}_\tau(x^0,\mathbf{y}) \psi_\sigma(x^0,\mathbf{x}) \psi_\tau(x^0,\mathbf{y}).$$

The connected amputated Green functions in position space are defined by the formal Taylor expansion of $\mathcal{G}$:

$$\mathcal{G}(\bar{\phi},\phi) = \sum_{m\geq 1} \sum_{\substack{\mathbf{x}_1,\ldots,\mathbf{x}_{2m}\in\mathbf{\Lambda}_L \\ \sigma_1,\ldots,\sigma_{2m}\in\{\uparrow,\downarrow\}}} \int dx_1^0 \cdots dx_{2m}^0 G_{2m}^{(L)}(x_1,\sigma_1,\ldots,x_{2m},\sigma_{2m}) \cdot \bar{\phi}_{\sigma_1}(x_1) \cdots \phi_{\sigma_{2m}}(x_{2m}),$$

where $x_i = (x_i^0, \mathbf{x}_i) \in \mathbb{R}\times\mathbf{\Lambda}_L$. The two-points (non-amputated) connected Green function is defined by

$$S_{\sigma\tau}^{(L)}(x,y) = C_{\sigma\tau}(x,y) + \sum_{\substack{\mathbf{z}_1,\mathbf{z}_2\in\mathbf{\Lambda}_L \\ \sigma_1,\sigma_2\in\{\uparrow,\downarrow\}}} \int dz_1^0 dz_2^0\, C_{\sigma\sigma_1}(x,z_1) G_{\sigma_1\sigma_2}^{(L)}(z_1,z_2) C_{\sigma_2\tau}(z_2,y),$$

where $G_{\sigma_1\sigma_2}^{(L)}(z_1,z_2)$ is the two-points connected, amputated Green function.

Remark 2.16: The two-points connected Green functions $G^{(L)}(x,y)$ and $S^{(L)}(x,y)$ are independent of the spin indices, such that

$$G_{\sigma\tau}^{(L)}(x,y) = \delta_{\sigma\tau} G^{(L)}(x,y) \quad \text{and} \quad S_{\sigma\tau}^{(L)}(x,y) = \delta_{\sigma\tau} S^{(L)}(x,y).$$

Definition 2.17: Let $f(x_1,\sigma_1;\ldots;x_{2m},\sigma_{2m})$ be a translation invariant function on $(\mathbb{R}\times\mathbf{\Lambda}_L^\sharp\times\{\uparrow,\downarrow\})^{2m}$. The Fourier transform $\hat{f}$ of f is defined by

$$\delta(\sum_{i=1}^{2m} k_i) \hat{f}(k_1,\sigma_1;\ldots;k_{2m-1},\sigma_{2m-1},\sigma_{2m}) = \sum_{\mathbf{x}_1,\ldots,\mathbf{x}_{2m}\in\mathbf{\Lambda}_L} \int dx_1^0 \cdots dx_{2m}^0\, f(x_1,\sigma_1,\ldots,x_{2m},\sigma_{2m}) e^{-i(<k_1,x_1>+\cdots+<k_{2m},x_{2m}>)},$$

where $< k, x >= -k_0 x^0 + \mathbf{kx}$, and $\delta(k-p) = \delta(k_0 - p_0)(2\pi)^d \delta_{\mathbf{kp}}$.

Theorem 2.18: For all $L \in \mathbb{N}$, there is a $\lambda_0^{(L)} > 0$ that depends on L and μ such that

(i) For all $u \in \mathcal{K}$, the connected Green functions $\hat{G}_{2m}^{(L)}(k_1, \ldots, k_{2m}, \lambda; u)$ are analytic in λ with analyticity radius $\lambda_0^{(L)}$.

(ii) For all $u \in \mathcal{K}$, the density of fermions $\rho^{(L)}(\lambda, u)$ is analytic in λ with analyticity radius $\lambda_0^{(L)}$.

Remark 2.19: In terms of Green functions, the density of fermions is given by

$$\rho^{(L)}(\lambda, u) = \lim_{x^0 \to 0_+} 2L^{-d} \sum_{\mathbf{k} \in \mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi} \hat{S}^{(L)}(k_0, \mathbf{k}; \lambda) e^{ik_0 x^0},$$

where $\hat{S}^{(L)}$ is the Fourier transform of the connected (non-amputated) two-points Green's function $S^{(L)}$.

Theorem 2.20: For $L < \infty$ and all $u \in \mathcal{K}$, the density of fermions is independent of the interaction:

$$\rho^{(L)}(\lambda; u) = \rho^{(L)}(0).$$

<u>Proof:</u> By lemma 2.14, the ground state $\Omega^{(L)}(\lambda)$ for $\overline{H}^{(L)}$ is non-degenerate, and

$$N^{(L)} \Omega^{(L)}(\lambda) = \rho^{(L)}(\lambda) \Omega^{(L)}(\lambda).$$

Since the spectrum of $N^{(L)}$ is in $\mathbb{N}$, $\rho^{(L)}(\lambda)$ is an analytic function that have value in a discrete set. There is therefore a $\lambda_0 > 0$ such that $\lambda \mapsto \rho^{(L)}(\lambda)$ is a constant for $|\lambda| < \lambda_0$. By the analyticity of the density, $\rho^{(L)}$ is constant for all $|\lambda| < \lambda_0^{(L)}$. □

2.5 The Infinite Volume Limit

In [7], the thermodynamic limit is controlled in the following way. In a first step, the propagator is regularized by an infrared cutoff at energy scale M^I with $I < 0$ and $M > 1$. As long as this cutoff is present, the regularized Green functions G_{2m}^I are analytic in the thermodynamic limit. In a second step, the limit $I \to -\infty$ is controlled using renormalization group ideas.

Definition 2.21:

(i) For $I \in \mathbb{Z}_-$ and $M > 1$, let $f \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} f \cap [0, M^{2I-4}) = \emptyset$. Define the regularized propagator

$$C^I(p) = \frac{f(p_0^2 + e^2(\mathbf{p}))}{ip_0 - e(\mathbf{p})}.$$

(ii) The connected, amputated Green functions $\hat{G}^I_{2m}$ with infrared cutoff at scale M^I are defined as the formal power series

$$\hat{G}^I_{2m} := \sum_{r \geq 1} \lambda^r G^I_{2m,r},$$

where $G^I_{2m,r}$ are the renormalized, connected amputated Green functions at order r in λ. The self-energy $\Sigma^I = \sum_{r \geq 1} \lambda^r \Sigma^I_r$ is given as a formal power series by the equation

$$\Sigma^I(p) = (1 - \hat{G}^I(p) C^I(p))^{-1} \hat{G}^I(p),$$

where C^I is the propagator in the infinite volume, with infrared cut-off at scale M^I.

The following theorem is proved in [7]:

Theorem 2.22: Assume that e and v verify **A1-3**. Then there is a formal power series

$$K^I(\mathbf{p}) = \sum_{r \geq 1} \lambda^r K^I_r(\mathbf{p})$$

such that the following statements hold. For all $m \in \mathbb{N}$, the infrared limit $I \to -\infty$ of $G^I_{2m,r}$ exists. More precisely, for every $r \geq 1$, there are $\Sigma_r \in C^1(\mathbb{R} \times \mathbb{T}, \mathbb{C})$, $K_r \in C^1(\mathbb{T}, \mathbb{C})$, and $G_{2m,r}$, such that as $I \to -\infty$,

(i) $G^I_{2,r} \to G_{2,r}$ in the $||\cdot||_0$−norm,

(ii) $G^I_{2m,r} \to G_{2m,r}$ in the $||\cdot||'$−norm,

(iii) $\Sigma^I_r \to \Sigma_r$ in the $||\cdot||_1$−norm, and the renormalization condition

$$\Sigma_r(0, \mathbf{p}) = 0$$

is satisfied for all $\mathbf{p} \in S$.

(vi) $K^I_r \to K_r$ in $||\cdot||_1$.

Moreover, there are constants $\Gamma_{2m,r}$, $\tilde{\Gamma}_{2,r}$, κ_r and σ_r such that

$$\begin{array}{lcl} ||G_{2,r}||_0 & \leq & \tilde{\Gamma}_{2,r} \\ ||\Sigma_r||_1 & \leq & \sigma_r \\ ||K_r||_1 & \leq & \kappa_r \\ ||G_{2m,r}||' & \leq & \Gamma_{2m,r}. \end{array}$$

Denote

$$\hat{G}_{2m} := \sum_{r\geq 1} \lambda^r G_{2m,r}$$

the formal amputated Green functions in the infinite volume, and let $\hat{S}(p)$ be the formal (non-amputated), two-points Green's function in momentum space, defined by

$$\hat{S}(p) = C(p) + C(p)\hat{G}(p)C(p).$$

Corollary 2.23: The density of Fermions in the infinite volume, defined by the formal power series

$$\rho(\lambda) = \sum_{r\geq 0} \lambda^r \rho_r$$

where

$$\rho_r := \lim_{\tau\to 0_+} 2\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \hat{S}_r(k) e^{ik_0\tau}$$

exists.

Theorem 2.24: Assume **A1-3**. Then there is a sequence of counterterms $\left(K^{(L)}(\mathbf{k},\lambda)\right)_{L\in\mathbb{N}}$ in $\mathcal{K}$ that converges uniformly in $\mathbf{k}\in\mathbb{T}$ to the formal power series $K(\mathbf{k},\lambda)$ of theorem 2.22, such that

(i) For $L\to\infty$, the two-points Green function $\hat{G}^{(L)}$ converges uniformly in $k = (k_0,\mathbf{k}) \in \mathbb{R}\times\mathbb{T}$ to the (formal) Green function $\hat{G}$ of the model in the infinite volume with dispersion relation $e_\mu(\mathbf{k})$ and counterterm $K(\mathbf{k},\lambda)$.

(ii) For each $m\geq 1$, the $2m-$points Green function $\hat{G}^{(L)}_{2m}$ converges in the limit $L\to\infty$ to the formal Green function of the model in the infinite volume in the L_1-norm.

(iii) The density of fermions $\rho^{(L)}(\lambda)$ converges in the sense of formal power series to the density of fermions in the infinite volume:

$$\rho^{(L)}(\lambda) \overset{L\to\infty}{\to} \rho(\lambda)$$

Since $\rho(\lambda)$ is the limit of the density $\rho^{(L)}(\lambda)$ in the finite volume, theorem 2.20 implies:

Corollary 2.25: Assume A1-3, and let $K(\mathbf{k}, \lambda)$ be the counterterm obtained in theorem 2.24. Then the density of fermions in the thermodynamic limit is independent of the interaction, that is formally,

$$\rho(\lambda) = \rho(0),$$

or, for all $r \geq 1$,

$$\rho_r = 0.$$

Further,

$$\rho(0) = 2\,\mathrm{Vol}(S).$$

Chapter 3

Analyticity of the Green Functions

In this chapter, we prove the analyticity of the Green functions for the model on the finite lattice, with the dispersion relation defined in 2.10. The analyticity of the density follows from the analyticity of the Green functions. This section is similar to the analysis of the insulator given in [12], and based on the techniques developed in [13].

Let A be the Grassman algebra generated by the fields $\phi(y,\tau)$ and $\bar{\phi}(y,\tau)$ with $(y,\tau) = (y^0, \mathbf{y}, \tau) \in \mathbb{R} \times \Lambda_L \times \{\uparrow, \downarrow\}$, and let $\mathcal{W}(\psi, \bar{\psi})$ be an even Grassman function. Then the generating functional for the connected amputated Green functions is formally defined as

$$\Omega_C(\mathcal{W})(\phi, \bar{\phi}) = \log \frac{1}{Z} \int e^{\mathcal{W}(\psi+\phi, \bar{\psi}+\bar{\phi})} d\mu_C(\bar{\psi}, \psi)$$

where $Z = \int e^{\mathcal{W}(\psi,\bar{\psi})} d\mu_C(\bar{\psi}, \psi)$. $d\mu_C(\bar{\psi}, \psi)$ is the Grassman Gaussian measure with covariance C on the Grassman algebra with coefficient in A, generated by the fields $\psi(x,\sigma)$ and $\bar{\psi}(x,\sigma)$ with $(x,\sigma) \in \mathbb{R} \times \Lambda_L \times \{\uparrow, \downarrow\}$.

In order to simplify the notation, let $\mathcal{B} = \mathbb{R} \times \mathbf{\Lambda}_L \times \{\uparrow, \downarrow\} \times \{0, 1\}$, and for $\xi = (x, \sigma, a) \in \mathcal{B}$,

$$\psi(\xi) = \begin{cases} \bar{\psi}(x,\sigma), \ a = 1 \\ \psi(x,\sigma), \ a = 0. \end{cases}$$

The connected, amputated Green functions are given by the Taylor expansion of Ω_C in the Grassman fields ϕ and $\bar{\phi}$:

$$\Omega_C(\mathcal{W})(\phi, \bar{\phi}) = \sum_{m \geq 1} \int d\xi_1 \cdots d\xi_{2m} \, G^{(L)}_{2m}(\xi_1, \ldots, \xi_{2m}) \, \bar{\phi}(\xi_1) \cdots \phi(\xi_{2m}),$$

where for $\xi = (x^0, \mathbf{x}, \sigma, a) \in \mathcal{B}$,

$$\int d\xi \cdot = \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{a \in \{0,1\}} \sum_{\mathbf{x} \in \mathbf{\Lambda}_L} \int_{-\infty}^{\infty} dx^0 \cdot$$

3.1 Contractions and Norms

Definition 3.1: Let $f : \mathcal{B}^m \times \mathcal{B}^n \to \mathbb{C}$.

(i) Define the L_1-L_∞−norm by

$$||f||_{1,\infty} = \begin{cases} \max\limits_{1\le i\le n} \sup\limits_{\xi_i\in\mathcal{B}} \displaystyle\int \prod_{j\neq i} d\xi_j \, |f(\xi_1,\ldots,\xi_n)|, & \text{for } m=0 \\ \sup\limits_{\eta_1,\ldots,\eta_m\in\mathcal{B}} \displaystyle\int \prod_{j=1}^{n} d\xi_j \, |f(\eta_1,\ldots,\eta_m;\xi_1,\ldots,\xi_n)|, & \text{for } m>0 \end{cases}$$

(ii) The supremum norm of f is defined by

$$||f||_\infty = \sup_{\substack{\eta_1,\ldots,\eta_m\in\mathcal{B} \\ \xi_1,\ldots,\xi_n\in\mathcal{B}}} |f(\eta_1,\ldots,\eta_m;\xi_1,\ldots,\xi_n)|.$$

Remark 3.2: Let f be a translation invariant function on $\mathcal{B}^n$, and $\hat{f}$ its Fourier transform, defined in 2.17. Then $\hat{f}$ is a function on $(\mathbb{R}\times\mathbf{\Lambda}_L^\sharp)^{n-1}\times(\{\uparrow,\downarrow\}\times\{0,1\})^n$, and

$$|\hat{f}|_0 \le ||f||_{1,\infty}.$$

Definition 3.3: Let $\mathcal{F}_m(n)$ be the space of all functions $f(\eta_1,\ldots,\eta_m;\xi_1,\ldots,\xi_n)$ on $\mathcal{B}^m\times\mathcal{B}^n$ that are antisymmetric in the η variables. For any function f in $\mathcal{B}^m\times\mathcal{B}^n$, its antisymmetrization in the external variables is

$$\mathrm{Ant}_{\mathrm{ext}}\, f(\eta_1,\ldots,\eta_m;\xi_1,\ldots,\xi_n) = \frac{1}{m!}\sum_{\pi\in S_m} \mathrm{sgn}(\pi) f(\eta_{\pi(1)},\ldots,\eta_{\pi(m)};\xi_1,\ldots,\xi_n).$$

Let f be a function on $\mathcal{B}^m\times\mathcal{B}^n$. For a permutation $\pi\in S_n$, let f^π be the function defined by

$$f^\pi(\eta_1,\ldots,\eta_m;\xi_1,\ldots,\xi_n) = f(\eta_1,\ldots,\eta_m;\xi_{\pi(1)},\ldots,\xi_{\pi(n)}).$$

A semi-norm $||\cdot||$ on $\mathcal{F}_m(n)$ is called symmetric, if for all permutations $\pi\in S_n$

$$||f^\pi|| = ||f||.$$

Definition 3.4: Let $C(\xi,\xi')$ be a skew symmetric function on $\mathcal{B}\times\mathcal{B}$, $m,n\ge 0$ and $1\le i<j\le n$. For $f\in\mathcal{F}_m(n)$, the contraction $\mathcal{C}\mathrm{on}_{ij} f\in\mathcal{F}_m(n-2)$ is defined as

$$\mathcal{C}\mathrm{on}_{ij} f(\eta_1,\ldots,\eta_m;\xi_1,\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_{j-1},\xi_{j+1},\ldots,\xi_n) =$$

$$(-1)^{j-i+1}\int d\zeta d\zeta' \, C(\zeta,\zeta')\, f(\eta_1,\ldots,\eta_m;\xi_1,\ldots,\xi_{i-1},\zeta,\xi_{i+1},\ldots,\xi_{j-1},\zeta',\xi_{j+1},\ldots,\xi_n).$$

Definition 3.5: Let $||\cdot||$ be a symmetric semi-norm on the spaces $\mathcal{F}_m(n)$. We say that $\mathcal{C} \geq 0$ is a contraction bound for the covariance C with respect to this semi-norm, if for all $m, n, m', n' \geq 0$, there exist i and j with $1 \leq i \leq n$ and $1 \leq j \leq n'$ such that

$$||\mathcal{C}\mathrm{on}_{ij}(\mathrm{Ant}_{\mathrm{ext}}(f \times f'))|| \leq \mathcal{C}\, ||f|| \cdot ||f'||.$$

Remark 3.6: The L_1-L_∞−norm of definition 3.1 accepts

$$\max\{||C||_{1,\infty}, ||C||_\infty\}$$

as a contraction bound for the covariance C.

Definition 3.7: We say that $b \in \mathbb{R}_+$ is an integral bound for the covariance C with respect to the semi-norm $||\cdot||$, if the following holds:

Let $m, n \geq 0$ and $1 \leq n' \leq n$. For $f \in \mathcal{F}_m(n)$, define $f' \in \mathcal{F}_m(n-n')$ by

$$f'(\eta_1, \ldots, \eta_n; \xi_{n'+1}, \ldots, \xi_n) =$$

$$= \int_{\mathcal{B}^{n'}} d\xi_1 \cdots d\xi_{n'} f(\eta_1, \ldots, \eta_n; \xi_1, \ldots, \xi_n) \int \psi(\xi_1) \cdots \psi(\xi_{n'}) d\mu_C(\psi).$$

Then

$$||f'|| \leq (b/2)^{n'} ||f||.$$

Remark 3.8: Suppose that

$$\left| \int \psi(\xi_1) \cdots \psi(\xi_n) d\mu_C(\psi) \right| \leq S^n$$

for a $S > 0$. Then $2S$ is an integral bound for C with respect to the L_1-L_∞−norm of definition 3.1.

Definition 3.9: We define $A_m[n]$ as the subspace of the Grassman algebra that consists of all elements of the form

$$Gr(f) = \int d\eta_1 \cdots d\eta_m d\xi_1 \cdots d\xi_n\, f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n) \phi(\eta_1) \cdots \phi(\eta_m) \psi(\xi_1) \cdots \psi(\xi_n),$$

for a function f on $\mathcal{B}^m \times \mathcal{B}^n$.

Every element of $A_m[n]$ has a unique representation of the form $Gr(f)$ with a function $f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n) \in \mathcal{F}_m(n)$ that is antisymmetric in its ξ variables. Hence a semi-norm $||\cdot||$ on $\mathcal{F}_m(n)$ defines a canonical semi-norm on $A_m[n]$, which we denote with the same symbol.

Definition 3.10: Let $||\cdot||$ be a symmetric semi-norm, and $\mathcal{W}(\phi,\psi)$ be a Grassman function. Write

$$\mathcal{W} = \sum_{m,n\geq 0} \mathcal{W}_{m,n}$$

with $\mathcal{W}_{m,n} \in A_m[n]$. For any $\mathcal{C} > 0$, $b > 0$ and $\alpha \geq 1$ set

$$N(\mathcal{W};\mathcal{C},b,\alpha) = \frac{1}{b^2}\,\mathcal{C} \sum_{m\geq 0,\, n>0} \alpha^n b^n ||\mathcal{W}_{m,n}||.$$

Definition 3.11: The Wick ordering with respect to the covariance C is the linear map $:\cdot:_{C,\psi} : A \to A$ with

$$: e^{(\eta,\psi)} :_{C,\psi} = e^{(\eta,C\eta)+(\eta,\psi)}.$$

In order to prove the analyticity of the model on the finite lattice, we use the following results of [13]:

Theorem 3.12: Let $||\cdot||$ be a symmetric semi-norm and let C be a covariance with contraction bound $\mathcal{C}$ and integral bound b. Then the formal Taylor series $\Omega_C(:\mathcal{W}:)$ converges to an analytic map on

$$\{\mathcal{W} \,|\, \mathcal{W}\, even,\ N(\mathcal{W};\mathcal{C},b,8\alpha) < \frac{\alpha^2}{4}\}.$$

Furthermore, if $\mathcal{W}(\phi,\psi)$ is an even Grassman function such that

$$N(\mathcal{W};\mathcal{C},b,8\alpha) < \frac{\alpha^2}{4},$$

then

$$N(\Omega_C(:\mathcal{W}:) - \mathcal{W};\mathcal{C},b,\alpha) \leq \frac{2}{\alpha^2}\frac{N(\mathcal{W};\mathcal{C},b,8\alpha)^2}{1-\frac{4}{\alpha^2}N(\mathcal{W};\mathcal{C},b,8\alpha)}.$$

Here $:\cdot:$ denotes the Wick ordering with respect to the covariance C.

Theorem 3.13: Let, for $s \in \mathbb{R}$ in a neighborhood of 0, C_s be an antisymmetric function on $\mathcal{B}\times\mathcal{B}$ and $\mathcal{W}_s$ an even Grassman function. Assume that $\alpha \geq 1$, $\mathcal{C} \leq \frac{\mathcal{C}^2}{\mu}$ and

$$N(\mathcal{W}_0;\mathcal{C},b,8\alpha) < \alpha^2.$$

Assume further that C_0 has contraction bound $\mathcal{C}$, $b/2$ is an integral bound for C_0, and $\mathcal{C}'$ is a contraction bound for $\left.\frac{d}{ds}\right|_{s=0} C_s$. Then

$$\begin{aligned} &N\left(\left.\frac{d}{ds}(\Omega_{C_s}(:\mathcal{W}_s:) - \mathcal{W}_s)\right|_{s=0};\mathcal{C},b,\alpha\right) \\ &\leq \frac{1}{2\alpha^2}\frac{N(\mathcal{W}_0;\mathcal{C},b,32\alpha)}{1-\frac{1}{\alpha^2}N(\mathcal{W}_0;\mathcal{C},b,32\alpha)}\left(N(\left.\frac{d}{ds}\right|_{s=0}\mathcal{W}_s;\mathcal{C},b,8\alpha) + N(\mathcal{W}_0;\mathcal{C},b,32\alpha)\frac{\mathcal{C}'}{4\mu}\right). \end{aligned}$$

3.2 Bounds for the Covariance

Definition 3.14:

(i) Let $e^{(L)}(\mathbf{k})$ be as in 2.10. Define

$$\hat{C}(k) := \frac{1}{ik_0 - e^{(L)}(\mathbf{k})}.$$

(ii) Define the covariance C on $\mathcal{B} \times \mathcal{B}$ as follows:

$$C(\xi, \xi') = \begin{cases} \delta_{\sigma\sigma'} L^{-d} \sum_{\mathbf{k} \in \mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi} e^{-i<k,x-x'>} \hat{C}(k) & \text{if } a = 0, a' = 1 \\ -\delta_{\sigma\sigma'} L^{-d} \sum_{\mathbf{k} \in \mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi} e^{-i<k,x'-x>} \hat{C}(k) & \text{if } a = 1, a' = 0 \\ 0 & \text{if } a = a', \end{cases}$$

where $< k, x >:= -k_0 x^0 + \mathbf{kx}$. The case $x_0 = x_0' = 0$ is defined through the limit $x_0 - x_0' \to 0_-$.

(iii) Let $u \in \mathcal{K}$. Set

$$\hat{C}_u(k) := \frac{1}{ik_0 - e^{(L)}(\mathbf{k}) - u(\mathbf{k})}.$$

(iv) For u and $\delta u \in \mathcal{K}$ and $s \in \mathbb{R}$ in a neighborhood of 0 such that $u + s\delta u \in \mathcal{K}$, let

$$\hat{C}_s(k) := \frac{1}{ik_0 - e^{(L)}(\mathbf{k}) - (u + s\delta u)(\mathbf{k})}.$$

(v) The covariances $C_u(\xi, \xi')$ and $C_s(\xi, \xi')$ are defined in the same way as the covariance $C(\xi, \xi')$ in (ii).

Remark 3.15: For the proof of the analyticity, the counterterm is considered as a change in the dispersion relation $e^{(L)}$. In order to prove the convergence (order by order in the coupling constant), the covariance $\hat{C}_u$ is expended in powers of u, and the counterterm is considered as a two-points interaction with vertex function $u(\mathbf{k})$.

Definition 3.16: For a skew symmetric function C on $\mathcal{B} \times \mathcal{B}$, define

$$S(C) = \sup_{m \in \mathbb{N}} \sup_{\xi_1, \ldots, \xi_{2m} \in \mathcal{B}} \left(\left| \int \psi(\xi_1) \cdots \psi(\xi_{2m}) d\mu_C(\psi) \right| \right)^{1/2m}.$$

Remark 3.17: For covariance C_1 and C_2,

$$S(C_1 + C_2) \leq S(C_1) + S(C_2).$$

Lemma 3.18: For all dispersion relations $e^{(L)}(\mathbf{k})$ with

$$\frac{A}{L} \leq |e^{(L)}(\mathbf{k})| \leq E$$

for all $\mathbf{k} \in \mathbf{\Lambda}_L^\sharp$, and $u, \delta u \in \mathcal{K}$ and $s \in \mathbb{R}$ in a neighborhood of 0 with $u + s\delta u \in \mathcal{K}$,

(i) $||C||_{1,\infty} \leq 2L^{d+1}/A$ and $||C_s||_{1,\infty} \leq 4L^{d+1}/A$.

(ii) $||C||_\infty \leq 1$ and $||C_s||_\infty \leq 1$.

(iii) $S(C) \leq 1 + \sqrt{\frac{2EL}{A}}$ and $S(C_s) \leq 1 + 2S(C)$.

(iv) $|| \frac{d}{ds}\big|_{s=0} C_s||_\infty \leq 1$ and $|| \frac{d}{ds}\big|_{s=0} C_s||_{1,\infty} \leq 4L^{d+1}/A$.

<u>Proof:</u>

(i) First perform the k_0 integral in the definition of C:

$$L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi} \frac{e^{-i<k,x>}}{ik_0 - e^{(L)}(\mathbf{k})} = L^{-d} \sum_{\substack{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp \\ x^0 e^{(L)}(\mathbf{k})<0}} e^{i\mathbf{k}\mathbf{x} - |e^{(L)}(\mathbf{k})x^0|},$$

Integrating over the x variable, one gets

$$\sum_{\mathbf{x}\in\mathbf{\Lambda}_L} \int dx^0 \left| L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi} \frac{e^{-i<k,x>}}{ik_0 - e^{(L)}(\mathbf{k})} \right|$$
$$\leq 2L^d \sup_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \int_0^\infty e^{-|e^{(L)}(\mathbf{k})|x^0}\, dx^0 \leq 2\frac{L^{d+1}}{A}.$$

Using

$$\sup_{\sigma\in\{\uparrow,\downarrow\}} \sum_{\sigma'\in\{\uparrow,\downarrow\}} \delta_{\sigma\sigma'} = 1$$

and the analog for the sum over the index a, we get the claim. The bounds for C_u and C_s follow in the same way, using

$$|e^{(L)}(\mathbf{k}) - u(\mathbf{k})| \geq \frac{A}{2L}$$

in order to get

$$\sum_{\mathbf{x}\in\mathbf{\Lambda}_L}\int dx^0\left|L^{-d}\sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp}\int\frac{dk_0}{2\pi}\frac{e^{-i<k,x>}}{ik_0-e^{(L)}(\mathbf{k})+u(\mathbf{k})}\right|\leq 4\frac{L^{d+1}}{A}.$$

(ii) In the supremum norm, we bound for C

$$\left|L^{-d}\sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp:\,x^0e^{(L)}(\mathbf{k})<0}e^{-i\mathbf{k}\mathbf{x}-|e^{(L)}(\mathbf{k})x^0|}\right|\leq 1.$$

The same bound can be apply to $\hat{C}_s$.

(iii) For $E>0$, decompose[1]

$$\frac{1}{ik_0-e^{(L)}(\mathbf{k})}=\frac{1}{ik_0-E}+\frac{e^{(L)}(\mathbf{k})-E}{(ik_0-E)(ik_0-e^{(L)}(\mathbf{k}))}$$

and let $C_1(\xi,\xi')$ and $C_2(\xi,\xi')$ be the covariances defined by $\frac{1}{ik_0-E}$ and $\frac{e^{(L)}(\mathbf{k})-E}{(ik_0-E)(ik_0-e^{(L)}(\mathbf{k}))}$. The first part of the propagator is given by

$$C_1(\xi,\xi')=L^{-d}e^{E(x_0-x_0')}\sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\delta_{\sigma\sigma'}\frac{(-1)^a-(-1)^{a'}}{2},$$

for $x_0-x_0'<0$, and for $x_0-x_0'\geq 0$,

$$C_1(\xi,\xi')=0.$$

Thus, for $x_0-x_0'<0$,

$$C_1(\xi,\xi')=e^{-E|x_0-x_0'|}\langle w_{\mathbf{x},\sigma},w_{\mathbf{x}',\sigma'}\rangle_{\mathcal{H}}\frac{(-1)^a-(-1)^{a'}}{2},$$

where $\mathcal{H}$ is the (finite dimensional) Hilbert Space of the functions on $\mathbf{\Lambda}_L^\sharp\times\{\uparrow,\downarrow\}$ with scalar product

$$\langle f,g\rangle_{\mathcal{H}}=L^{-d}\sum_{\substack{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp\\ \sigma\in\{\uparrow,\downarrow\}}}f^*(\mathbf{k},\sigma)g(\mathbf{k},\sigma),$$

and

$$w_{\mathbf{x},\sigma}(\mathbf{k},\tau)=e^{i\mathbf{k}\mathbf{x}}\delta_{\sigma\tau}$$

[1]See [12], lemma IV.4

Using the Proposition B.1 of [13], we get

$$S(C_1) \leq 1,$$

since

$$||w_{\mathbf{x},\sigma}||_{\mathcal{H}} = 1.$$

We turn now to C_2. Let $\mathcal{H} = L_2(\mathbb{R} \times \mathbf{\Lambda}_L^\sharp \times \{\uparrow, \downarrow\})$, with the scalar product

$$\langle f, g\rangle_{\mathcal{H}} = L^{-d} \sum_{\substack{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp \\ \sigma\in\{\uparrow,\downarrow\}}} \int \frac{dk_0}{2\pi} f(k,\sigma) g^*(k,\sigma).$$

Then

$$C_2(x,\sigma,a;x',\sigma',a') = \begin{cases} \langle f_{x\sigma}, g_{x',\sigma'}\rangle_{\mathcal{H}}, & a = 0 \text{ and } a' = 1 \\ -\langle f_{x\sigma}, g_{x',\sigma'}\rangle_{\mathcal{H}}, & a = 1 \text{ and } a' = 0 \\ 0, & a = a', \end{cases}$$

where

$$f_{x,\sigma}(k,\tau) = \delta_{\sigma\tau} \frac{e^{i<k,x>}}{|(ik_0 - E)(ik_0 - e^{(L)}(\mathbf{k}))|^{1/2}} \frac{e^{(L)}(\mathbf{k}) - E}{\sqrt{|e^{(L)}(\mathbf{k}) - E|}}$$

and

$$g_{x,\sigma}(k,\tau) = \delta_{\sigma\tau} e^{i<k,x>} \frac{(ik_0 - E)(ik_0 - e^{(L)}(\mathbf{k}))}{|(ik_0 - E)(ik_0 - e^{(L)}(\mathbf{k}))|^{3/2}} \sqrt{|e^{(L)}(\mathbf{k}) - E|}$$

such that

$$||f_{x,\sigma}||_{\mathcal{H}}^2 = ||g_{x,\sigma}||_{\mathcal{H}}^2 = L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \int \frac{k_0}{2\pi} \frac{|e^{(L)}(\mathbf{k}) - E|}{\sqrt{(k_0^2 + E^2)(k_0^2 + e^{(L)}(\mathbf{k})^2)}}.$$

We bound the k_0-integral setting $E = \max\{||e||_0, 3\}$:

$$\begin{aligned} \int \frac{1}{\sqrt{(k_0^2 + E^2)(k_0^2 + e^{(L)}(\mathbf{k})^2)}} \frac{dk_0}{2\pi} &\leq \frac{1}{\pi} \int_0^\infty \frac{1}{k_0^2 + e^{(L)}(\mathbf{k})^2} dk_0 \\ &\leq \frac{1}{|e^{(L)}(\mathbf{k})|}. \end{aligned}$$

Hence

$$||f_x||_{\mathcal{H}}^2 \leq L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \frac{|e^{(L)}(\mathbf{k}) - E|}{|e^{(L)}(\mathbf{k})|} \leq \frac{2EL}{A}.$$

By the Gram bound,

$$S(C_2) \leq \sqrt{\frac{2EL}{A}}.$$

The bound on $S(C)$ follows from the remark 3.17. The claim for C_u and C_s follows in the same way, with now

$$||f_x||_{\mathcal{H}}^2 \leq \frac{4EL}{A},$$

since $|e^{(L)}(\mathbf{k}) - u(\mathbf{k})| \geq A/2L$.

(iv) First observe that

$$\frac{d}{ds}\bigg|_{s=0} \hat{C}_s(k) = \frac{\delta u(\mathbf{k})}{(ik_0 - e^{(L)}(\mathbf{k}) - u(\mathbf{k}))^2}.$$

By Cauchy formula,

$$\begin{aligned}\int \frac{dk_0}{2\pi} \frac{e^{ik_0x^0}}{(ik_0 - a)^{n+1}} &= \Theta(-x^0a)\frac{1}{n!}\frac{d^n}{dk_0^n}\bigg|_{k_0=-ia} e^{ik_0x^0} \\ &= \frac{1}{n!}\Theta(-x^0a)(x^0)^n e^{-|ax^0|}.\end{aligned}$$

The integral over x^0 of such an expression can be bounded by

$$\int_0^\infty dx^0 \frac{1}{n!}|x^0|^n e^{-|ax^0|} \leq a^{-n-1}.$$

Using this remark, we bound

$$\sup_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \left| \int \frac{dk_0}{2\pi} \frac{e^{ik_0x^0}}{(ik_0 - e^{(L)}(\mathbf{k}) - u(\mathbf{k}))^2} \right| \leq |x^0| e^{-|x^0|\frac{A}{2L}}.$$

Hence,

$$|| \frac{d}{ds}\bigg|_{s=0} C_s||_{1,\infty} \leq 4\frac{L^{d+1}}{A}\sum_{n\geq 1}\frac{n}{2^n} = 4\frac{L^{d+1}}{A}.$$

Further,

$$|| \frac{d}{ds}\bigg|_{s=0} C_s||_{\infty} \leq \frac{2}{e} \leq 1.$$

□

Remark 3.19:

(i) In the $||\cdot||_{1,\infty}$−norm, the covariance C_s has contraction bound

$$\mathcal{C} = \max\{||C_s||_{1,\infty}, ||C_s||_\infty\} \leq \frac{4L^{d+1}}{A}.$$

The integral bound b for the covariance C_s is bounded by

$$b^2 = 4S(C_s)^2 \leq \frac{2^5EL}{A}.$$

The contraction bound for the derivative of the covariance $\frac{d}{ds}\big|_{s=0} C_s$ is

$$\mathcal{C}' = \max\{|| \frac{d}{ds}\bigg|_{s=0} C_s||_{1,\infty}, || \frac{d}{ds}\bigg|_{s=0} C_s||_\infty\} \leq \frac{4L^{d+1}}{A}.$$

(ii) In the bounds $\mathcal{C}$ and $\mathcal{C}'$, the factor L^d is not relevant. A better bound can be found, since the propagator in position space is integrable.

3.3 The Analyticity of the Green Functions

Definition 3.20: Let $\mathcal{V} \in A_0[4]$ be

$$\mathcal{V}(\psi) = \frac{\lambda}{2} \int d\xi_1 \cdots d\xi_4 \, V(\xi_1, \ldots, \xi_4) \psi(\xi_1) \cdots \psi(\xi_4)$$

where

$$\begin{aligned} &V(x_1, \sigma_1, a_1, \ldots, x_4, \sigma_4, a_4) \\ &= \delta_{a_1 1} \delta_{a_2 1} \delta_{a_3 0} \delta_{a_4 0} \delta_{\sigma_1 \sigma_3} \delta_{\sigma_2 \sigma_4} \delta(x_1 - x_3) \delta(x_2 - x_4) V_{\sigma_1 \sigma_2}(x_1 - x_2), \end{aligned}$$

with

$$V_{\sigma_1 \sigma_2}(x_1 - x_2) = \delta(x_1^0 - x_2^0) V^{(L)}_{\sigma_1 \sigma_2}(\mathbf{x}_1 - \mathbf{x}_2).$$

Remark 3.21:

(i) $||V||_{1,\infty} = \sup\limits_{\sigma \in \{\uparrow,\downarrow\}} \sum\limits_{\substack{\mathbf{x} \in \mathbf{\Lambda}_L \\ \tau \in \{\uparrow,\downarrow\}}} |V^{(L)}_{\sigma\tau}(\mathbf{x})| \leq 2L^d |v|_0.$

(ii) $N(\mathcal{V}, \alpha) = |\lambda| \alpha^4 b^2 \mathcal{C} \sup\limits_{\sigma \in \{\uparrow,\downarrow\}} \sum\limits_{\substack{\mathbf{x} \in \mathbf{\Lambda}_L \\ \tau \in \{\uparrow,\downarrow\}}} |V^{(L)}(\mathbf{x})| \leq 2|\lambda| \alpha^4 b^2 \mathcal{C} L^d |v|_0.$

Definition 3.22: Let $\mathcal{W}_s$ be the Grassman function

$$\mathcal{W}_s = \mathcal{V} + \mathcal{U}_s,$$

where

$$\mathcal{U}_s = \lambda \int d\xi_1 d\xi_2 \, U_s(\xi_1, \xi_2) \psi(\xi_1) \psi(\xi_2)$$

and

$$\begin{aligned} &U_s(x_1, \sigma_1, a_1, x_2, \sigma_2, a_2) = \\ &= 2\delta_{a_1 1} \delta_{a_2 0} \Bigg(-C_s(x_1 - x_2) \delta_{\sigma_1 \sigma_2} \frac{V_{\sigma_1 \sigma_1}(x_1 - x_2) + V_{\sigma_1 \sigma_1}(x_2 - x_1)}{2} \\ &+ 2\delta(x_1 - x_2) \sum_{\substack{\mathbf{z} \in \mathbf{\Lambda}_L \\ \tau \in \{\uparrow,\downarrow\}}} \int dz^0 \frac{V_{\sigma_1 \tau}(z - x_2) + V_{\tau \sigma_1}(x_2 - z)}{2} C_s(z, z) \Bigg). \end{aligned}$$

Remark 3.23:

(i) $: \mathcal{W}_s := \mathcal{V} + \mathbb{C} -$ number.

(ii) $\Omega_C(\mathcal{V}) = \Omega_C(:\mathcal{W}_s:)$.

(iii) $||U_s||_{1,\infty} \leq 4||V||_{1,\infty}$.

(iv) If $\alpha \geq 1$ and $b \geq 1$, $N(\mathcal{U}_s, \alpha) \leq 4N(\mathcal{V}, \alpha)$ and $N(\mathcal{W}_s, \alpha) \leq 5N(\mathcal{V}, \alpha)$

Proof: In order to prove the point (i), remark that

$$\begin{aligned} :\mathcal{V}: \;=\; & \frac{\lambda}{2}\int d\xi_1 \cdots d\xi_4\, V(\xi_1, \ldots, \xi_4)\, \psi(\xi_1) \cdots \psi(\xi_4) \\ & -\lambda \int d\xi_1 d\xi_2\, U_s(\xi_1, \xi_2)\, \psi(\xi_1)\psi(\xi_2) \\ & +\mathbb{C} - \text{number}, \end{aligned}$$

and

$$:\mathcal{U}_s: = \mathcal{U}_s + \mathbb{C} - \text{number}.$$

Hence

$$:\mathcal{W}_s: =: \mathcal{V}: + :\mathcal{U}_s: = \mathcal{V} + \mathbb{C} - \text{number}.$$

For point (iv),

$$N(\mathcal{U}_s, \alpha) = \mathcal{C}\alpha^2 ||U_s||_{1,\infty} \leq 4|\lambda|\mathcal{C}\frac{1}{b^2}\alpha^4 b^4 ||V||_{1,\infty} \leq 4N(\mathcal{V}, \alpha).$$

□

Theorem 3.24: Let $u \in \mathcal{K}$. Assume that

$$N(\mathcal{V}, 8\alpha) \leq \frac{\alpha^2}{20}.$$

Then the generating functional $\Omega_C(\mathcal{V})$ is analytic in λ, for λ satisfying

$$|\lambda| < \frac{A^2}{2^{22} E\alpha^2 L^{d+2} ||V||_{1,\infty}}.$$

Further

$$N(\Omega_C(\mathcal{V}) - \mathcal{V}, \alpha) \leq N(\mathcal{V}, 8\alpha)\left(1 + \frac{2^7}{\alpha^2}\frac{N(\mathcal{V}, 8\alpha)}{1 - \frac{20}{\alpha^2}N(\mathcal{V}, 8\alpha)}\right)$$

Proof: Applying successively remark 3.23(ii) and (i), theorem 3.12 and remark 3.23(iv),

we get

$$\begin{aligned} N(\Omega_C(\mathcal{V}) - \mathcal{V}, \alpha) &\leq N(\Omega_C(: \mathcal{W} :) - \mathcal{W}, \alpha) + N(\mathcal{V} - \mathcal{W}, \alpha) \\ &\leq \frac{2}{\alpha^2} \frac{N(\mathcal{W}, 8\alpha)^2}{1 - \frac{4}{\alpha^2} N(\mathcal{W}, 8\alpha)} + N(\mathcal{V} - \mathcal{W}, \alpha) \\ &\leq 4N(\mathcal{V}, \alpha) + \frac{2^7}{\alpha^2} \frac{N(\mathcal{V}, 8\alpha)^2}{1 - \frac{20}{\alpha^2} N(\mathcal{V}, 8\alpha)} \\ &\leq N(\mathcal{V}, 8\alpha) \left(1 + \frac{2^7}{\alpha^2} \frac{N(\mathcal{V}, 8\alpha)}{1 - \frac{20}{\alpha^2} N(\mathcal{V}, 8\alpha)} \right). \end{aligned}$$

By the definition of the norm N and remark 3.19,

$$N(\mathcal{V}, 8\alpha) \leq 2^{12} |\lambda| \mathcal{C} \alpha^4 b^2 ||V||_{1,\infty} \leq 2^{19} \alpha^4 |\lambda| \frac{E ||V||_{1,\infty} L^{d+2}}{A^2}.$$

If $|\lambda| < A^2/(2^{22} E \alpha^2 L^{d+2} ||V||_{1,\infty})$, then

$$N(\mathcal{V}, 8\alpha) \leq \frac{\alpha^2}{20}.$$

□

Corollary 3.25:

(i) For $|\lambda| < \lambda_0$ where

$$\lambda_0 = \frac{A^2}{2^{25} E \alpha^2 L^{d+2} ||V||_{1,\infty}},$$

the amputated $2m-$points Green functions $G_{2m}^{(L)}$ are analytic functions of λ. Further,

$$||G_2^{(L)}||_{1,\infty} < 2^{22} |\lambda| \frac{||V||_{1,\infty} \alpha^2 E L}{A}.$$

For the four-points function,

$$||G_4^{(L)}||_{1,\infty} < (1 + 2^{17}) |\lambda| \, ||V||_{1,\infty},$$

and for $m > 2$,

$$||G_{2m}^{(L)}||_{1,\infty} < 2^{17} |\lambda| \, ||V||_{1,\infty}.$$

(ii) The connected two-points Green function $S_2^{(L)}(x, y)$ is analytic in λ with

$$||S_2^{(L)}||_\infty \leq ||C||_\infty (1 + ||C||_{1,\infty} ||G_2^{(L)}||_{1,\infty})$$

and

$$||S_2^{(L)}||_{1,\infty} \leq ||C||_{1,\infty} (1 + ||C||_{1,\infty} ||G_2^{(L)}||_{1,\infty}).$$

(iii) The Fourier transform $\hat{G}_{2m}^{(L)}$ of $G_{2m}^{(L)}$, as well as the Fourier transform $\hat{S}_2^{(L)}$ of $S_2^{(L)}$ are analytic functions of λ, bounded in the sup-norm.

Proof: Observe that by hypothesis, $N(\mathcal{V}, 8\alpha) < \alpha^2/64$, such that for $m \neq 2$,

$$\begin{aligned}
\alpha^{2m} b^{2m} ||G_{2m}^{(L)}||_{1,\infty} &\leq \frac{b^2}{\mathfrak{C}} N(\Omega_{C_u}(\mathcal{V}) - \mathcal{V}, \alpha) \\
&\leq \frac{b^2}{\mathfrak{C}} N(\mathcal{V}, 8\alpha)(1 + \frac{2^7}{\alpha^2} \frac{N(\mathcal{V}, 8\alpha)}{1 - \frac{20}{\alpha^2} N(\mathcal{V}, 8\alpha)}) \\
&\leq \frac{b^2}{\mathfrak{C}} N(\mathcal{V}, 8\alpha)(1 + \frac{2^8}{\alpha^2} N(\mathcal{V}, 8\alpha)) \\
&\leq 2^3 \frac{b^2}{\mathfrak{C}} N(\mathcal{V}, 8\alpha) \\
&\leq 2^{17} \alpha^4 b^4 |\lambda| \, ||V||_{1,\infty}.
\end{aligned}$$

Thus, for $m = 1$ we have

$$\begin{aligned}
||G_2^{(L)}||_{1,\infty} &\leq 2^{17} \alpha^2 b^2 |\lambda| \, ||V||_{1,\infty} \\
&\leq 2^{22} \alpha^2 |\lambda| \frac{||V||_{1,\infty} EL}{A},
\end{aligned}$$

and for $m > 2$, since $\alpha \geq 1$ and $b \geq 1$ for L big enough,

$$\begin{aligned}
||G_{2m}^{(L)}||_{1,\infty} &\leq 2^{17} \alpha^{4-2m} b^{4-2m} |\lambda| \, ||V||_{1,\infty} \\
&\leq 2^{17} |\lambda| \, ||V||_{1,\infty}.
\end{aligned}$$

For $m = 2$, in the same way,

$$\alpha^4 b^4 ||G_4^{(L)} - \lambda V||_{1,\infty} \leq 2^{17} \alpha^4 b^4 |\lambda| \, ||V||_{1,\infty},$$

such that

$$||G_4^{(L)}||_{1,\infty} \leq (1 + 2^{17}) |\lambda| \, ||V||_{1,\infty}.$$

The point (ii) is trivial, and the last point follows from remark 3.2. □

Remark 3.26: For $|\lambda| < \lambda_0$,

$$|\hat{G}_2^{(L)} \hat{C}_u|_0 < \frac{A}{4L^{d+1}} |\hat{C}_u|_0 < \frac{1}{2L^d}.$$

Theorem 3.27: For each dispersion relation $e^{(L)}$ defined in 2.10 and $u \in \mathcal{K}$, the occupation number defined by

$$n_\sigma^{(L)}(\mathbf{k}, \lambda) = \lim_{x^0 \to 0_+} \int \frac{dk_0}{2\pi} \hat{S}_2^{(L)}(k_0, \mathbf{k}, \lambda) e^{ix^0 k_0}$$

is an analytic function of λ for $|\lambda| < \lambda_0$. Further, the density

$$\begin{aligned}\rho^{(L)}(\lambda) &= L^{-d} \sum_{\substack{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp \\ \sigma\in\{\uparrow,\downarrow\}}} n_\sigma^{(L)}(\mathbf{k},\lambda) \\ &= 2 \lim_{x^0\to 0_+} L^{-d} \int \frac{dk_0}{2\pi} \sum_{\substack{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp \\ \sigma\in\{\uparrow,\downarrow\}}} \hat{S}_2^{(L)}(k_0,\mathbf{k},\lambda) e^{ix^0k_0}\end{aligned}$$

is analytic in λ for $|\lambda| < \lambda_0$.

<u>Proof:</u> By the corollary 3.25, the functions

$$\int \frac{dk_0}{2\pi} \hat{S}_2^{(L)}(k_0,\mathbf{k},\lambda) e^{ix^0k_0} = \sum_{\mathbf{x}\in\mathbf{\Lambda}_L} S_2^{(L)}(x^0,\mathbf{x},\lambda) e^{-i\mathbf{x}\mathbf{k}}$$

are well-defined and analytic in λ. We prove that the limit

$$\lim_{x^0-y^0\to 0_+} S_2^{(L)}(x,y,\lambda)$$

exists. Since $\lim_{x^0\to 0_+} C(x^0,\mathbf{x})$ exists, we have to verify it for the functions

$$\begin{aligned}S_2^{(L)}(x,y) - C(x,y) &= \sum_{\mathbf{z},\mathbf{z}'\in\mathbf{\Lambda}_L} \int dz^0 dz'^0\, C(x,z) G_2^{(L)}(z,z') C(z',y) \\ &= \sum_{\mathbf{z},\mathbf{z}'\in\mathbf{\Lambda}_L} \int dz^0 dz'^0\, C(x-y,z) G_2^{(L)}(z,z') C(z',0),\end{aligned}$$

by translation invariance. Note that

$$|C(x-y,z) G_2^{(L)}(z,z') C(z',0)| \leq ||C||_\infty |G_2^{(L)}(z,z') C(z',0)|,$$

such that by dominated convergence,

$$\begin{aligned}&\lim_{x^0\to 0_+} (S_2^{(L)}(x,y) - C(x,y)) = \\ &= \sum_{\mathbf{z},\mathbf{z}'\in\mathbf{\Lambda}_L} \int dz^0 dz'^0 \lim_{x^0\to 0_+} C(x,z) G_2^{(L)}(z,z') C(z',0),\end{aligned}$$

which is well-defined.

The sum over $\mathbf{\Lambda}_L$ and over $\{\uparrow,\downarrow\}$ is finite, such that the occupation number is well-defined. The sum over $\mathbf{\Lambda}_L^\sharp$ is finite as well, hence the density is well-defined and analytic. □

Remark 3.28: The theorem 2.18 follows from the corollary 3.24 and the theorem 3.27.

Chapter 4

The Renormalization

As we saw in last chapter, the Green functions on the finite lattice are analytic, with a radius of analyticity that shrinks to zero as L goes to infinity. Further, it turns out that the Green functions at each order in λ diverge as L tends to infinity. In order to control the Green functions at given orders in λ, we have to perform renormalization, which cures the infrared divergences.

In the diagrammatic analysis, the divergences appear in graphs containing two-legged insertions. The renormalization procedure consists in subtracting to each two-legged insertion its value, projected onto the Fermi surface. This corresponds to a particular choice of the counterterm, which is determined by the condition that the Fermi surface is held fixed.

In this section, we first describe the main tool of the renormalization, namely the localization operator, following the construction given in [7]. On the dual lattice $\mathbf{\Lambda}_L^\sharp$, the localization operator has to be modified in such a way, that points in $\mathbf{\Lambda}_L^\sharp$ are projected onto points of the lattice that are close to the Fermi surface.

In the second part of this section, we prove that for each finite L, there is a counterterm satisfying the renormalization condition, which preserves the analyticity obtained in the previous chapter.

4.1 Norms in Momentum Space

Definition 4.1: Let $T : \mathbf{\Lambda}_L^\sharp \to \mathbb{C}$ be a function on the lattice. For $1 \leq \alpha \leq d$, we define the "derivative" on the dual lattice by

$$\nabla_\alpha T(\mathbf{k}) := \frac{L}{2\pi}\left(T(\mathbf{k} + \frac{2\pi}{L}\mathbf{e}_\alpha) - T(\mathbf{k})\right),$$

where $\mathbf{e}_\alpha$ is one of the vector of the standard basis in $\mathbb{R}^d$. If $\alpha = 0$, set

$$\nabla_0 := \partial_{k_0},$$

the usual derivative with respect to k_0.

Definition 4.2: Let $u : (\mathbb{R} \times \mathbf{\Lambda}_L^\sharp)^n \to \mathbb{C}$ be a function of n variables. The $|\cdot|_1-$norm of u is defined by

$$|u|_1 := |u|_0 + \underset{\alpha=0,\dots,d}{max_{i=1,\dots,n}} |\nabla_{i\alpha} u|_0.$$

Here $\nabla_{i\alpha}$ is the derivative on the lattice defined above with respect to the α-component of the $i-$th variable.

Remark 4.3:

(i) For a differentiable function $T : \mathbb{T} \to \mathbb{C}$, we have

$$|\nabla_\alpha T(\mathbf{k})| \leq ||T||_1,$$

where $||\cdot||_1$ is the derivative norm defined in 2.8.

(ii) For two functions T_1 and T_2 on $\mathbf{\Lambda}_L^\sharp$, the following "Leibniz product rule" in the supremum norm yields

$$|\nabla_\alpha(T_1 \cdot T_2)|_0 \leq |T_1|_0 |\nabla_\alpha T_2|_0 + |\nabla_\alpha T_1|_0 |T_2|_0.$$

4.2 The Localization Operator

The localization operator l, defined in [7], implements the projection onto the Fermi surface for functions defined on $\mathbb{R} \times \mathbb{T}$. On the finite lattice $\mathbf{\Lambda}_L^\sharp$, we need a localization operator $l^{(L)}$ which projects points of the lattice onto other points of the lattice, that are close to the Fermi surface. This is necessary in order to preserve the periodic boundary conditions of the model in finite volume.

4.2.1 The Projection onto the Fermi Surface

For the projection l onto the Fermi surface, we give here the main results of [7], and refer to it for the proofs.

Remark 4.4: By assumption **A2**, S is a compact $(d-1)-$dimensional C^r-submanifold of $\mathbb{T}$. There is $\delta > 0$ such that

$$G_0 = \sup\{|\nabla e(\mathbf{p})|, \mathbf{p} \in U_{2\delta}(S)\}$$

is finite, and $g_0 = \inf\{|\nabla e(\mathbf{p})|, \mathbf{p} \in U_{2\delta}(S)\} > 0$. Here $U_{2\delta}(S) = \{\mathbf{p} \in \mathbb{T} \,|\, d(\mathbf{p}, S) < 2\delta\}$.

Definition 4.5: Let u be a $C^\infty-$vector field on a neighborhood $U_\eta(S)$ of S. u is transversal to S, if there is $u_0 > 0$ such that for all $\mathbf{p} \in S$, $\nabla e(\mathbf{p}) \cdot u(\mathbf{p}) \geq u_0 > 0$. Denote the integral curve of u through $\mathbf{p} \in S$ by $\gamma_\mathbf{p}$:

$$\begin{array}{rccc} \gamma_\mathbf{p}: & (-t_0, t_0) & \to & \mathbb{T} \\ & t & \mapsto & \gamma_\mathbf{p}(t), \end{array}$$

with $\gamma_\mathbf{p}(0) = \mathbf{p}$ and $\dot{\gamma}_\mathbf{p}(t) = u(\gamma_\mathbf{p}(t))$.

Lemma 4.6: Assume **A2**. Then:

(i) There is a $C^\infty-$vector field u transversal to S, and $t_0 > 0$ such that

$$\begin{array}{rccc} \Psi: & S \times (-t_0, t_0) & \to & \Psi(S \times (-t_0, t_0)) \subset \mathbb{T} \\ & (\mathbf{p}, t) & \mapsto & \Psi(\mathbf{p}, t) = \gamma_\mathbf{p}(t) \end{array}$$

is a C^r-diffeomorphism.

(ii) There are $\delta > 0$ and $u_0 \in (0, 1)$ such that

$$\overline{U_{2\delta}(S)} \subset \Psi(S \times (-t_0, t_0)),$$

and such that for all $\mathbf{q} \in U_{2\delta}(S)$:

$$0 < \frac{g_0}{2} \leq u_0 \leq \nabla e(\mathbf{q}) \cdot u(\mathbf{q}) \leq G_0.$$

(iii) Define the functions

$$\tau : \overline{U_{2\delta}(S)} \to \mathbb{R}$$

and

$$\omega : \overline{U_{2\delta}(S)} \to S$$

as follows. For $\mathbf{q} \in \overline{U_{2\delta}(S)}$,

$$(\omega(\mathbf{q}), \tau(\mathbf{q})) = \Psi^{-1}(\mathbf{q}).$$

In other words, $\gamma_{\omega(\mathbf{q})}(\tau(\mathbf{q})) = \mathbf{q}$. Then

$$\mathbf{q} = \omega(\mathbf{q}) + \int_0^{\tau(\mathbf{p})} u(\gamma_{\omega(\mathbf{q})}(t))dt$$

so $|\mathbf{q} - \omega(\mathbf{q})| \leq |\tau(\mathbf{q})|$ and

$$|\mathbf{q} - \omega(\mathbf{q})| \leq \frac{1}{u_0}|e(\mathbf{q})|.$$

Furthermore, $u_0 \leq e(\mathbf{q})/\tau(\mathbf{q}) \leq G_0$.

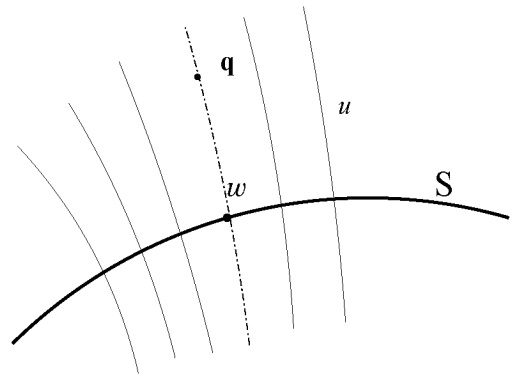

Figure 4.1: *The projection onto the Fermi surface*

(iv) Let $\mathbf{p} \in U_\delta(S)$ and $\rho = e(\mathbf{p})$. The map

$$\phi : \mathbf{p} \mapsto (\rho, \omega)$$

is a C^r-diffeomorphism from $U_\delta(S)$ to a subset of $\mathbb{R} \times S$. Denoting its inverse map by $\mathbf{p}(\rho, \omega)$, there are constants A_0 and A_1 such that the Jacobian $J(\rho, \omega) = \det \frac{\partial \mathbf{p}}{\partial(\rho,\omega)}$ obeys

$$\sup_{\mathbf{p} \in U_\delta(S)} |J(\rho, \omega)| \leq \frac{A_0}{u_0},$$

and its derivative ∂J obeys

$$\sup_{\mathbf{p} \in U_\delta(S)} |\partial J(\rho, \omega)| \leq \frac{A_1}{u_0^2}.$$

A_0 depends on δ, u_0, and $||u||_1$; A_1 also depends on the second derivative of u.

Definition 4.7: Let $\chi \in C^\infty(\mathbb{T}, [0, 1])$ be such that $\chi|_{U_\delta(S)} = 1$ and $\chi|_{\mathbb{T} \setminus U_{2\delta}(S)} = 0$. Let (ρ, ω) denotes the coordinates defined in the preceding lemma.

For a function $T : \mathbb{R} \times \mathbb{T} \mapsto X$, where X is a linear space, we define the projection operator l as follow:

$$(lT)\,(q^0, \mathbf{q}) = \begin{cases} 0, & \mathbf{q} \notin U_{2\delta}(S) \\ T(0, \omega(\mathbf{q}))\chi(\mathbf{q}), & \mathbf{q} \in U_{2\delta}(S) \end{cases}$$

Lemma 4.8: For each differentiable function on $T : \mathbb{R} \times U_\delta(S) \to \mathbb{C}$,

$$|(1 - l)T(q)| \leq \frac{\sqrt{2}}{u_0} |iq^0 - e(\mathbf{q})| \cdot ||T||_1,$$

for $\mathbf{q} \in U_{2\delta}(S)$. Further

$$||lT||_0 \leq ||T||_0$$

and

$$||lT||_1 \leq (1 + d||\omega||_1)||T||_1,$$

where

$$||\omega||_1 := \max_{\alpha=1,\ldots,d} ||\omega_\alpha||_1,$$

ω_α denoting the $\alpha-$component of the vector function ω. A proof of this lemma is given in [7].

4.2.2 The Projection on the Lattice

We turn now to the projection on the lattice $\mathbf{\Lambda}_L^\sharp$. Let $\{\mathbf{e}_\alpha\}_{\alpha=1}^d$ be the standard basis of $\mathbb{R}^d$, and for $\mathbf{k} \in \mathbf{\Lambda}_L^\sharp$ let

$$\mathbf{k} = \frac{2\pi}{L} \sum_{\alpha=1}^{d} k_\alpha \mathbf{e}_\alpha,$$

with $k_\alpha \in \mathbb{Z}$.

Definition 4.9: Let $A \geq 4\pi \sup\{1, ||e||_1\}$. Define

$$\mathcal{S}^{(L)} = \left\{ \mathbf{k} \in \mathbf{\Lambda}_L^\sharp \,\middle|\, |e(\mathbf{k})| \leq \frac{A}{L} \right\}.$$

Lemma 4.10: For $r \geq \frac{A}{L}$, let $N(L,r) := |\{\mathbf{k} \in \mathbf{\Lambda}_L^\sharp \,|\, |e(\mathbf{k})| \leq r\}|$ be the number of points of $\mathbf{\Lambda}_L^\sharp$ in the shell of width r around the Fermi surface. Then there are constants $c_2 > c_1 > 0$ independent of L, r and $|S|$, such that

$$c_1 |S|\, r\, L^d \leq N(L,r) \leq c_2 |S|\, r\, L^d,$$

where $|S|$ is the $(d-1)-$area of the Fermi surface S.

<u>Proof:</u> Let $\chi \in C_0^\infty(\mathbb{T}, [0,1])$ be such that $\chi(\mathbf{k}) = 1$ for $\mathbf{k}$ with $|e(\mathbf{k})| \leq r$, and $\chi(\mathbf{k}) = 0$ for $\mathbf{k}$ with $|e(\mathbf{k})| \geq 2r$. Further, assume that $||\chi||_1 \leq C/r$ for a constant $C > 1$. Then

$$\begin{aligned} L^{-d} \sum_{\mathbf{k} \in \mathbf{\Lambda}_L^\sharp} 1_{|e(\mathbf{k})| \leq r} &\leq L^{-d} \sum_{\mathbf{k} \in \mathbf{\Lambda}_L^\sharp} \chi(\mathbf{k}) \\ &\leq \int \frac{d^d p}{(2\pi)^d} \chi(\mathbf{p}) + \left| L^{-d} \sum_{\mathbf{k} \in \mathbf{\Lambda}_L^\sharp} \chi(\mathbf{k}) - \int \frac{d^d p}{(2\pi)^d} \chi(\mathbf{p}) \right|. \end{aligned}$$

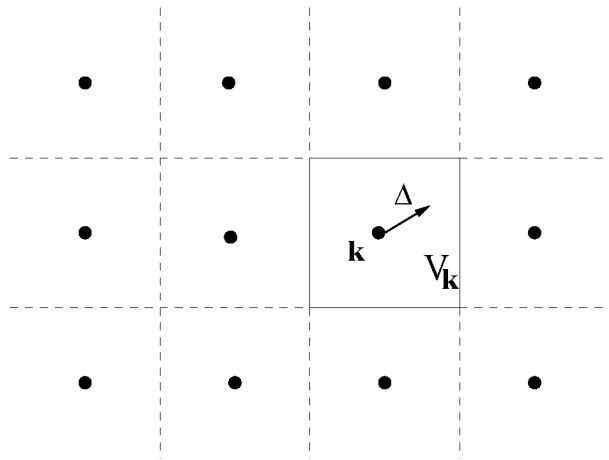

Figure 4.2: *The lattice and its fundamental zone.*

The last term is bounded by

$$\left| L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \chi(\mathbf{k}) - \int \frac{d^dp}{(2\pi)^d}\chi(\mathbf{p}) \right| \leq \frac{2\pi}{L}||\chi||_1 \int \frac{d^dp}{(2\pi)^d} 1_{|e(\mathbf{p})|\leq 2r},$$

such that

$$\begin{aligned} L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} 1_{|e(\mathbf{k})|\leq r} &\leq (1 + C\frac{2\pi}{Lr}) \int \frac{d^dp}{(2\pi)^d} 1_{|e(\mathbf{p})|\leq 2r} \\ &\leq \frac{1+C/2}{(2\pi)^d} \int_{-2r}^{2r} d\rho \int_S d\omega J(\rho,\omega) \\ &\leq \frac{2C}{(2\pi)^d} ||J||_0 \, |S| \, r. \end{aligned}$$

It follows that

$$N(r,L) = \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} 1_{|e(\mathbf{k})|\leq r} \leq \frac{2C||J||_0}{(2\pi)^d} |S| \, L^d \, r.$$

We turn to the first inequality. For each point $\mathbf{k} \in \mathbf{\Lambda}_L^\sharp$, let $V_\mathbf{k}$ denotes the fundamental cell of the lattice that contains $\mathbf{k}$. The volume of $V_\mathbf{k}$ is $(2\pi)^d/L^d$. Define the set $S_r := \{\mathbf{p} \in \mathbb{T} \,|\, |e(\mathbf{p})| \leq r\}$. Remark that for $\mathbf{k} \in \mathbf{\Lambda}_L^\sharp$, if $V_\mathbf{k} \cap S_{r-\frac{A}{2L}} \neq \emptyset$, then $\mathbf{k} \in S_r$:

Let $\mathbf{k} + \mathbf{\Delta} \in V_\mathbf{k}$ such that $|e(\mathbf{k}+\mathbf{\Delta})| \leq r - A/2L$ (See Fig. 4.2). Then

$$\begin{aligned} |e(\mathbf{k})| &\leq |e(\mathbf{k}+\mathbf{\Delta})| + |e(\mathbf{k}+\mathbf{\Delta}) - e(\mathbf{k})| \\ &\leq r - \tfrac{A}{2L} + ||e||_1 \tfrac{2\pi}{L} \\ &\leq r. \end{aligned}$$

It follows

$$L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} 1_{|e(\mathbf{k})|\leq r} \geq \int \frac{d^dp}{(2\pi)^d} 1_{|e(\mathbf{p})|\leq r-\frac{A}{2L}}.$$

On the other hand,

$$\begin{aligned}\int \frac{d^d p}{(2\pi)^d} 1_{|e(\mathbf{p})|\leq r-\frac{A}{2L}} &\geq \frac{2}{(2\pi)^d} \inf_{(\rho,\omega)\in U_\delta(S)} |J(\rho,\omega)|\,|S|\,(r-\frac{A}{2L}) \\ &\geq \frac{1}{(2\pi)^d} \inf_{(\rho,\omega)\in U_\delta(S)} |J(\rho,\omega)|\,|S|\,r.\end{aligned}$$

Hence

$$N(r,L) \geq (2\pi)^{-d} \inf_{\mathbf{p}\in U_\delta(S)} |J(\mathbf{p})|\,|S|\,r\,L^d.$$

□

Remark 4.11:

(i) $\mathbf{k} \in \mathcal{S}^{(L)}$ implies that $d(\mathbf{k}, S) \leq \frac{\sqrt{2}}{u_0}\frac{A}{L}$.

(ii) $\mathcal{S}^{(L)} \neq \emptyset$. The number of points in $\mathcal{S}^{(L)}$ grows like L^{d-1}.

(iii) For L big enough, $\mathcal{S}^{(L)} \subset U_\delta(S)$.

<u>Proof:</u> For point (i),

$$d(\mathbf{k}, S) = \inf_{\mathbf{p}\in S} |\mathbf{k}-\mathbf{p}| \leq |\mathbf{k} - l\mathbf{k}| \leq \frac{\sqrt{2}}{u_0}|e^{(L)}(\mathbf{k})| \leq \frac{\sqrt{2}}{u_0}\frac{A}{L}.$$

We turn to point (ii), and apply the lemma 4.11, with $r = A/L$. Then

$$Ac_1|S|L^{d-1} \leq |\mathcal{S}^{(L)}| \leq Ac_2|S|L^{d-1}.$$

□

Definition 4.12: For $\mathbf{k} \in \mathbf{\Lambda}_L^\sharp$, define the projection $l^{(L)}\mathbf{k}$ as follows:

- If $\mathbf{k} \in \mathcal{S}^{(L)}$, then $l^{(L)}\mathbf{k} = \mathbf{k}$.
- If $\mathbf{k} \notin \mathcal{S}^{(L)}$ and $\mathbf{k} \in U_{2\delta}(S)$, then $l^{(L)}\mathbf{k}$ is one of the points in $\mathcal{S}^{(L)}$, with $d(l^{(L)}\mathbf{k}, l\mathbf{k}) = \min_{\mathbf{k}'\in\mathcal{S}^{(L)}} d(\mathbf{k}', l\mathbf{k})$, and $\mathrm{sgn}(e^{(L)}(\mathbf{k})) = \mathrm{sgn}(e(l^{(L)}\mathbf{k}))$. If several points fulfill this condition, choose one of them arbitrarily.
- Finally, if $\mathbf{k} \notin U_{2\delta}(S)$, then $l^{(L)}\mathbf{k} = \mathbf{k}$.

Remark 4.13: If $\mathbf{k} \notin \mathcal{S}^{(L)}$, then $|l\mathbf{k} - l^{(L)}\mathbf{k}| \leq \frac{4\pi}{L}$

<u>Proof:</u> Let $B_{\frac{4\pi}{L}}(l\mathbf{k})$ denotes the ball of radius $4\pi/L$ around $l\mathbf{k}$. For each $\mathbf{p} \in B_{\frac{4\pi}{L}}(l\mathbf{k})$,

$$|e(\mathbf{p})| = |e(\mathbf{p}) - e(l\mathbf{k})| \leq ||e||_1 \, |\mathbf{p} - l\mathbf{k}| \leq ||e||_1 \frac{4\pi}{L} \leq \frac{A}{L},$$

such that $B_{\frac{4\pi}{L}}(l\mathbf{k}) \cap \mathbf{\Lambda}_L^\sharp \subset \mathcal{S}^{(L)}$, and $B_{\frac{4\pi}{L}}(l\mathbf{k}) \cap \mathbf{\Lambda}_L^\sharp \neq \emptyset$. Hence for $\mathbf{k}' \in S^{(L)}$, $d(\mathbf{k}', l\mathbf{k}) = \min_{\mathbf{k}'' \in \mathcal{S}^{(L)}} d(\mathbf{k}'', l\mathbf{k})$ implies $\mathbf{k}' \in B_{\frac{4\pi}{L}}(l\mathbf{k})$. In particular, $l^{(L)}\mathbf{k} \in B_{\frac{4\pi}{L}}(l\mathbf{k})$, such that $|l\mathbf{k} - l^{(L)}\mathbf{k}| \leq \frac{4\pi}{L}$.

□

Definition 4.14: For a function $T : \mathbb{R} \times \mathbf{\Lambda}_L^\sharp \to X$, define the projection operator $l^{(L)}$ as

$$\left(l^{(L)}T\right)(k_0, \mathbf{k}) = \begin{cases} 0, & \mathbf{k} \notin U_{2\delta}(S) \\ T(0, l^{(L)}\mathbf{k})\chi(\mathbf{k}), & \mathbf{k} \in U_{2\delta}(S), \end{cases}$$

where χ was defined in 4.7.

Lemma 4.15: Let $T : \mathbb{R} \times \mathbf{\Lambda}_L^\sharp \to \mathbb{C}$ be a function with $|T|_1 < \infty$. Then for $\mathbf{k} \in U_\delta(S)$,

(i) $\left|(1 - l^{(L)})T(k)\right| \leq \alpha |ik_0 - e^{(L)}(\mathbf{k})||T|_1$.

(ii) $|l^{(L)}T|_0 \leq |T|_0$.

(iii) $|l^{(L)}T|_1 \leq \beta |T|_1$.

The constant α and β are given by

$$\alpha = \sqrt{2}(1 + \sqrt{d}(1 + \tfrac{\sqrt{2}}{u_0})) \quad \text{and} \quad \beta = \sqrt{d}(4 + A\tfrac{\sqrt{2}}{u_0} + d||\omega||_1).$$

In order to prove this lemma, we need the following definition:

Definition 4.16: Let $\mathbf{k}$ and $\mathbf{k}'$ be two points in $\mathbf{\Lambda}_L^\sharp$.

(i) A path γ of length $n \in \mathbb{N}$ between $\mathbf{k}$ and $\mathbf{k}'$ is a finite sequence

$$\gamma = (\mathbf{k} = \mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{p}_n = \mathbf{k}')$$

of points $\mathbf{p}_i \in \mathbf{\Lambda}_L^\sharp$ such that for all $0 \leq i \leq n-1$,

$$|\mathbf{p}_i - \mathbf{p}_{i+1}| = \frac{2\pi}{L}$$

and for all $1 \leq i < j \leq n$, $\mathbf{p}_i \neq \mathbf{p}_j$. Here

$$|\mathbf{p}_i - \mathbf{p}_{i+1}| = \sqrt{(\mathbf{p}_i - \mathbf{p}_{i+1}) \cdot (\mathbf{p}_i - \mathbf{p}_{i+1})}$$

is the Euclidean distance in $\mathbb{R}^d$. We say that γ has length $|\gamma| = n$.

(ii) A path γ between $\mathbf{k}$ and $\mathbf{k}'$ is said to be minimal, if

$$|\gamma| = \min\{|\gamma'| : \gamma' \text{ is a path from } \mathbf{k} \text{ to } \mathbf{k}'\}.$$

Remark 4.17: Given $\mathbf{k}$ and $\mathbf{k}'$ in $\mathbf{\Lambda}_L^\sharp$, there always exists at least one minimal path γ_0 between $\mathbf{k}$ and $\mathbf{k}'$.

If $\mathbf{k} = \frac{2\pi}{L}\sum_{\alpha=1}^{d} k_\alpha \mathbf{e}_\alpha$ and $\mathbf{k}' = \frac{2\pi}{L}\sum_{\alpha=1}^{d} k'_\alpha \mathbf{e}_\alpha$, then

$$|\gamma_0| = \frac{L}{2\pi}\sum_{\alpha=1}^{d} |k_\alpha - k'_\alpha|.$$

By Hölder's inequality,

$$|\gamma_0| \leq \sqrt{d}\frac{L}{2\pi}|\mathbf{k} - \mathbf{k}'|,$$

where $|\mathbf{k} - \mathbf{k}'| = \sqrt{(\mathbf{k} - \mathbf{k}') \cdot (\mathbf{k} - \mathbf{k}')}$.

<u>Proof of 4.15:</u>

(i) By definition of χ, $\chi(\mathbf{k}) = 1$ for $\mathbf{k} \in U_\delta(S)$. Observe that

$$\left|(1 - l^{(L)})T(k)\right| \leq |T(k_0, \mathbf{k}) - T(0, \mathbf{k})| + \left|T(0, \mathbf{k}) - T(0, l^{(L)}\mathbf{k})\right|.$$

The first term is bounded by

$$|T(k_0, \mathbf{k}) - T(0, \mathbf{k})| \leq |\partial_0 T|_0 \cdot |k_0| \leq |k_0|\,|T|_1.$$

If $|e^{(L)}(\mathbf{k})| \leq A/L$, then $l^{(L)}\mathbf{k} = \mathbf{k}$, and the claim is trivial. In the case $|e^{(L)}(\mathbf{k})| > A/L$, let γ be a minimal path between $\mathbf{k}$ and $l^{(L)}\mathbf{k}$. Then

$$\begin{aligned}
\left|T(0, l^{(L)}\mathbf{k}) - T(0, \mathbf{k})\right| &\leq \sum_{i=1}^{|\gamma|} |T(0, \mathbf{p}_i) - T(0, \mathbf{p}_{i-1})| \\
&\leq \sum_{i=1}^{|\gamma|} \frac{2\pi}{L}|\nabla_{\alpha_i} T(\mathbf{p}_{i-1})| \\
&\leq \frac{2\pi}{L}|\gamma| \cdot |T|_1 \\
&\leq \sqrt{d}\,|\mathbf{k} - l^{(L)}\mathbf{k}| \cdot |T|_1.
\end{aligned}$$

Further,

$$|\mathbf{k} - l^{(L)}\mathbf{k}| \leq |\mathbf{k} - l\mathbf{k}| + |l\mathbf{k} - l^{(L)}\mathbf{k}|,$$

and by the construction of l, $|\mathbf{k} - l\mathbf{k}| \leq \sqrt{2}/u_0 \cdot |e^{(L)}(\mathbf{k})|$. We use the remark 4.13 and

$$|e^{(L)}(\mathbf{k})| \geq \frac{A}{L} \geq \frac{4\pi}{L}$$

in order to bound the second term with $|e^{(L)}(\mathbf{k})|$. Hence

$$|\mathbf{k} - l^{(L)}\mathbf{k}| \leq (1 + \frac{\sqrt{2}}{u_0})|e^{(L)}(\mathbf{k})|,$$

and the first point follows with

$$\alpha = \sqrt{2}(1 + \sqrt{d}(1 + \frac{\sqrt{2}}{u_0})).$$

(ii) Trivial.

(iii) By hypothesis $\mathbf{k} \in U_\delta(S)$, and $\chi(\mathbf{k}) = 1$. Since the projection is constant with respect to k_0, we consider only the derivative with respect to the spacial variables, defined by

$$|\nabla_\alpha l^{(L)} T(\mathbf{k})| = \frac{L}{2\pi}|T(l^{(L)}(\mathbf{k} + \frac{2\pi}{L}\mathbf{e}_\alpha)) - T(l^{(L)}\mathbf{k})|.$$

If $\mathbf{k}$ and $\mathbf{k} + \frac{2\pi}{L}\mathbf{e}_\alpha$ are both in $\mathcal{S}^{(L)}$, then the claim is trivial. If $\mathbf{k}$ or $\mathbf{k} + \frac{2\pi}{L}\mathbf{e}_\alpha$ is not in $\mathcal{S}^{(L)}$, we choose a minimal path γ between $l^{(L)}(\mathbf{k} + \frac{2\pi}{L}\mathbf{e}_\alpha)$ and $l^{(L)}\mathbf{k}$. Then

$$|\nabla_\alpha l^{(L)} T(\mathbf{k})| \leq |T|_1|\gamma| \leq \sqrt{d}\frac{L}{2\pi}|T|_1|l^{(L)}(\mathbf{k} + \frac{2\pi}{L}\mathbf{e}_\alpha) - l^{(L)}\mathbf{k}|.$$

The last term can be bounded as follows

$$\begin{aligned} |l^{(L)}(\mathbf{k} + \frac{2\pi}{L}\mathbf{e}_\alpha) - l^{(L)}\mathbf{k}| \quad &\leq \quad |l^{(L)}(\mathbf{k} + \tfrac{2\pi}{L}\mathbf{e}_\alpha) - l(\mathbf{k} + \tfrac{2\pi}{L}\mathbf{e}_\alpha)| \\ &\quad + |l^{(L)}\mathbf{k} - l\mathbf{k}| + |l(\mathbf{k} + \frac{2\pi}{L}\mathbf{e}_\alpha) - l\mathbf{k}|. \end{aligned}$$

Using

$$|l\mathbf{k} - l\mathbf{k}'| \leq d||\omega||_1|\mathbf{k} - \mathbf{k}'|,$$

we bound the last term with $d||\omega||_1\frac{2\pi}{L}$. If both $\mathbf{k}$ and $\mathbf{k} + \frac{2\pi}{L}\mathbf{e}_\alpha$ are not in $\mathcal{S}^{(L)}$, then use the remark 4.13 in order to get

$$|l^{(L)}(\mathbf{k} + \frac{2\pi}{L}\mathbf{e}_\alpha) - l^{(L)}\mathbf{k}| \leq \frac{2\pi}{L}(4 + d||\omega||_1).$$

If one of the points is in $\mathcal{S}^{(L)}$, i.e. $\mathbf{k} \in \mathcal{S}^{(L)}$, then $l^{(L)}\mathbf{k} = \mathbf{k}$, and

$$|\mathbf{k} - l\mathbf{k}| \leq \frac{\sqrt{2}}{u_0}|e^{(L)}(\mathbf{k})| \leq \frac{\sqrt{2}}{u_0}\frac{A}{L}.$$

In that case,

$$|l^{(L)}(\mathbf{k}+\frac{2\pi}{L}\mathbf{e}_\alpha)-l^{(L)}\mathbf{k}|\leq\frac{2\pi}{L}(2+A\frac{\sqrt{2}}{u_0}+d||\omega||_1).$$

Finally, we get

$$|\nabla_\alpha l^{(L)}T(\mathbf{k})|\leq\sqrt{d}(4+A\frac{\sqrt{2}}{u_0}+d||\omega||_1)|T|_1.$$

We get therefore the last claim, with $\beta=\sqrt{d}(4+A\frac{\sqrt{2}}{u_0}+d||\omega||_1)$.

□

Lemma 4.18: Let $T:\mathbb{R}\times\mathbb{T}\to\mathbb{C}$ be a C^1-function. Then

$$\left|l^{(L)}T(k)-lT(k)\right|_0\leq\frac{4\pi}{L}||T||_1.$$

Proof:

$$|l^{(L)}T(k)-lT(k)|=|\chi(\mathbf{k})T(0,l^{(L)}\mathbf{k})-\chi(\mathbf{k})T(0,l\mathbf{k})|\leq||T||_1\,|l^{(L)}\mathbf{k}-l\mathbf{k}|.$$

The remark 4.13 implies the claim.

□

4.3 The Renormalization Condition

Lemma 4.19: For each generic dispersion relation $e^{(L)}$, $u\in\mathcal{K}$, and $|\lambda|<\lambda_0$, the self-energy $\Sigma^{(L)}(k,\lambda;u)$ defined by the equation

$$\Sigma^{(L)}(k,\lambda;u)=(1+\hat{G}_2^{(L)}(k;\lambda)\hat{C}_u(k))^{-1}\hat{G}_2^{(L)}(k;\lambda)$$

is analytic in λ. Further,

$$|\Sigma^{(L)}(k,\lambda;u)|<2^{23}\alpha^2||V||_{1,\infty}\,E|\lambda|L/A.$$

Proof: Since for$|\lambda|<\lambda_0$, by remark 3.26

$$|\hat{G}_2^{(L)}|_0|\hat{C}_u|_0<1/2,$$

the self-energy is analytic in λ for $|\lambda| < \lambda_0$. We bound the self-energy as follows:

$$\begin{aligned} |\Sigma^{(L)}(k,\lambda;u)| &\leq |(1+\hat{G}_2^{(L)}(k;\lambda)\hat{C}_u(k))^{-1}| \cdot |\hat{G}_2^{(L)}|_0 \\ &\leq \frac{2^{23}\alpha^2||V||_{1,\infty}EL|\lambda|}{A}. \end{aligned}$$

□

Remark 4.20:

(i) For $|\lambda| < \lambda_0$, $|\Sigma^{(L)}(k,\lambda;u)| < \frac{1}{4L^{d+1}}$, such that

(ii) $|\hat{C}_u(k)\Sigma^{(L)}(k,\lambda)|_0 < 1$.

Definition 4.21: The renormalization condition is given by the equation

$$l^{(L)}u(\mathbf{k},\lambda) - l^{(L)}\Sigma^{(L)}(k_0,\mathbf{k},\lambda;u) = 0.$$

Remark 4.22: We will prove that it is always possible to find a counterterm $K^{(L)} \in \mathcal{K}$ such that $l^{(L)}K^{(L)}(\mathbf{k};\lambda) = K^{(L)}(\mathbf{k};\lambda)$.

Theorem 4.23: Let $e^{(L)}$ be a dispersion relation defined in 2.10, and u, $\delta u \in \mathcal{K}$. Further, let $s \in \mathbb{R}$ be close to 0, such that $u + s\delta u \in \mathcal{K}$. If

$$N(\mathcal{V}, 32\alpha) < \frac{\alpha^2}{5}$$

then

$$N\left(\frac{d}{ds}\bigg|_{s=0}\Omega_{C_s}(\mathcal{V});\alpha\right) \leq \frac{2}{\alpha^2}\frac{N(\mathcal{V},32\alpha)^2}{1-\frac{5}{\alpha^2}N(\mathcal{V},32\alpha)}(1+\mathcal{C}') + 4N(\mathcal{V},\alpha)$$

<u>Proof:</u> First note that for L big enough, $\mathcal{C} \leq \mathcal{C}^2$, such that we set $\mu = 1$ in the theorem 3.13. Further

$$N\left(\frac{d}{ds}\bigg|_{s=0}\Omega_{C_s}(\mathcal{V});\alpha\right) \leq N\left(\frac{d}{ds}\bigg|_{s=0}(\Omega_{C_s}(:\mathcal{W}_s:) - \mathcal{W}_s);\alpha\right) + N\left(\frac{d}{ds}\bigg|_{s=0}\mathcal{W}_s\right).$$

Since

$$||\frac{d}{ds}\bigg|_{s=0}U_s||_{1,\infty} \leq 4||V||_{1,\infty}||\frac{d}{ds}\bigg|_{s=0}C_s||_\infty \leq 4||V||_{1,\infty},$$

the second term is bounded by

$$N\left(\frac{d}{ds}\bigg|_{s=0}\mathcal{W}_s\right) = N\left(\frac{d}{ds}\bigg|_{s=0}\mathcal{U}_s\right) \leq 4\mathcal{C}\alpha^2|\lambda|\,||V||_{1,\infty} \leq 4N(\mathcal{V},\alpha).$$

The first term is bounded by theorem 3.13:

$$
\begin{aligned}
& N(\tfrac{d}{ds}\big|_{s=0}(\Omega_{C_s}(:\mathcal{W}_s:)-\mathcal{W}_s);\alpha) \\
& \leq \frac{1}{2\alpha^2}\frac{N(\mathcal{W}_0,32\alpha)}{1-\frac{1}{\alpha^2}N(\mathcal{W}_0,32\alpha)}\left(N(\frac{d}{ds}\bigg|_{s=0}\mathcal{W}_s,8\alpha)+\frac{\mathcal{C}'}{4}N(\mathcal{W}_0,32\alpha)\right) \\
& \leq \frac{2}{\alpha^2}\frac{N(\mathcal{V},32\alpha)}{1-\frac{5}{\alpha^2}N(\mathcal{V},32\alpha)}\left(N(\mathcal{V},8\alpha)+\mathcal{C}'N(\mathcal{V},32\alpha)\right) \\
& \leq \frac{2}{\alpha^2}\frac{N(\mathcal{V},32\alpha)^2}{1-\frac{5}{\alpha^2}N(\mathcal{V},32\alpha)}(1+\mathcal{C}').
\end{aligned}
$$

□

Remark 4.24: For $|\lambda|<\lambda_0'$, where

$$\lambda_0'=\frac{A^2}{2^{29}\alpha^2L^{d+2}E||V||_{1,\infty}},$$

the hypothesis of the theorem is satisfied, and $N(\mathcal{V},32\alpha)<\alpha^2/10$.

Corollary 4.25: For $|\lambda|<\lambda_0'$,

$$||\frac{d}{ds}\bigg|_{s=0}G_2^{(L)}||_{1,\infty}\leq|\lambda|\cdot\frac{2^{27}L^{d+2}E\alpha^2||V||_{1,\infty}}{A^2}.$$

<u>Proof:</u>

$$
\begin{aligned}
||\tfrac{d}{ds}\big|_{s=0}G_2^{(L)}||_{1,\infty} &\leq \frac{1}{\mathcal{C}\alpha^2}N(\frac{d}{ds}\bigg|_{s=0}\Omega_{C_s}(\mathcal{V}),\alpha) \\
&\leq \frac{4\mathcal{C}'}{\mathcal{C}\alpha^4}\frac{N(\mathcal{V},32\alpha)^2}{1-\frac{4}{\alpha^2}N(\mathcal{V},32\alpha)}+\frac{4}{\mathcal{C}\alpha^2}N(\mathcal{V},\alpha) \\
&\leq \frac{4}{\mathcal{C}\alpha^2}N(\mathcal{V},32\alpha)(1+\frac{2\mathcal{C}'}{\alpha^2}N(\mathcal{V},32\alpha)) \\
&\leq \frac{8\mathcal{C}'}{\mathcal{C}\alpha^2}N(\mathcal{V},32\alpha) \\
&\leq 2^{25}\mathcal{C}'\alpha^2b^2|\lambda|\,||V||_{1,\infty} \\
&\leq 2^{27}\alpha^2\frac{E||V||_{1,\infty}L^{d+2}}{A^2}|\lambda|.
\end{aligned}
$$

□

Lemma 4.26: For $|\lambda| < \lambda_0$,

$$|\left.\frac{d}{ds}\right|_{s=0} \Sigma^{(L)}(k,\lambda;u)| < |\lambda| \frac{2^{28}\alpha^2||V||_{1,\infty}EL^{d+2}}{A^2}.$$

Proof: First remark that

$$\left.\frac{d}{ds}\right|_{s=0} \hat{C}_s(k) = -\hat{C}_u^2(k)\delta u(\mathbf{k}).$$

Thus,

$$\begin{aligned}\left.\frac{d}{ds}\right|_{s=0} \Sigma^{(L)}(k,\lambda;u+s\delta u) &= \left.\frac{d}{ds}\right|_{s=0} \hat{G}_2^{(L)}(k,\lambda)\frac{1-\Sigma^{(L)}(k,\lambda)\hat{C}_u(k)}{1+\hat{G}_2^{(L)}(k,\lambda)\hat{C}_u(k)} \\ &\quad +\Sigma^{(L)}(k,\lambda)^2\hat{C}_u^2(k)\delta u(\mathbf{k}).\end{aligned}$$

Thus, by remarks 4.20 and 3.26,

$$\begin{aligned}\left|\left.\frac{d}{ds}\right|_{s=0} \Sigma^{(L)}(k,\lambda;u+s\delta u)\right| &\leq 4\left|\left.\frac{d}{ds}\right|_{s=0} \hat{G}_2^{(L)}(k,\lambda)\right| + \left|\Sigma^{(L)}(k,\lambda,u)\right| \\ &\leq |\lambda|\cdot 2^{23}\frac{\alpha^2||V||_{1,\infty}E|\lambda|L}{A}\left(\frac{2^4L^{d+1}}{A}+1\right) \\ &\leq |\lambda|\frac{2^{28}\alpha^2||V||_{1,\infty}EL^{d+2}}{A^2}\end{aligned}$$

□

Corollary 4.27: If $|\lambda| < \lambda_0''$, where

$$\lambda_0'' = \frac{A^3}{2^{32}\alpha^2||V||_{1,\infty}EL^{d+3}}$$

then

$$|\left.\frac{d}{ds}\right|_{s=0} \Sigma^{(L)}(k,\lambda;u)| < \frac{A}{4L}.$$

Theorem 4.28: For dispersion relations $e^{(L)}$ defined in 2.10, there is a unique counterterm $K^{(L)} \in \mathcal{K}$ with $l^{(L)}K^{(L)} = K^{(L)}$ such that the renormalization condition is satisfied for $|\lambda| < \lambda_0''$.

Proof: We have to solve the equation

$$u(\mathbf{k}) - l^{(L)}\Sigma^{(L)}(0,\mathbf{k},\lambda;u) = 0$$

for $u \in \mathcal{K}$. By the corollary 4.27, the map $u \mapsto l^{(L)}\Sigma^{(L)}(\cdot; u)$ is a contracting map. We define the sequence $u^{(n)} \in \mathcal{K}$ as follow. Let $u^{(0)} = 0$, and

$$u^{(n)}(\mathbf{k}) := l^{(L)}\Sigma^{(L)}(0, \mathbf{k}, \lambda; u^{(n-1)}).$$

By the remark 4.20(i), the sequence is in $\mathcal{K}$. Further,

$$\begin{aligned} |u^{(n)} - u^{(n-1)}| &= \left|l^{(L)}\Sigma^{(L)}(0, \mathbf{k}, \lambda; u^{(n-1)}) - l^{(L)}\Sigma^{(L)}(0, \mathbf{k}, \lambda; u^{(n-2)})\right| \\ &\leq \left|\left.\frac{d}{ds}\right|_{s=0} \Sigma^{(L)}(0, \mathbf{k}, \lambda; u^{(n-2)} + s\delta u^{(n-1)})\right| |u^{(n-1)} - u^{(n-2)}| \end{aligned}$$

where $\delta u^{(n-1)} = u^{(n-1)} - u^{(n-2)}$. Hence by corollary 4.27,

$$|u^{(n)} - u^{(n-1)}| \leq \frac{A}{4L}|u^{(n-1)} - u^{(n-2)}|.$$

Iterating the bound, we get

$$|u^{(n)} - u^{(n-1)}| \leq \left(\frac{A}{4L}\right)^n.$$

The sequence of the $u^{(n)}$ is therefore a Cauchy sequence in the Banach space of the function on $\mathbf{\Lambda}_L^\sharp \times \mathbb{C}$ that are analytic in λ. Hence

$$K^{(L)}(\mathbf{k}; \lambda) := \lim_{n\to\infty} u^{(n)}(\mathbf{k}, \lambda) \in \mathcal{K}$$

is the counterterm.

Suppose that two functions K_1 and $K_2 \in \mathcal{K}$ solve the renormalization equation. Then for all $\mathbf{k} \in \mathbf{\Lambda}_L^\sharp$ and λ,

$$\begin{aligned} |K_1(\mathbf{k}; \lambda) - K_2(\mathbf{k}; \lambda)| &= |\Sigma^{(L)}(0, l^{(L)}\mathbf{k}, \lambda; K_1) - \Sigma^{(L)}(0, l^{(L)}\mathbf{k}, \lambda; K_2)| \\ &\leq \sup_{\delta u \in \mathcal{K}} \left|\left.\frac{d}{ds}\right|_{s=0} \Sigma^{(L)}(0, \mathbf{k}, \lambda; K_1 + s\delta u)\right| |K_1 - K_2|_0. \end{aligned}$$

Since $A/4L < 1$ for L big enough, this leads to a contradiction unless $K_1 = K_2$. By definition of $K^{(L)}$ as the solution of the equation

$$u_\sigma(\mathbf{k}) - l^{(L)}\Sigma_\sigma^{(L)}(0, \mathbf{k}, \lambda; u) = 0,$$

we have

$$\begin{aligned} l^{(L)}K^{(L)}(\mathbf{k}, \sigma, \lambda) &= l^{(L)}\left(l^{(L)}\Sigma_\sigma^{(L)}(0, \mathbf{k}, \lambda; K^{(L)})\right) \\ &= l^{(L)}\Sigma_\sigma^{(L)}(0, \mathbf{k}, \lambda; K^{(L)}) \\ &= K^{(L)}(\mathbf{k}, \sigma, \lambda). \end{aligned}$$

□

Chapter 5

The Thermodynamic Limit

5.1 The Graphs Expansion

Definition 5.1:

(i) The amputated Green functions $G^{(L)}_{2m,r}$ at order r in λ are defined through Taylor expansion,

$$\hat{G}^{(L)}_{2m} = \sum_{r\geq 0} \lambda^r G^{(L)}_{2m,r}.$$

For the two-point (non-amputated) Green function,

$$\hat{S}^{(L)}(k) = C(k) + \sum_{r\geq 1} \lambda^r S^{(L)}_r(k),$$

where $S^{(L)}_r(k) = C(k)G^{(L)}_{2,r}(k)C(k)$.

(ii) The conterterm $K^{(L)}_r$ at order r in λ is defined in the same way:

$$K^{(L)}(\mathbf{k};\lambda) = \sum_{r\geq 1} \lambda^r K^{(L)}_r(\mathbf{k}).$$

(iii) The self-energy $\Sigma^{(L)}_r$ at order r in λ is defined by:

$$\Sigma^{(L)}(k,\lambda) = \sum_{r\geq 1} \lambda^r \Sigma^{(L)}_r(k).$$

Remark 5.2:

(i) It is well-known that

$$G^{(L)}_{2m,r} = \sum_{G\in\mathcal{G}^{2m}(r)} \mathrm{Val}_L\, G,$$

where the sum runs over all the $2m-$legged, connected, amputated graphs of order r, with n four-legged vertices V and n_i two-legged vertices $K_i^{(L)}$, for $i = 1,\dots,r$, such that $n + n_1 + 2n_2 + \cdots + rn_r = r$.

The value of the graph G is given by

$$\begin{aligned}\delta(\sum_{i=1}^{2m} k_i)\,\mathrm{Val}_L(G) = \; & L^{-d|L(G)|} \sum_{\mathbf{k}_l\in\mathbf{\Lambda}_L^\sharp,\, l\in L(G)} \int \prod_{l\in L(G)} \frac{dk_{l0}}{2\pi} C^{(L)}(k_l) \\ & \cdot \prod_{v\in V_4(G)} \left(\prod_{l_1,\dots,l_4\in L(v)} \delta(\sum_{i=1}^{4} k_{l_i}) V(k_{l_1},\dots,k_{l_4}) \right) \\ & \cdot \prod_{w\in V_2(G)} \left(\prod_{l_1,l_2\in L(w)} \delta(k_{l_1}+k_{l_2}) K_w^{(L)}(k_{l_1}) \right),\end{aligned}$$

where

$$\delta(k-k') = (2\pi L)^d \delta_{\mathbf{k}\mathbf{k}'}\delta(k_0-k_0').$$

We define $L(G)$ to be the set of internal lines of G^J and $E(G^J)$ the set of external legs of G^J. $V(G)$ denotes the set of all vertices of G^J, and $V_2(G)$ resp. $V_4(G)$ denotes the set of the two- resp. four-legged vertices of G.

The propagator in momentum space is

$$C^{(L)}(k) = \frac{e^{ik_0 0_+}}{ik_0 - e^{(L)}(\mathbf{k})}.$$

(ii) For the self-energy,

$$\Sigma_r^{(L)} = \sum_{G\in\tilde{\mathcal{G}}^2(r)} \mathrm{Val}_L\, G,$$

where $\tilde{\mathcal{G}}^2(r)$ is the set of all two-legged, 1PI graphs built up from n four-legged vertices V, and n_i two-legged vertices $K_i^{(L)}$, for $i = 1,\dots,r$, such that $n + n_1 + \cdots + rn_r = r$.

Remark 5.3:

(i) We consider the counterterm $K_r^{(L)}(\mathbf{k})$ as a 1PI graph build up from one two-legged vertex. The counterterm $K_r^{(L)}$ is therefore absorbed in the self-energy. Thus, the renormalization condition reads now:

$$l^{(L)}\Sigma_r^{(L)}(k) = 0.$$

(ii) It is useful to solve the renormalization equation explicitly in terms of graphs. For that purpose, remark that

$$l^{(L)}\Sigma_r^{(L)}(k) = K_r^{(L)}(\mathbf{k}) + l^{(L)} \sum_{G\in\tilde{\mathcal{G}}'} \mathrm{Val}_L\, G,$$

where $\tilde{\mathcal{G}}'$ is the set of all graphs in $\tilde{\mathcal{G}}^2(r)$ that contain at least one four-legged vertex.

It follows that the two-legged vertex $K_r^{(L)}$ cannot enter the composition of the second term. We get an inductive procedure to determine the counterterm order by order in λ, which is given by

$$K_r^{(L)}(\mathbf{k}) = -l^{(L)} \sum_{G\in\tilde{\mathcal{G}}'} \mathrm{Val}_L\, G,$$

the right hand side containing only counterterms at order $r' < r$ in λ.

5.2 The Convergence in Thermodynamic Limit

Theorem 5.4: Suppose that the assumption **A1-A3** are verified. Then:

(i) For each $L \in \mathbb{N}$, there are constants A_0, B_0 and B_1 that are independent of L but depend on r such that

$$|G_{2,r}^{(L)}|_0 \leq A_0,$$

and

$$|\Sigma_r^{(L)}|_0 \leq B_0 \quad \text{and} \quad |\Sigma_r^{(L)}|_1 \leq B_1.$$

(ii) In thermodynamic limit,

$$G_{2,r}^{(L)} \overset{L\to\infty}{\to} G_{2,r} \text{ in the } ||\cdot||_0 - \text{norm},$$

and

$$\Sigma_r^{(L)} \overset{L\to\infty}{\to} \Sigma_r \text{ in the } ||\cdot||_0 - \text{norm},$$

where $G_{2,r}$ and Σ_r are defined in 2.22.

(iii) For $L \to \infty$, the counterterms $K_r^{(L)}$ converges to the counterterms K_r in the supremum norm $||\cdot||_0-$norm.

(iv) In the thermodynamic limit,

$$G_{2m,r}^{(L)} \overset{L\to\infty}{\to} G_{2m,r} \text{ in the } ||\cdot||' - \text{norm}.$$

(v) The same is true for the two-points Green function at order $r \geq 1$ in λ:

$$S_r^{(L)} \overset{L\to\infty}{\to} S_r \text{ in the } ||\cdot||' - \text{norm},$$

where $G_{2m,r}$ and S_r are the Green functions defined in 2.22.

Corollary 5.5: In thermodynamic limit, the density $\rho^{(L)}(\lambda)$ converges in the sense of formal power series to $\rho(\lambda)$, that is for all $r \geq 0$,

$$\rho_r^{(L)} \overset{L\to\infty}{\to} \rho_r$$

and

$$\rho_0^{(L)} \overset{L\to\infty}{\to} 2\,\mathrm{Vol}(S).$$

<u>Proof:</u> By remark 2.19,

$$|\rho_r^{(L)}| \leq L^{-d} \sum_{\substack{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp \\ \sigma\in\{\uparrow,\downarrow\}}} \int \frac{dk_0}{2\pi} |S_r^{(L)}(k)|.$$

By point (v) of the theorem 5.4,

$$L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi} |S_r^{(L)}(k)| \overset{L\to\infty}{\to} \int \frac{d^{d+1}k_0}{(2\pi)^{d+1}} |S_r(k)|.$$

Hence,

$$\rho_r^{(L)} \overset{L\to\infty}{\to} \sum_{\sigma\in\{\uparrow,\downarrow\}} \int \frac{d^{d+1}k_0}{(2\pi)^{d+1}} S_r(k) e^{ik_0 0_+} = \rho_r.$$

Further,

$$\rho_0^{(L)} = 2L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi} \frac{e^{ik_0 0_+}}{ik_0 - e^{(L)}(\mathbf{k})} = 2L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \Theta(-e^{(L)}(\mathbf{k})),$$

which tends to twice the volume enclosed in the Fermi surface, since $\Theta(-e^{(L)}(\mathbf{k})) - \Theta(-e(\mathbf{k}))$ is non vanishing on a set of volume bounded by $const\, L^{-1}$. □

Remark 5.6: The theorem 2.24 follows directly from theorem 5.4 and its corollary 5.5.

Chapter 6

Proof of the Convergence

In this section, we prove the convergence of the Green functions in thermodynamic limit at each order in λ. The proof given here follows the proof given in [7] on the continuum momentum space, performing the same scale decomposition, and using the same power counting. The difficulties appear at energy scales lower than the dual lattice spacing. The effective cutoff implement in definition 2.10 allows to apply the same bound as in the continuous case.

6.1 Scale Decomposition

In order to bound the value of a graph, we decompose the propagator in a sum over energy scales. The problem of computing a possibly divergent integrals is replaced by the question of the convergence of series. In order to perform the decomposition over the scales, we first define a $C^\infty-$partition of the unity.

Definition 6.1: Let ϵ be defined by the volume improvement estimate, u_0 and δ defined in 4.5, and $M > \max\{4^{1/\epsilon}, (u_0\delta)^{-1}\}$. Then $|e(\mathbf{p})| < M^{-1}$ implies $\mathbf{p} \in U_\delta(S)$. Let $a \in C^\infty(\mathbb{R}_+, [0,1])$ be such that

$$a(x) = \begin{cases} 0 & \text{for} \quad x \leq M^{-4} \\ 1 & \text{for} \quad x \geq M^{-2} \end{cases}$$

and $a'(x) > 0$ for all $x \in (M^{-4}, M^{-2})$. Set

$$f(x) = a(x) - a(x/M^2) = \begin{cases} 0 & \text{for} & x \leq M^{-4} \\ a(x) & \text{for} & M^{-4} \leq x \leq M^{-2} \\ 1 - a(x/M^2) & \text{for} & M^{-2} \leq x \leq 1 \\ 0 & \text{for} & x \geq 1 \end{cases}$$

so that for all $x > 0$, $f(x) \geq 0$, and

$$1 - a(x) = \sum_{j=-\infty}^{-1} f(M^{-2j}x).$$

With $f_j(x) = f(M^{-2j}x)$, we have

$$\operatorname{supp} f_j \subset [M^{2j-4}, M^{2j}]$$

and for all $x \geq 0$,

$$f_j(x) f_{j'}(x) = 0 \text{ if } |j - j'| \geq 2.$$

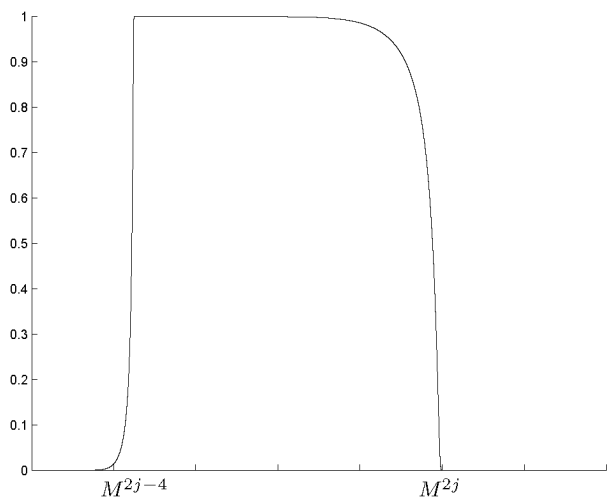

Figure 6.1: *The $j-$shell*

Definition 6.2:

(i) The propagator at scale $j < 0$ of the model on the finite lattice is defined by

$$C_j^{(L)}(k_0, \mathbf{k}) = \frac{f_j\left(k_0^2 + e^2(\mathbf{k})\right)}{ik_0 - e^{(L)}(\mathbf{k})}.$$

(ii) The propagator at scale $j < 0$ of the model in the infinite volume is defined by

$$C_j(p_0, \mathbf{p}) = \frac{f_j\left(p_0^2 + e^2(\mathbf{p})\right)}{ip_0 - e(\mathbf{p})}.$$

(iii) The propagator at scale 0 is defined to be the UV-part of the propagator:

$$C_0^{(L)}(p) = C_0(p) = \frac{1}{ip_0 - e(\mathbf{p})} a((p_0)^2 + e^2(\mathbf{p})).$$

Remark 6.3:

(i) Let

$$j_L := \left[\frac{\ln A - \ln L}{\ln M}\right] + 2,$$

where for $x \in \mathbb{R}$, $[x] \in \mathbb{Z}$ with $[x] - 1 \leq x \leq [x]$. Then for $j \geq j_L$, $\operatorname{supp} f_j \subset (\frac{A^2}{L^2}, M^{-4})$, and $C_j^{(L)}(k) = C_j(k)$.

(ii) By definition,

$$\frac{e^{ip_0 0_+}}{ip_0 - e(\mathbf{p})} = \frac{e^{ip_0 0_+}}{ip_0 - e(\mathbf{p})} a((p_0)^2 + e^2(\mathbf{p})) + e^{ip_0 0_+} \sum_{j=-\infty}^{-1} C_j(p_0, \mathbf{p}),$$

and

$$\frac{e^{ik_0 0_+}}{ik_0 - e^{(L)}(\mathbf{k})} = \frac{e^{ik_0 0_+}}{ik_0 - e(\mathbf{k})} a((k_0)^2 + e^2(\mathbf{k})) + e^{ik_0 0_+} \sum_{j=-\infty}^{-1} C_j^{(L)}(k_0, \mathbf{k}).$$

(iii) On the finite lattice, for dispersion relations with $\mu \in \mathcal{M} \setminus M$, there is a natural cut-off at scale

$$I(L) := \left[\frac{\ln c_L}{\ln M}\right] + 2.$$

Lemma 6.4:

(i) For $j \leq 0$, $|C_j|_0 \leq M^{2-j}$ and $||C_j^{(L)}||_0 \leq M^{2-j}$. More precisely,

$$|C_j(k_0, \mathbf{k})| \leq M^{2-j} 1_{|ik_0 - e(\mathbf{k})| \in [M^{j-2}, M^j]}$$

and

$$|C_j^{(L)}(k_0, \mathbf{k})| \leq M^{2-j} 1_{|ik_0 - e(\mathbf{k})| \in [M^{j-2}, M^j]}.$$

(ii) There is a constant $K_0 > 0$ such that for $j < 0$,

$$|C_j^{(L)}|' = L^{-d} \sum_{\mathbf{k} \in \mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi} |C_j^{(L)}(k)| \leq K_0 M^j$$

and

$$||C_j||' = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |C_j(k)| \leq K_0 M^j.$$

(iii) With the same constant $K_0 > 0$,

$$\sup_{p\in\mathbb{R}\times\mathbb{T}} \sup_{v\in\{+,-\}} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |C_0(k)C_0(p+vk)| \leq K_0 M^2,$$

and

$$\sup_{p\in\mathbb{R}\times\mathbf{\Lambda}_L^\sharp} L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi} |C_0(k)C_0(p\pm k)| \leq K_0 M^2.$$

(vi) For each $0 \leq \alpha \leq d$, there is a constant $W > 0$ such that

$$|\nabla_\alpha C_j^{(L)}(k)| \leq W M^{2-2j} 1_{|ik_0 - e(\mathbf{k})|\in[M^{j-2},M^j]},$$

and

$$|\partial_\alpha C_j(k)| \leq W M^{2-2j} 1_{|ik_0 - e(\mathbf{k})|\in[M^{j-2},M^j]}$$

(v) Volume improvement bound: For all $j_1, j_2, j_3 < 0$,

$$\sup_{\substack{p\in\mathbb{R}\times\mathbb{T}\\ v,v'\in\{+,-\}}} |\int \frac{dk^{d+1}}{(2\pi)^{d+1}} \frac{dk'^{d+1}}{(2\pi)^{d+1}} C_{j_1}(k)C_{j_2}(k')C_{j_3}(vk+v'k'+p)| \leq K_1 M^{j_1+j_2+(\epsilon-1)j_3},$$

and

$$\sup_{\substack{p\in\mathbb{R}\times\mathbf{\Lambda}_L^\sharp\\ v,v'\in\{+,-\}}} |L^{-2d} \sum_{\mathbf{k},\mathbf{k}'\in\mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi}\frac{dk_0'}{2\pi} C_{j_1}^{(L)}(k)C_{j_2}^{(L)}(k')C_{j_3}^{(L)}(vk+v'k'+p)| \leq K_1 M^{j_1+j_2+(\epsilon-1)j_3}.$$

<u>Proof:</u>

(i) This claim follows trivially from the support properties of C_j. For $j < j_L$, we use $\frac{L}{A} \leq M^{2-j}$.

(ii) We begin with the continuous case, using (i):

$$\begin{aligned}
\int \frac{d^{d+1}k}{(2\pi)^{d+1}} |C_j(k)| &\leq M^{2-j} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} 1_{|ik_0-e(\mathbf{k})|\in[M^{j-2},M^j]} \\
&\leq \frac{2M^2}{2\pi} \int \frac{d^dk}{(2\pi)^d} 1_{|e(\mathbf{k})|\leq M^j} \\
&\leq \frac{2M^2}{(2\pi)^{d+1}} \int_{-M^j}^{M^j} d\rho \int_S d\omega |J(\rho,\omega)| \\
&\leq \frac{4M^2}{(2\pi)^{d+1}} ||J||_0 |S| M^j.
\end{aligned}$$

On the finite lattice $\mathbf{\Lambda}_L^\sharp$, we distinguish the cases $j \geq j_L$ and $j < j_L$. In the first case, using lemma 4.10

$$\begin{aligned} L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi} |C_j(k)| &\leq |C_j|_0 L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi} 1_{|ik_0 - e(\mathbf{k})|\in[M^{j-2},M^j]} \\ &\leq |C_j|_0 \int \frac{dk_0}{2\pi} 1_{|k_0|\leq M^j} L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} 1_{|e(\mathbf{k})|\leq M^j} \\ &\leq \frac{2M^2}{2\pi} c_2 |S| M^j. \end{aligned}$$

If $j < j_L$, then

$$\begin{aligned} L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi} |C_j^{(L)}(k)| &\leq \int \frac{dk_0}{2\pi} 1_{|k_0|\leq M^j} L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \frac{L}{A} 1_{|e(\mathbf{k})|\leq \frac{A}{L}} \\ &\leq \frac{L}{A\pi} M^j c_2 |S| \frac{A}{L} \\ &\leq \frac{c_2|S|}{\pi} M^j. \end{aligned}$$

We set $K_0 = \sup\{\frac{4M^2}{(2\pi)^{d+1}} ||J||_0 |S|, \frac{M^2}{\pi} c_2 |S|\}$.

(iii) The integral over the torus $\mathbb{T}$ of the propagators is harmless:

$$\begin{aligned} \int \frac{dk^{d+1}}{(2\pi)^{d+1}} |C_0(k) C_0(p \pm k)| &\leq \sup_{\mathbf{k}\in\mathbb{T}} \int \frac{dk_0}{2\pi} |C_0(k) C_0(p \pm k)| \\ &\leq \int \frac{dk_0}{2\pi} \frac{1_{|k_0|>M^{-1}} 1_{|k_0 \pm p_0|>M^{-1}}}{\sqrt{(k_0^2 + M^{-2})((k_0+p_0)^2 + M^{-2})}} \\ &\leq I, \end{aligned}$$

where $I = \sup_{y\in\mathbb{R}} I(y)$ with

$$I(y) = \int_{|x|,|x+y|>M^{-1}} \frac{dx}{\sqrt{(x^2 + M^{-2})((x+y)^2 + M^{-2})}}.$$

If $|y| < 1/2M$, then for $|x| \geq M^{-1}$, $|y|/|x| \leq 1/2$ and

$$(x+y)^2 + M^{-2} \geq x^2 (1 + \frac{y}{x})^2 \geq x^2 (1 - \left|\frac{y}{x}\right|)^2 \geq x^2/4$$

Hence

$$\sup_{|y|<(2M)^{-1}} I(y) \leq 2 \int_{M^{-1}}^\infty \frac{dx}{x^2} \leq 2M \leq 2M^2.$$

We turn to the case $|y| \geq 1/2M$. First observe that $I(y) = I(-y)$:

$$\begin{aligned}
&I(y) = \\
&= \int_{|x+y/2|,|x-y/2|>M^{-1}} \frac{dx}{\sqrt{((x+\frac{y}{2})^2+M^{-2})((x-\frac{y}{2})^2+M^{-2})}} \\
&= 2\int_{|x\pm y/2|>M^{-1}} \frac{dx}{\sqrt{((x+\frac{y}{2})^2+M^{-2})((x-\frac{y}{2})^2+M^{-2})}} \\
&= I(-y),
\end{aligned}$$

such that $I = \sup_{y>0} I(y)$. We bound the denominator with

$$(x^2 - \frac{y^2}{4})^2 + 2M^{-2}(x^2 + \frac{y^2}{4}) + M^{-4} \geq (x^2 - \frac{y^2}{4})^2,$$

and get

$$I(y) \leq 2\int_{M^{-1}+\frac{y}{2}}^{\infty} \frac{dx}{(x+\frac{y}{2})(x-\frac{y}{2})} = \frac{1}{y}\ln\frac{y+M^{-1}}{M^{-1}}.$$

Thus, the function $I(y)$ is bounded by a decreasing function of $y \geq 1/2M$. Thus, for $y \geq 1/2M$,

$$I(y) \leq \frac{M}{2}\ln(3/2) \leq 2M^2,$$

and

$$\int \frac{dk^{d+1}}{(2\pi)^{d+1}} |C_0(k)C_0(p \pm k)| \leq 2M^2.$$

The same bound holds on the lattice $\mathbf{\Lambda}_L^\sharp$.

(vi) We first bound the derivative of the propagator on $\mathbb{R} \times \mathbb{T}$. In that case,

$$|\partial_\alpha C_j(p)| \leq \frac{|\partial_\alpha f_j(p_0^2 + e^2(\mathbf{p}))|}{|ip_0 - e(\mathbf{p})|} + \frac{|f_j(p_0^2 + e^2(\mathbf{p}))|}{p_0^2 + e^2(\mathbf{p})} \sup\{1, ||e||_1\}.$$

Since by definition,

$$\begin{aligned}
|\partial_\alpha f_j(k_0^2 + e^2(\mathbf{k}))| &= |\partial_\alpha(k_0^2 + e^2(\mathbf{k}))M^{-2j} f'(M^{-2j}(k_0^2 + e^2(\mathbf{k})))| \\
&\leq 2\sup\{1, ||e||_1\}|ik_0 - e(\mathbf{k})|\, M^{-2j} \\
&\qquad \cdot |f'(M^{-2j}(k_0^2 + e^2(\mathbf{k})))| 1_{|ik_0 - e(\mathbf{k})| \leq M^j} \\
&\leq 2\sup\{1, ||e||_1\}||f||_1 M^{-j},
\end{aligned}$$

we get the claim with $W = \sup\{1, ||e||_1\}(1 + 2||f||_1)M^2$. On $\mathbb{R} \times \mathbf{\Lambda}_L^\sharp$, for $j \geq j_L$, the remark 4.3 implies the bound. For $j < j_L$,

$$\left| \partial_0 \frac{f_j(k_0^2 + e^2(\mathbf{k}))}{ik_0 - \frac{A}{L}} \right| \leq W M^{2-2j},$$

in the same way as for C_j, using $L/A \leq M^{2-j}$. Further, for $\alpha = 1, \ldots, d$,

$$\left| \nabla_\alpha \frac{f_j(k_0^2 + e^2(\mathbf{k}))}{ik_0 - \frac{A}{L}} \right| \leq \frac{||f_j||_1}{|ik_0 - \frac{A}{L}|},$$

by the remark 4.3. The claim follows.

(v) We prove first the bound on $\mathbb{R} \times \mathbb{T}$. Performing the k_0-integral, we get

$$\begin{aligned}
&\left| \int \frac{dk^{d+1}}{(2\pi)^{d+1}} \frac{dk'^{d+1}}{(2\pi)^{d+1}} C_{j_1}(k) C_{j_2}(k') C_{j_3}(vk + v'k' + p) \right| \\
&\leq \frac{4M^{4-j_3}}{(2\pi)^2} \int \frac{dk^d}{(2\pi)^d} \frac{dk'^d}{(2\pi)^d} 1_{|e(\mathbf{k})| \leq M^{j_1}} 1_{|e(\mathbf{k}')| \leq M^{j_2}} 1_{|e(v\mathbf{k} + v'\mathbf{k}' + \mathbf{p})| \leq M^{j_3}} \\
&= \frac{4M^{4-j_3}}{(2\pi)^2} I(M^{j_1}, M^{j_2}, M^{j_3}).
\end{aligned}$$

Applying the volume improvement bound, we get the claim. We turn now to the lattice case. Observe that without lost of generality, one can set $j_3 = \min\{j_1, j_2, j_3\}$. If $j_3 \geq j_L$, then performing the k_0-integrals, we get

$$\begin{aligned}
&\left| L^{-2d} \sum_{\mathbf{k}, \mathbf{k}' \in \mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi} \frac{dk_0'}{2\pi} C_{j_1}(k) C_{j_2}(k') C_{j_3}(vk + v'k' + p) \right| \\
&\leq \frac{4M^{4-j_3}}{(2\pi)^2} L^{-2d} \sum_{\mathbf{k}, \mathbf{k}' \in \mathbf{\Lambda}_L^\sharp} 1_{|e(\mathbf{k})| \leq M^{j_1}} 1_{|e(\mathbf{k}')| \leq M^{j_2}} 1_{|e(v\mathbf{k} + v'\mathbf{k}' + \mathbf{p})| \leq M^{j_3}}.
\end{aligned}$$

We bound the sums over the lattice points using lemma 4.10. For this purpose, define the set

$$\begin{aligned}
A(M^{j_1}, M^{j_2}, M^{j_3}) = \ & \big\{ (\mathbf{p}, \mathbf{p}') \in \mathbb{T} \times \mathbb{T} : |e(\mathbf{p})| < M^{j_1}, |e(\mathbf{p}')| < M^{j_2} \text{ and} \\
& |e(v\mathbf{P} + v'\mathbf{p}' + \mathbf{q})| < M^{j_3} \big\}.
\end{aligned}$$

Thus,

$$1_{|e(\mathbf{k})| \leq M^{j_1}} 1_{|e(\mathbf{k}')| \leq M^{j_2}} 1_{|e(v\mathbf{k} + v'\mathbf{k}' + \mathbf{p})| \leq M^{j_3}} = 1_{(\mathbf{k}, \mathbf{k}') \in A(M^{j_1}, M^{j_2}, M^{j_3})}.$$

Further, define $\chi \in C^\infty(\mathbb{T} \times \mathbb{T}, [0, 1])$ with

$$A(M^{j_1}, M^{j_2}, M^{j_3}) \subset \operatorname{supp} \chi \subset A(2M^{j_1}, 2M^{j_2}, 2M^{j_3}),$$

and

$$\chi|_{A(M^{j_1}, M^{j_2}, M^{j_3})} = 1.$$

We choose χ such that $||\chi||_1 \leq DM^{-\inf\{j_1,j_2\}} \leq DL/2\pi$, for a constant $D > 1$. Then

$$\begin{aligned}
&L^{-2d} \sum_{\mathbf{k},\mathbf{k}' \in \mathbf{\Lambda}_L^\sharp} 1_{(\mathbf{k},\mathbf{k}') \in A(M^{j_1},M^{j_2},M^{j_3})} \\
&\leq L^{-2d} \sum_{\mathbf{k},\mathbf{k}' \in \mathbf{\Lambda}_L^\sharp} \chi(\mathbf{k},\mathbf{k}') \\
&\leq \int \frac{d^d\mathbf{p}}{(2\pi)^d}\frac{d^d\mathbf{p}'}{(2\pi)^d}\chi(\mathbf{p},\mathbf{p}') + \left| \int \frac{d^d\mathbf{p}}{(2\pi)^d}\frac{d^d\mathbf{p}'}{(2\pi)^d}\chi(\mathbf{p},\mathbf{p}') - L^{-2d} \sum_{\mathbf{k},\mathbf{k}' \in \mathbf{\Lambda}_L^\sharp} \chi(\mathbf{k},\mathbf{k}') \right| \\
&\leq \int \frac{d^d\mathbf{p}}{(2\pi)^d}\frac{d^d\mathbf{p}'}{(2\pi)^d} 1_{(\mathbf{k},\mathbf{k}') \in A(2M^{j_1},2M^{j_2},2M^{j_3})} \left(1 + ||\chi||_1 \frac{2\pi}{L}\right) \\
&\leq (1+D) I(2M^{j_1}, 2M^{j_2}, 2M^{j_3}).
\end{aligned}$$

Applying the volume improvement, one get the claim with

$$K_1 = (1+D) C_{vol} 2^{4+\epsilon} \frac{M^4}{(2\pi)^2}.$$

If $j_3 < j_L$, we proceed essentially in the same way, replacing M^{j_i} by A/L, if $j_i \leq j_L$, for $i = 1, 2$ or 3. We present here the case where j_1, j_2 and $j_3 < j_L$. Performing the k_0-integrals, we obtain

$$\begin{aligned}
&\left| L^{-2d} \sum_{\mathbf{k},\mathbf{k}' \in \mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi}\frac{dk_0'}{2\pi} C_{j_1}^{(L)}(k) C_{j_2}^{(L)}(k) C_{j_3}^{(L)}(vk + v'k' + p) \right| \\
&\leq M^{j_1+j_2} \frac{L^3}{A^3} L^{-2d} \sum_{\mathbf{k},\mathbf{k}' \in \mathbf{\Lambda}_L^\sharp} 1_{|e(\mathbf{k})| \leq \frac{A}{L}} 1_{|e(\mathbf{k}')| \leq \frac{A}{L}} 1_{|e(v\mathbf{k}+v'\mathbf{k}'+\mathbf{q})| \leq \frac{A}{L}},
\end{aligned}$$

where the factors M^{j_1} and M^{j_2} are the length of the integration range of the variables k_0 and k_0', and $L^3/A^3 \geq M^{-j_1-j_2-j_3}$ bounds the supremum of the propagators. We then proceed as in the case $j_3 \geq j_L$, and get

$$\begin{aligned}
&\left| L^{-2d} \sum_{\mathbf{k},\mathbf{k}' \in \mathbf{\Lambda}_L^\sharp} \int \frac{dk_0}{2\pi}\frac{dk_0'}{2\pi} C_{j_1}^{(L)}(k) C_{j_2}^{(L)}(k') C_{j_3}^{(L)}(vk + v'k' + p) \right| \\
&\leq (1+D) M^{j_1+j_2} \frac{L^3}{A^3} I(2\frac{A}{L}, 2\frac{A}{L}, 2\frac{A}{L}) \\
&\leq D 2^{3+\epsilon} M^{j_1+j_2} \left(\frac{A}{L}\right)^{\epsilon-1} \\
&\leq D 2^{3+\epsilon} M^{j_1+j_2} M^{(\epsilon-1)j_3}.
\end{aligned}$$

□

6.2 The Tree Expansion

Let $G \in \mathcal{G}^{2m}(r)$ be one of the connected, amputated graphs contributing to $G^{(L)}_{2m,r}$. We decompose all the propagators of the graph G using the scale decomposition. Each line of G gets therefore an additional scale labeling. We denote a labeled graphs by G^J, where $J = \{j_l \,|\, l \in L(G)\}$, and sum over all j_l from $I(L)$ to 0.

Definition 6.5:

(i) Let G^J be a connected labeled graph. We define the tree $t(G^J)$ associated to G^J inductively as follows. If G^J consists in a single vertex v, then

$$t(G^J) := \{G^J\}.$$

Consider now a graph G^J with N vertices, $N > 1$. Assume that the trees corresponding to all graphs with N' vertices, $N' < N$ have been constructed. Let $j := \inf_{l \in L(G^J)} j_l$ be the lowest scale of the graph G^J. Suppose that the graph $G/\{l \in L(G) \,|\, j_l = j\}$, obtained cutting all the lines at scale j of G^J, has n connected components G_1,...,G_n, with associated trees $t(G_1), \ldots, t(G_n)$. Then

$$t(G^J) := \{G^J\} \cup \bigcup_{i=1}^{n} t(G_i).$$

Hence the tree $t(G^J)$ is the set of all connected subgraphs of G^J, obtained cutting recursively all lines at a given scale.

For f and $g \in t(G^J)$,

$$f \leq g \Leftrightarrow g \subset f$$

define a partial ordering on $t(G^J)$. If $f \leq g$ and $f \neq g$, we write $f < g$.

(ii) Let f be a fork of the tree $t(G^J)$. We denote G_f the connected subgraph of G^J corresponding to the fork f. $2m_f$ is the number of external legs of G_f, and $j_f := \inf_{l \in L(G_f)} j_l$ is the scale of the fork f. u_f is the number of upward branches of the fork f in $t(G^J)$.

Remark that for f and $g \in t(G^J)$,

$$f \leq g \Rightarrow j_f \leq j_g.$$

For simplicity, we call "two-legged forks" the forks f with $m_f = 1$ and "four-legged forks" the forks f with $m_f = 2$, referring to the number of branches of G_f rather than the number of legs of f in t.

(iii) The root ϕ of the tree $t(G^J)$ is the fork corresponding to the graph G^J. In particular, $f \leq \phi \Rightarrow f = \phi$. The scale $j = j_\phi$ is the root scale of $t(G^J)$ or G^J.

A fork $v \in t(G^J)$ with no upwards branches, i.e. $u_v = 0$, is called a leaf of $t(G^J)$. The scale of a leaf v is $j_v := 1$, and the graph corresponding to the leaf v is a single vertex in $V(G^J)$. In particular, $v \leq f$ implies $v = f$.

We call "two-legged leaves" the leaves that corresponds to two-legged vertices of G^J, and "four-legged leaves" the leaves corresponding to four-legged vertices of G^J.

(iv) Let $f \in t(G^J)$. We call $\pi(f)$ the fork preceding f in $t(G^J)$:
$\pi(f) \in t(G^J)$ with $\pi(f) \leq f$ such that there is no $g \in t(G^J)$ with $\pi(f) \leq g < f$.

(v) The graph $\tilde{G}(f)$ is obtained collapsing all the subgraphs G_g with $\pi(g) = f$ into vertices. The tree $t(\tilde{G}(f))$ is obtained from the tree $t(G^J)$, replacing the forks g with $\pi(g) = f$ by a leaf. In particular, the graph $\tilde{G}(\phi)$ contains only lines at scale j, and generalized vertices at scales $\geq j + 1$.

Remark 6.6: It is possible to show (See [7] and [6]) that the sum over the graphs of the remark 5.2 can be replaced by the sum over the trees:

$$\sum_{G \in \mathcal{G}^{2m}(r)} \sum_J \mathrm{Val}_L\, G^J = \sum_{j \geq I} \sum_t \prod_{f \in t} \frac{1}{u_f!} \sum_J \sum_{G \in \mathcal{G}(t)} \mathrm{Val}_L\, G^J,$$

where the sum over t runs over all the rooted trees with less than $\sup_{G \in \mathcal{G}^{2m}(r)} |L(G)|$ forks and less than r leaves. The sum over the scales runs over all the scale sets $J = \{j_f, f \in t\}$ such that $j_g < j_f$ if $g < f$, the root scale $j_\phi = j$ remaining fixed. The last sum over the graphs is taken over the graphs $G \in \mathcal{G}^{2m}(r)$ such that $t(G^J) = t$.

Remark 6.7: Let consider a graph G^J. We introduce the following labeling of the two-legged forks of $t(G^J)$. If $f \in t(G^J)$ is a two-legged fork, G_f is 1PI, and f is not a leaf of $t(G^J)$, then we say that f is a $r - fork$ of $t(G^J)$. If f is a two-legged fork and G_f is one particle reducible, then f is a $s - fork$.

Let f be one of the two-legged leaves of $t(G^J)$, corresponding to the counterterm

$$K_r^{(L)}(\mathbf{k}) = -l^{(L)} \sum_{\bar{G} \in \tilde{\mathcal{G}}'} \sum_{\bar{J}} \mathrm{Val}_L\, \bar{G}^{\bar{J}},$$

where $\tilde{\mathcal{G}}'$ is the set of all the two-legged, 1PI graphs that have at least one four-legged vertex. (See remark 5.3) Let $\bar{G}^{\bar{J}} \in \tilde{\mathcal{G}}'$ be one of the graph contributing to $K_r^{(L)}$, with associated tree $t(\bar{G}^{\bar{J}})$ and root scale $\bar{j} \geq I$.

The tree $t(G^J)$ can be expanded, replacing the leaf f with the two-legged root of the tree $t(\bar{G}^{\bar{J}})$. The fork f is now a two-legged fork that corresponds to a two-legged, 1PI

graph. Remark that the scale of the fork f is independent of the root scale of G^J. A two-legged fork obtained in this way will be called a *c−fork*.

Following this procedure, all the two-legged leaves of the graph $t(G^J)$ can be replaced by $c-$forks, ending with a new tree t that has only four-legged leaves. The value of the graph G associated to the root of f has to be computed inductively, replacing first in G each subgraph G_f associated to a $c-$fork f by a vertex with the vertex function $-l^{(L)}\,\mathrm{Val}_L\, G_f$.

Hence, we have

$$\sum_{G\in\mathcal{G}^{2m}(r)}\sum_{J}\mathrm{Val}_L\, G^J=\sum_{j\geq I}\sum_{t}\prod_{f\in t}\frac{1}{u_f!}\sum_{J}\sum_{G}\mathrm{Val}_L\, G^J,$$

where now the sum over the trees runs over all the trees with less than $2r+m$ lines. The sum runs also over the labeling of the two-legged forks in $s-$, $r-$, and $c-$forks. The sum over the scales is taken over the sets $J=\{j_f, f\in t\}$ such that $j_g<j_f$ if $g<f$, if g is not a $c-$fork. If g is $c-$fork, then $j_g\geq I$.

Observe that each tree t containing a two-legged fork corresponding to a 1PI subgraph contributes once to the sum with the f fork labeled with r and once with the f fork labeled with c. Both those contributions can be grouped together. The contribution of a 1PI, two-legged fork f to the value of the graph is given by

$$\sum_{j_f>j_{\pi(f)}}\sum_{J_f\in\mathcal{J}(t_f,j_f)}\mathrm{Val}_L\, G_f^{J_f}-l^{(L)}\sum_{j_f\geq I}\sum_{J_f\in\mathcal{J}(t_f,j_f)}\mathrm{Val}_L\, G_f^{J_f}=$$
$$=(1-l^{(L)})\sum_{j_f>j_{\pi(f)}}\sum_{J_f\in\mathcal{J}(t_f,j_f)}\mathrm{Val}_L\, G_f^{J_f}-l^{(L)}\sum_{j_f=I}^{j_f}\sum_{J_f\in\mathcal{J}(t_f,j_f)}\mathrm{Val}_L\, G_f^{J_f}$$

Resuming the contribution in that way, we see that a $r-$fork f gives a contribution at scale bigger than $j_{\pi(f)}$, projected by the operator $1-l^{(L)}$, and the $c-$fork f gives a contribution at scales less or equal $j_{\pi(f)}$, projected onto $\mathcal{S}^{(L)}$ by $l^{(L)}$.

We get finally

$$G^{(L)}_{2m,r}=\sum_{j\geq I(L)}\sum_{t}\prod_{f\in t}\frac{1}{u_f!}\sum_{G}\sum_{J\in\mathcal{J}(j,t)}\mathrm{Val}_L\, G^J,$$

where the sum over the trees runs over all planar trees with r leaves. The sum over the graphs runs over all connected graphs G with $2m$ external legs, and r vertices. There is also a sum over the labeling in $r-$, $s-$ and $c-$forks. The scales label are in the set

$$\mathcal{J}(j,t)=\{(j_f, f\in t)\,|\, j_f>j_{\pi(f)} \text{ if } 2m_f>2 \text{ or } f \text{ is a } r- \text{ or a } s-\text{fork}$$
$$\text{and } j\geq j_f\geq I \text{ if } f \text{ is a } c-\text{fork}\}.$$

Remark 6.8: Integrating out successively the different energy shell, one would obtain a semi-group structure associated with renormalization. The results described above can be as well obtained using this idea, see [7].

Definition 6.9: Let G^J be a labeled graph with tree $t = t(G^J)$ and root ϕ.

(i) A graph G is called *overlapping* if it contains two loops that share at least a common line. If the graph $\tilde{G}(\phi)$ defined in 6.5(v) is overlapping, we say that G^J overlaps at the root scale. We denote with $\mathcal{O}$ the set of all overlapping graphs.[1]

(ii) Let G^J be a non-overlapping graph at the root scale. We construct the following subtree τ of the tree t, rooted at ϕ. A fork of t is in τ if and only if G_f is non-overlapping, and $\pi(f) \in \tau$. Define the non-overlapping graph $\tilde{G}(\tau)$ to be the graph obtained from G^J collapsing all leaves of τ to vertices. Define the overlapping scale of G to be $j^* = \inf_{f \in t \text{ and } f \notin \tau} j_f$. τ is the maximal subtree of t such that $\tilde{G}(\tau)$ is non-overlapping.

Definition 6.10:

(i) We construct inductively the spanning tree $T(G^J)$ of G as follows. Suppose that we have constructed the spanning trees $T(G_f)$ for all the forks f directly above ϕ in t. Construct then the spanning tree $\tilde{T}$ of the graph $\tilde{G}(\phi)$. If $\tilde{G}(\phi)$ is overlapping, then there are two loops in $\tilde{G}(\phi)$ that share at least one line l^*. Choose a spanning tree that contains the line l^*. The momenta in $\tilde{G}(\phi)$ can be set in such a way that each line of the spanning tree carries an external momentum.

The tree $T(G)$ is then obtained replacing in $\tilde{T}$ all the vertices by the trees $T(G_f)$.

(ii) The value of a labeled graph G^J with two-legged vertices $\Theta_w(k)$ and four-legged vertices $U_v(k_1, k_2, k_3)$ is given by

$$\mathrm{Val}_L(G^J) = L^{-d|L(G)\setminus L(T)|} \sum_{\mathbf{k}_l \in \mathbf{\Lambda}_L^\sharp,\, l \in L(G)\setminus L(T)} \int \prod_{l \in L(G)\setminus L(T)} \frac{dk_{l0}}{2\pi} \prod_{l \in L(G)} C_{j_l}(k_l) \prod_{v \in V_4(G)} U_v(k_1^v, k_2^v, k_3^v) \prod_{w \in V_2(G)} \Theta_w(k_w),$$

where T is the spanning tree of G^J defined above. For $l \in L(T)$, k_l is a linear combination of external and loop momenta.

In order to construct the thermodynamic limit of the Green functions, we defined

Definition 6.11: Let $I \in \mathbb{Z}_-$. Then

$$G^I_{2m,r} = \sum_{j \geq I(L)} \sum_t \prod_{f \in t} \frac{1}{u_f!} \sum_G \sum_{J \in \mathcal{J}(j,t)} \mathrm{Val}_\infty G^J,$$

[1] See [7] for a rigorous definition of overlapping graphs.

where the value Val_∞ of a graph is defined by

$$\delta(\sum_{i=1}^{2m} k_i)\,\mathrm{Val}_\infty(G^J) = \int \prod_{l\in L(G)} \frac{d^{d+1}k_l}{(2\pi)^{d+1}} C_{j_l}(k_l) \cdot \prod_{v\in V_4(G)} \left(\prod_{l_1,\dots,l_4\in L(v)} \delta(\sum_{i=1}^{4} k_{l_i}) V(k_{l_1},\dots,k_{l_4}) \right) \cdot \prod_{w\in V_2(G)} \left(\prod_{l_1,l_2\in L(w)} \delta(k_{l_1}+k_{l_2}) K_w(k_{l_1}) \right).$$

The projection operator $l^{(L)}$ is replaced by the projection operator l, and the root scale is $j \geq I$.

6.3 The Power Counting

Remark 6.12: It follows from lemma 6.4 that the same power counting applies for $\mathrm{Val}_L(G)$ and $\mathrm{Val}_\infty(G)$. Hence, we simply denote $\mathrm{Val}(G)$ the value of a graph when no distinction between the lattice case and the continuous case is needed.

If no distinction is needed, we further denote with $|\cdot|_0$ and $|\cdot|_1$ the supremum and derivative norms, meaning $||\cdot||_0$ and $||\cdot||_1$ on $\mathbb{R}\times\mathbb{T}$, and $|\cdot|_0$ and $|\cdot|_1$ on $\mathbb{R}\times\mathbf{\Lambda}_L^\sharp$. Further, we abusively replace the Riemann sum over the lattice $\mathbf{\Lambda}_L^\sharp$ by an integral, in order to simplify the notation.

Definition 6.13: Let G^J be a labeled graph. For a fork $f \in t(G^J)$, we define

$$D_f = |L(G_f)| - 2(|V(G_f)| - 1).$$

Remark 6.14: Counting the lines of the graph G_f, one get 4 half-lines for each four-legged vertex, and 2 half-line for each two-legged vertex, minus $2m$ half-lines from the external lines:

$$4|V_4(G_f)| + 2|V_2(G_f)| = \text{ number of half-lines of } G_f \ = 2|L(G_f)| + 2m_f.$$

Using this equality, we get

$$D_f = \frac{1}{2}(4 - 2m_f) - |V_2(G_f)|.$$

Theorem 6.15: Let G^J be a $2m-$legged connected, amputated labeled graphs. We denote with U_v the four-legged vertices of G^J, and with Θ_w its two-legged vertices. Then

(i)

$$|\operatorname{Val}(G^J)|_0 \leq (4K_0)^{|L(G)|} K_1 M^{\epsilon j^*} M^{D_\phi j} \prod_{f>\phi} M^{D_f(j_f - j_{\pi(f)})} \prod_{v\in V_2(G)} |\Theta_v|_0 \prod_{v\in V_4(G)} |U_v|_0,$$

where j^* is the scale at which G^J overlaps. If G^J does not overlap, then $j^* = 0$.

(ii) For the integral norm,

$$|\operatorname{Val}(G^J)|' \leq (4K_0)^{|L(G)|} \prod_{w\in V_2(G)} (|\Theta_w|_0 M^{-j_{\pi(w)}}) \prod_{v\in V_4(G)} |U_v|_0 \, S_{int} \, S_{f,ext} \, S_{v,ext},$$

where

$$S_{int} = \prod_{f>\phi,\, internal} M^{D_f^R(j_f - j_{\pi(f)})}$$

where the product is only over those forks of $t(G^J)$ such that G_f^J does not contains any external vertices, and $D_f^R = D_f + |V_2(G_f)|$.

$$S_{f,ext} = \prod_{f>\phi,\, external} M^{\Delta_f(j_f - j_{\pi(f)})}$$

where the product is only over those forks of $t(G^J)$ such that G_f^J contains an external vertex of G, and

$$\Delta_f = -\frac{1}{2}\left|\{l : l \text{ internal line of } G^J \text{ and external line of } G_f^J\}\right|.$$

Finally,

$$S_{v,ext} = \prod_{v,\, external} M^{\Delta_v(0 - j_{\pi(v)})}.$$

The product is over the vertices of G to which an external leg of G^J is joined, and

$$\Delta_v = -\frac{1}{2}\left|\{l : l \text{ internal line of } G^J,\, l \in v\}\right|.$$

Remark 6.16:

(i) The products run only over the fork of the tree $t(G^J)$ that are not leaves of $t(G^J)$.

(ii) Observe that since $\pi(f)$ denotes the fork preceding f in the tree $t(G)$, it holds

$$j_f - j_{\pi(f)} \geq 0.$$

Proof of 6.15:

(i) Let T be the spanning tree of G^J defined in 6.10(i). We first bound all the vertex functions by their sup-norms, and get

$$|\operatorname{Val}(G^J)|_0 \leq \prod_{v\in V_2(G)} |\Theta_v|_0 \prod_{v\in V_4(G)} |U_v|_0\, X,$$

where X is given by

$$X = \left| \int \prod_{l\in L(G)/L(T)} \frac{d^{d+1}k_l}{(2\pi)^{d+1}} |C_{j_l}(k_l)| \prod_{t\in L(T)} |C_{j_t}(P_t)| \right|_0,$$

where the P_t are linear combinations of momenta k_l, $l \in L(G)/L(T)$ and of external momenta. In the lattice case, the integrals over the Brillouin zone have to be replaced by sums. We prove now that

$$X \leq (2K_0)^{|L(G)|} (K_1 M^{\epsilon j})^{1_{\tilde{G}(\phi)\in\mathcal{O}}} M^{jD_\phi} \prod_{f>\phi} M^{(j_f - j_{\pi(f)})D_f},$$

which directly implies the claim. If the root scale j of G^J is strictly negative, decompose $L(G^J)$ in the set of the lines of $\tilde{G}(\phi)$, which are at scale j, and the set of the lines of subgraphs corresponding to forks in

$$\sigma(\phi) = \{f \in t \,|\, \pi(f) = \phi\}.$$

We obtain:

$$L(G) = L(\tilde{G}(\phi)) \cup \bigcup_{f\in\sigma(\phi)} L(G_f).$$

Let $\tilde{T}(\phi)$ be the spanning tree of $\tilde{G}(\phi)$. We get then

$$X \leq \int \prod_{l\in L(\tilde{G}(\phi))/L(\tilde{T}(\phi))} dp_l |C_j(p_l)| \prod_{t\in L(\tilde{T}(\phi))} |C_j(P_t)| \prod_{f\in\sigma(\phi)} X_f,$$

where X_f is the corresponding of X for the graph G_f. The integration are taken only over propagator with scale $j < 0$, such that we can bound the integral with the point (ii) of lemma 6.4. If $\tilde{G}(\phi)$ is overlapping, there is a line in $\tilde{T}(\phi)$ belonging to two independent loops. Such a line brings a factor

$$\int dp_1\, dp_2 |C_j(p_1)||C_j(p_2)||C_j(p_1 \pm p_2 + q)| \leq K_0^2 K_1 M^{(\epsilon-1)j} M^{2j},$$

due to the volume improvement estimate. The other lines are bounded by the naive power counting, such that

$$X \leq (K_1 M^{j\epsilon})^{1_{\tilde{G}(\phi)\in\mathcal{O}}} K_0^{|L(\tilde{G}(\phi))|} \prod_{l\in L(\tilde{G}(\phi))} M^{-j_l} \prod_{t\in L(\tilde{G}(\phi))/L(\tilde{T}(\phi))} M^{2j_t} \prod_{f\in\sigma(\tau_\phi)} X_f.$$

If the root scale j of G^J is zero, then all the propagators are UV-propagator. The integral is performed over the loop momenta, which enter at least two lines. We use the point (iii) of lemma 6.4 in order to bound the integrals.

We apply the induction hypothesis in order to bound X_f:

$$X_f \leq (K_1 M^{j_f \epsilon})^{1_{G_f \in \mathcal{O}}} K_0^{|L(G_f)|} \prod_{l \in L(G_f)} M^{-j_l} \prod_{t \in L(G_f)/L(T_f)} M^{2j_t}.$$

and get

$$X \leq K_0^{|L(G)|} K_1 M^{j^* \epsilon} \prod_{l \in L(G)} M^{-j_l} \prod_{t \in L(G)/L(T)} M^{2j_t},$$

where j^* is the overlapping scale defined above. If G^J does not overlaps at any scales, set $j^* = 0$. In order to obtain the final bound, we use the telescope sum

$$j_l = j + \sum_{\substack{f > \phi \\ G_f \ni l}} (j_f - j_{\pi(f)}),$$

such that

$$\begin{aligned} \prod_{l \in L(G)} M^{-j_l} \prod_{t \in L(G)/L(T)} M^{2j_t} &= M^{j(2|L(G)|-2|L(T)|-|L(G)|)}. \\ &\quad \cdot \prod_{f > \phi} M^{(j_f - j_{\pi(f)})(2|L(G_f)|-2|L(T_f)|-|L(G_f)|)} \\ &= M^{D_\phi j} \prod_{f > \phi} M^{(j_f - j_{\pi(f)}) D_f}, \end{aligned}$$

by recursive construction of the tree T from the trees T_f.

(ii) In order to bound the integral norm, we construct the graph G^*, with $2m^*$ external legs, satisfying:

- G^* has the same vertices as G, with one $2(m+m^*)-$legged additional vertex v^*, with vertex function $\delta(k_1 + \cdots + k_{2m+2m^*})$.
- G^* has the same lines as G^J, and $2m$ additional lines that join the external legs of G to v^*. This lines carry a propagator C^* at scale 1, with supremum $|C^*|_0 \leq 1$.
- The other propagators of G^* are given by $|C_j^{(L)}(k)|$, resp. $|C_j(k)|$ in the infinite volume, and the other vertices functions by $|U_v(k)|$ and $|\Theta_v(k)|$.

The fact that the propagator $C^*(k)$ is not integrable is harmless, since we can choose a spanning tree T^* of G^* that contains all the lines l^* of $G^* \setminus G$. Since $|C^*|_0 \leq 1$, the lines l^* do not contribute to the power counting.

The conservation of the external momenta of G^* imposes the condition

$$\sum_{i=1}^{2m^*} k_{2m+i} = 0,$$

such that

$$|\operatorname{Val}_L(G^J)|' \leq |\operatorname{Val}_L(G^*)|_0.$$

We apply now the theorem 6.15(i) on $|\operatorname{Val}_L(G^*)|_0$:

$$|\operatorname{Val}_L(G^*)|_0 \leq (4K_0)^{|L(G)|} M^{jD^*_{\phi^*}} \prod_{v\in V_2(G)} |\Theta_v|_0 \prod_{v\in V_4(G)} |U_v|_0 \prod_{f>\phi^*} M^{D^*_f(j_f - j_{\pi(f)})}.$$

Let $\{j_0 = j < j_1 < \cdots < j_N = 0\}$ be the set of the scales of G^J. The product over the forks of $t(G^*)$ can be rewritten as

$$\begin{aligned}
\prod_{f>\phi^*} M^{D^*_f(j_f - j_{\pi(f)})} &= \prod_{i=1}^{N} \prod_{f\in t(G^*),\, j_f = j_i} M^{D^*_f(j_i - j_{\pi(f)})} \\
&= \prod_{i=1}^{N} \prod_{k=0}^{i-1} \prod_{\substack{f\in t(G^*) \\ j_f = j_i,\, j_{\pi(f)} = j_k}} M^{D^*_f(j_i - j_k)} \\
&= \prod_{i=1}^{N} \prod_{k=0}^{i-1} \prod_{\substack{f\in t(G^*) \\ j_f = j_i,\, j_{\pi(f)} = j_k}} \prod_{j'=j_k+1}^{j_i} M^{D^*_f} \\
&= \prod_{j'=j+1}^{0} \prod_{\substack{f\in t(G^*) \\ j_{\pi(f)} < j' \leq j_f}} M^{D^*_f} \\
&= \prod_{j'=j+1}^{0} \prod_{f\in C^*_{j'}} M^{D^*_f}.
\end{aligned}$$

Here $C^*_{j'}$ is the set of the connected components of $\{l \in G^* \mid j_l \geq j'\}$. $C^*_{j'}$ is composed from subgraphs of G^J at scale j' that do not contain any external vertices of G^J, and one subgraph G_{f^*} of G^* that contains the vertex v^*, and all external vertices of G^J. Since for internal forks f of G^J, i.e. the forks of G^J that contains no external vertices of G^J, $D^*_f = D_f$,

$$\prod_{j'=j+1}^{0} \prod_{f\in C^*_{j'}} M^{D^*_f} = \prod_{j'=j+1}^{0} \left(M^{D^*_{f^*}} \prod_{\substack{f\in C_{j'} \\ f\, internal}} M^{D_f} \right),$$

where $C_{j'}$ is the set of the connected components of $\{l \in G^J \mid j_l \geq j'\}$.

The subgraphs G_{f^*} is composed of the vertex v^*, r external subgraphs $G_{f_1}, \ldots, G_{f_r} \in C_{j'}$ of G^J, and s external vertices $v_1, \ldots, v_s$ of G^J, that are not in any G_{f_i}, $i =$

$1, \ldots, r$. Hence

$$\begin{aligned}
D^*_{f^*} &= |L(G_{f^*})| - 2(|V(G_{f^*})| - 1) \\
&= \sum_{i=1}^{r} (|L(G_{f_i})| - 2|V(G_{f_i})|) - 2 + 2m - 2s + 2 \\
&= -\sum_{i=1}^{r} (|V_2(G_{f_i})| + m_{f_i}) + 2m - 2s \\
&= -\sum_{i=1}^{r} |V_2(G_{f_i})| + m - 2s \\
&\quad + \frac{1}{2} \sum_{i=1}^{r} (|\{l^* \in G^* \setminus G \,|\, l^* \text{ hooked to } G_{f_i}\}| - 2m_{f_i}) \\
&\quad + \frac{1}{2} \sum_{i=1}^{s} |\{l^* \in G^* \setminus G \,|\, l^* \text{ hooked to } v_i\}| \\
&= -\sum_{i=1}^{r} |V_2(G_{f_i})| + \sum_{i=1}^{r} \Delta_{f_i} + m - 2s \\
&\quad + \frac{1}{2} \sum_{v \in V_4(G) \cap \{v_1, \ldots, v_s\}} (4 - |\{l \text{ internal line of } G \text{ hooked to } v\}|) \\
&\quad + \frac{1}{2} \sum_{v \in V_2(G) \cap \{v_1, \ldots, v_s\}} (2 - |\{l \text{ internal line of } G \text{ hooked to } v\}|) \\
&= -\sum_{i=1}^{r} |V_2(G_{f_i})| + \sum_{i=1}^{r} \Delta_{f_i} + m + \sum_{i=1}^{s} \Delta_{v_i} - |V_2(G) \cap \{v_1, \ldots, v_s\}|.
\end{aligned}$$

Therefore

$$\prod_{j'=j+1}^{0} \left(\prod_{f \in C_{j'},\, f\, int} M^{D_f} \cdot M^{D^*_{f^*}} \right) = \prod_{j'=j+1}^{0} \left(\prod_{f \in C_{j'},\, f\, int} M^{D_f} \prod_{f \in C_{j'},\, f\, ext} M^{(\Delta_f - |V_2(G_f)|)} \right)$$
$$\cdot \prod_{j'=j+1}^{0} M^{\sum_{i=1}^{s} \Delta_{v_i} - |V_2(G) \cap \{v_1, \ldots, v_s\}|} \cdot \prod_{j'=j+1}^{0} M^m.$$

Further

$$\prod_{j'=j+1}^{0} M^m = M^{-jm},$$

and

$$\prod_{j'=j+1}^{0} M^{-|V_2(G) \cap \{v_1, \ldots, v_s\}|} \leq 1.$$

The product over the vertices can be rewritten in a more convenient form, using

$$\begin{aligned}
\sum_{j'=j+1}^{0}\sum_{i=1}^{s}\Delta_{v_i} &= \sum_{j'=j+1}^{0}\sum_{v\,ext:\,j_{\pi(v)}<j'}\Delta_v \\
&= \sum_{i=0}^{N}\sum_{\substack{v\,ext:\\ j_{\pi(v)}=j_i}}\sum_{j'=j_i+1}^{0}\Delta_v \\
&= \sum_{i=0}^{N}\sum_{\substack{v\,ext:\\ j_{\pi(v)}=j_i}}\Delta_v(0-j_{\pi(v)}) \\
&= \sum_{v\,ext}\Delta_v(0-j_{\pi(v)}).
\end{aligned}$$

Remark that

$$D^*_{\phi^*} = |L(G^*)| - 2(|V(G^*)|-1) = |L(G)| + 2m - 2|V(G)| = -|V_2(G)| + m,$$

and

$$-j|V_2(G)| = \sum_{f>\phi}|V_2(G_f)|(j_f - j_{\pi(f)}) - \sum_{w\in V_2(G)} j_{\pi(w)}.$$

With $D^R_f = D_f + |V_2(G_f)|$, we get therefore

$$\begin{aligned}
M^{D^*_{\phi^*}}\prod_{f>\phi^*}M^{D^*_f(j_f-j_{\pi(f)})} &\le M^{-|V_2(G)|j}\prod_{f,\,int}M^{D_f(j_f-j_{\pi(f)})} \\
&\quad\cdot\prod_{f,\,ext}M^{(\Delta_f-|V_2(G_f)|)(j_f-j_{\pi(f)})}\prod_{v,\,ext}M^{\Delta_v(0-j_{\pi(v)})} \\
&\le \prod_{f,\,int}M^{D^R_f(j_f-j_{\pi(f)})}\prod_{f,\,ext}M^{\Delta_f(j_f-j_{\pi(f)})} \\
&\quad\cdot\prod_{v,\,ext}M^{\Delta_v(0-j_{\pi(v)})}\prod_{v\in V_2(G)}M^{-j_{\pi(v)}}.
\end{aligned}$$

□

6.4 The 1PI Graphs

6.4.1 Non-Overlapping two-legged Graphs

Definition 6.17: Let G be a connected, two-legged graph with N vertices, all having even incidence number.

(i) If $G_1,\ldots,G_n$ are 1PI, two-legged graphs, the graph obtained by connecting G_{i-1} to G_i for i running from 2 to n by a propagator is called a string.

(ii) G is called a self-contracted two-legged (ST) diagram if G consists only of one two-legged vertex with two external legs, or if G has exactly one vertex v_1 with incidence number bigger than 2, to which both external legs of G connect, all remaining legs of v_1 being joined pairwise by strings of two-legged vertices to form a loop.

(iii) A generalized ST (GST) diagram with N vertices is defined recursively: if $N = 1$, then G is a ST diagram. If $N > 1$, and GST diagrams are defined for $N' < N$, a GST diagram with N vertices is a graph such that G has exactly one external vertex v_1 to which the two external legs of G join, and all other legs of v_1 are joined pairwise by strings of GST with at most $N - 1$ vertices, to form a loop.

Lemma 6.18: Let G be a connected, two-legged graph, all vertices of G having even incidence number. If G is non-overlapping, then G is a string of GST graphs.

Proof: First note that if the statement is true for 1PI-graphs, then it is true as well for all connected, two-legged graphs. Let therefore G be a two-legged, 1PI graph. We prove the lemma by induction over the number of vertices of G.

If G consists only in one vertex v of incidence number $2m$, then G is obtained connecting $2(m-1)$ legs of v together. By definition, G is a ST-graph.

Let consider now a graph G with n vertices. Distinguish two cases:

- G has only one external vertex v. Then G consists in the vertex v connected to the external legs of G, and of N 1PI-graphs G_1,..., G_N, that are connected to v by strings of two-legged graphs, and pairwise disjoint. By the induction hypothesis, the strings connecting v and the G_i are GST.

  Suppose that for a i with $1 \leq i \leq N$, G_i has an external vertex v_i, $m_i > 2$ legs of which are connected to v. In that case, G contains two loops that share at least one line, that is, G is overlapping.

  It follows that all the G_i are two-legged, 1PI. By induction hypothesis, they are all ST graphs, and consequently, G as well.

- Consider the case where G has two external vertices v_1 and v_2. By the same argument as before, v_1 and v_2 cannot be connected to connected subgraphs of G with more than two strings. Using the fact that the incidence numbers of v_1 and v_2 are even, it follows that v_1 and v_2 are connected by one string of two-legged subgraphs of G, which is not possible, since G is 1PI.

□

6.4.2 The Graph G'

In order to bound the value of a graph G^J inductively, we define the graph G', which is obtained collapsing all 1PI two- and four-legged subgraphs of G^J into vertices. We define t' to be the tree corresponding to G'. The graph G' has the following properties:

- G' has only two- and four-legged vertices, with vertex functions that are either interaction vertices or values of 1PI two- or four-legged subgraphs of G.
- The only non-trivial two-legged subgraphs of G' are strings of two-legged vertices of G', and any non-trivial four-legged subgraphs of G' consists of a single four-legged vertex with strings of two-legged vertices appended.

Definition 6.19: Let G^J be a labeled graph, and $t = t(G^J)$ its corresponding tree. We construct the graph G' corresponding to G^J as follows.

Let ϕ be the root of t, and let $f_1, \ldots, f_r$ be all forks of t that satisfy:

$$\forall k \in \{1, \ldots, r\}\,,\ E(G_{f_k}) \in \{2, 4\} \text{ and } f_k \text{ is minimal,}$$

that is $\nexists\, g$ with $\phi < g < f$ such that $E(G_g) \in \{2, 4\}$. Let $\tilde{t}$ be the tree rooted at ϕ and obtain from t trimming t at $f_1, \ldots, f_r$, that is replacing the forks $f_1, \ldots, f_r$ by leaves, with vertex functions $\mathrm{Val}(G_{f_k})$, $k = 1, \ldots, r$.

The graph $\tilde{G}$ corresponding to the tree $\tilde{t}$ is obtained collapsing all the two- or four-legged subgraphs of G into vertices. $\tilde{t}$ has no fork that corresponds to a non-trivial two- or four-legged subdiagram, but it is not the tree with the stated properties, because the leaves of $\tilde{t}$ do not need to correspond to 1PI subgraphs of G.

In order to construct t' from $\tilde{t}$ we proceed as follow. Pick up $f \in \{f_1, \ldots, f_r\}$.

- If f is a 2-legged, $r-$ or $c-$ fork, then G_f is 1PI by definition, and f will be a leaf of t'.
- If f is a two-legged $s-$fork, then it consists in a string of two-legged vertices. The external legs of G_f are at scale $j_{\pi(f)}$ or above, and by the support property of the propagators, the scales of the propagators of the string are in $\{j_{\pi(f)}, j_{\pi(f)}+1\}$. Hence G_f is a string of two-legged graphs, joined by propagators at scale $j_f = j_{\pi(f)} + 1$.

  The vertex function corresponding to f is given by

  $$\mathrm{Val}(G_f^J)(p) = \prod_{i=1}^{n-1} T_i(p) C_{j_f}(p) T_n(p),$$

  where $T_i = \mathcal{P}_{\Theta_i} \mathrm{Val}_L(\Theta_i)$. The Θ_i are two-legged graphs, which can correspond to a $c-$, $r-$ or a $s-$fork g directly above f, or it can be a two-legged subgraph of G at scale j_f. We call the latter case a same scale insertion.

If Θ_i corresponds to a $c-$fork, then $\mathcal{P}_{\Theta_i} = l^{(L)}$, Θ_i is 1PI, g is added to $\tilde{t}$, and becomes a leave of t'.

If Θ_i correspond to a $r-$fork, then $\mathcal{P}_{\Theta_i} = 1 - l^{(L)}$. g is added to $\tilde{t}$, and becomes a leave of t'.

In the case where Θ_i is a same scale insertion, set $\mathcal{P}_{\Theta_i} = 1$. In that case add a fork g, with corresponding subgraph Θ_i to the tree $\tilde{t}$. If Θ_i is 1PI, then g will be a leave of t'.

The case where Θ_i corresponds to a $s-$fork is treated in the same way as the same scale insertion.

- If f is four-legged and one particle reducible, remove the strings attached to G_f, and add a leaf above the fork f for the 1PI core of G_f, as well as for each 1PI 2-legged subdiagram Θ_i of the strings. The strings have the same properties as the ones discussed in the $s-$fork case.

- Follow this procedure, extending $\tilde{t}$ to a larger tree until all the leaves correspond to 1PI subdiagrams. The final tree is t'.

Observe that if G is 1PI, then G' is also 1PI.

The vertices of G' carry a scale, which has to be summed over. The relation between the sums over the scales for G and G' is

$$\sum_{J\in\mathcal{J}(t,j)} \mathrm{Val}(G^J) = \sum_{J'\in\mathcal{J}(t',j)} \mathrm{Val}(G'^{J'}).$$

Note that the set $\mathcal{J}$ is the same in both sums, but some scales in J' correspond to vertices. Let denote with j_w the scale index of the vertex w. If $j_w = 1$, then w is also a vertex of G, and the associated function is v. Otherwise, j_w is the root scale of a subgraph of G, whose value is a vertex function in G', and j_w is summed over. For fixed $j_{\pi(w)}$, the summed vertex function is

$$F_w = \mathcal{P}_w \sum_{j_w} \sum_{J\in\mathcal{J}(t_w,j_w)} \mathrm{Val}(\tilde{G}(t_w)),$$

where $\mathcal{P}_w \in \{1 - l^{(L)}, l^{(L)}\}$ for 2-legged vertices associated to $r-$ or $c-$forks. For same scale insertion, $s-$forks or for 4-legged vertices, $\mathcal{P}_w = 1$. The range of summation for j_w is

$$\begin{array}{ll} I \leq j_w \leq j_{\pi(w)} & \text{for } c-\text{forks}, \\ j_{\pi(w)} < j_w \leq 0 & \text{for } r- \text{ or } s-\text{forks or 4-legged vertices}, \\ j_w = j_{\pi(w)} + 1 & \text{for same scale insertions.} \end{array}$$

Remark 6.20: By construction, all the two-legged vertices of G' have label r or c, but not s, since the last would correspond to one particle reducible graphs.

6.4.3 Bounding the Value of 1PI Graphs

Definition 6.21: For $n \in \mathbb{N}$, $j \in \mathbb{Z}_-$ and $\epsilon \in (0,1)$, let

$$\lambda_n(j,\epsilon) = \sum_{p=1}^{\infty} (|j| + p + 1)^n M^{-p\epsilon}.$$

Lemma 6.22: λ is monotonically increasing in $|j|$ and in n, and

$$(|j| + 1)^m \lambda_n(j,\epsilon) \leq \lambda_{n+m}(j,\epsilon).$$

For $\epsilon > 0$, $M \geq 2^{4/\epsilon}$, $a \geq \epsilon$, $m, n \in \mathbb{Z}$ and $j \in \mathbb{Z}_-$,

(i) $\lambda_n(j_1,\epsilon)\lambda_m(j_2,\epsilon) \leq \lambda_{n+m}(j,\epsilon)$, where $j = \min\{j_1, j_2\}$.

(ii) $\displaystyle\sum_{h=j+1}^{0} \lambda_n(h,\epsilon) \leq \lambda_{n+1}(j,\epsilon)$.

(iii) $\displaystyle\sum_{l=-\infty}^{j} (|l| + 1)^m M^{al} \lambda_n(l,\epsilon) \leq \frac{\lambda_{n+m}(j,\epsilon)}{1 - M^{-\epsilon}} M^{aj} \leq 2\lambda_{n+m}(j,\epsilon) M^{aj}$.

(iv) $\displaystyle\sum_{h=j+1}^{0} (|h| + 1)^m M^{h\epsilon} \lambda_n(h,\epsilon) \leq 2\lambda_{n+m}(j,\epsilon)$.

(v) $\displaystyle\sum_{h=j+1}^{0} (|h| + 1)^m M^{-a(h-j)} \lambda_n(h,\epsilon) \leq 2\lambda_{n+m}(j,\epsilon)$.

(vi) $\lambda_n(j,\epsilon) \leq a_n j^n + b_n$,
where $a_n = 2^n/(M^{2\epsilon} - 1)$ and $b_n = \displaystyle\sum_{p \geq 1} (2p+1)^n M^{-2\epsilon p}$.

See [7] for a proof of this Lemma.

Theorem 6.23: Let G be a graph with $2m$ external legs, and t be a planar tree rooted at ϕ compatible with G such that the pair (t, G) contributes to the $2m-$points renormalized Green function at order r in λ at the scale $j \geq I$. For $J \in \mathcal{J}(t,j)$, let $\mathrm{Val}(G^J)$ be the value of the labeled graph G^J with root scale j. For each fork $j \in t$, let

$$n_f = |\{f' \in t : \ f' > f, \ \tilde{G}(t') \text{ non-overlapping }, \ E(G_{f'}) = 4 \text{ and } G_{f'} \text{ 1PI}\}|.$$

Then for $s = 0$ or $s = 1$,

$$\sum_{J \in \mathfrak{J}(t,j)} |\operatorname{Val}(G^J)|_s \leq Q_0^{|L(G)|} |v|_1^{|V(G)|} \lambda_{n_\phi}(j, \epsilon/2) M^{jY_s(G)}.$$

The power counting function $Y_s(G)$ is given by

$$Y_s(G) = \begin{cases} 2 - m - s & \text{if } E(G) = 2m \text{ and } \tilde{G}(\phi) \text{ non-overlapping} \\ 2 - m - s + \epsilon & \text{if } E(G) = 2m \text{ and } \tilde{G}(\phi) \text{ is overlapping} \\ \epsilon & \text{if } E(G) = 2 \text{ and } s = 1, \end{cases}$$

where ϵ is the volume improvement exponent.

<u>Proof:</u> This theorem is proved by induction over the depth of the pair (t, G), which is defined as

$$P = \max\{k \in \mathbb{N} \,|\, \exists\, f_1 > f_2 > \cdots > f_k > \phi \text{ with } E(G_{f_i}) \in \{2, 4\}\},$$

that is, given any leaf of t, there are at most P two- or four-legged forks on the unique path between this leaf and ϕ. We first prove the naive power counting bounds with volume improvement using the theorem 6.15(i).

<u>$P = 0$</u>

Case 1: $s = 0$, $E(G) \geq 2$.

In this case, we bound the value of G using the naive power counting given by theorem 6.15(i):

$$\sum_{J \in \mathfrak{J}(t,j)} |\operatorname{Val}(G^J)|_0 \overset{\text{Thm. 6.15(i)}}{\leq} (4K_0)^{|L(G)|} \left(K_1 M^{\epsilon j}\right)^{1_{\tilde{G}(\phi) \in \mathfrak{O}}} M^{D_\phi j} \cdot \sum_{J \in \mathfrak{J}(t,j)} \prod_{f > \phi} M^{D_f (j_f - j_{\pi(f)})} \prod_{v \in V_4(G)} |v|_1.$$

In order to bound the sum over the scales, we use

$$\sum_{j_f = j_{\pi(f)}+1}^{0} M^{D_f (j_f - j_{\pi(f)})} = \sum_{l=1}^{|j_{\pi(f)}|} M^{D_f l} \leq \sum_{l=1}^{\infty} M^{-l} = \frac{1}{1 - M^{-1}},$$

since $D_f \leq -1$. The sum is therefore bounded by $(1 - M^{-1})^{-|L(G)|}$, because the number of forks in t is bounded by $|L(G)|$. Using $D_\phi = 2 - m - |V_2(G)|$ and $|V_2(G)| = 0$, we get

$$\sum_{J \in \mathfrak{J}(t,j)} |\operatorname{Val}(G^J)|_0 \leq \left(\frac{4K_0 K_1}{1 - M^{-1}}\right)^{|L(G)|} |v|_0^{|V(G)|} M^{j(2 - m + \epsilon 1_{\tilde{G}(\phi) \in \mathfrak{O}})}.$$

Case 2: $s = 1$, $E(G) = 4$ or $E(G) = 2$ and $\tilde{G}(\phi)$ is overlapping

The derivative can act on interaction lines, in which case its effect is bounded by $|v|_1 \leq |v|_1 M^{-j}$. The derivative can also hit a line of the spanning tree $T(G)$. In this case, a factor $WM^{-j_1} \leq WM^{-j}$ appear for each derivative in addition to the bound obtained in the *case 1*. The number of terms obtained by the Leibnitz rule is bounded by:

$$\underbrace{|V(G)|}_{\text{Vertices}} + \underbrace{|V(G)| - 1}_{\text{Lines of the tree}} \leq 1 + 2|L(G)| \leq e^{2|L(G)|}.$$

The sum over the scales is performed exactly like in the *case 1*, and we obtain

$$\sum_{J \in \mathcal{J}(t,j)} |\operatorname{Val}(G^J)|_1 \leq \left(\frac{4K_0K_1(1+W)e^2}{1-M^{-1}} \right)^{|L(G)|} |v|_1^{|V(G)|} M^{j(1-m+\epsilon 1_{\tilde{G}(\phi)\in\mathcal{O}})}.$$

Case 3: $s = 1$, $E(G) = 2$ and $\tilde{G}(\phi)$ is non-overlapping.

Let consider the graph $\tilde{G}(\tau)$, which is a ST-graph. The derivative with respect to the external momentum can only act on the external vertex of $\tilde{G}(\tau)$, which is an overlapping subgraph of G at scale j^*.

If the derivative hits an interaction line, then bound its effect with $|v|_1 \leq M^{(\epsilon-1)j}|v|_1$, since $\epsilon < 1$. If the derivative acts on a fermionic line, then it produces a factor $WM^{-j_1} \leq WM^{-j^*}$, since $j_1 \geq j^*$. The overlapping loop brings an additional factor $K_1 M^{\epsilon j^*}$. The effect of the derivative is therefore bounded by

$$WK_1M^{(\epsilon-1)j^*} \leq WK_1M^{(\epsilon-1)j}.$$

The number of terms obtained by the product rule is bounded like in the *case 2*, and apart from the derivative and from the overlapping loop, we use the naive power counting in order to prove the claim.

Set

$$Q_0 \geq \frac{4K_0K_1M(1+W)e^2}{1-M^{-1}}(M^\epsilon - 1),$$

in order to take into account the factor $\lambda_0(j,\epsilon) = (M^\epsilon - 1)^{-1}$.

<u>$P \geq 1$</u>

Consider a graph G with $P \geq 1$. Construct the graph G' obtained collapsing all the two- and four-legged 1PI subgraphs of G into vertices. By construction, $P' = 0$, and

for each vertex $w \in V(G')$, the depth P_w of the corresponding 1PI subgraph G_w of G satisfies $P_w < P$. We can therefore use the inductive hypothesis.

We first bound the vertex functions F_w associated to $w \in V(G')$.

- If w is a $c-$fork, then

$$\begin{aligned}
|F_w|_0 &= \left| l^{(L)} \sum_{j_w=I}^{j_{\pi(w)}} \sum_{J \in \mathfrak{J}(t_w, j_w)} \mathrm{Val}\,(G_w^J) \right|_0 \\
&\leq \sum_{j_w=I}^{j_{\pi(w)}} \sum_{J \in \mathfrak{J}(t_w, j_w)} |\,\mathrm{Val}\,(G_w^J)|_0 \\
&\overset{\text{I.H.}}{\leq} Q_0^{|L(G_w)|} |v|_1^{|V(G_w)|} \sum_{j_w=I}^{j_{\pi(w)}} \lambda_{n_w}(j_w, \epsilon/2) M^{j_w} \\
&\overset{6.22(\text{iii})}{\leq} 2Q_0^{|L(G_w)|} |v|_1^{|VG_w)|} \lambda_{n_w}(j_{\pi(w)}, \epsilon/2) M^{j_{\pi(w)}}.
\end{aligned}$$

Using the lemma 4.15, the derivative is bounded by

$$\begin{aligned}
|F_w|_1 &\leq |l^{(L)} \sum_{j_w=I}^{j_{\pi(w)}} \sum_{J \in \mathfrak{J}(t_w, j_w)} \mathrm{Val}\,(G_w^J)|_1 \\
&\leq (1+\beta) \sum_{j_w=I}^{j_{\pi(w)}} \sum_{J \in \mathfrak{J}(t_w, j_w)} |\,\mathrm{Val}\,(G_w^J)|_1 \\
&\leq (1+\beta) Q_0^{|L(G_w)|} |v|_1^{|V(G_w)|} \sum_{j_w=I}^{j_{\pi(w)}} \lambda_{n_w}(j_w, \epsilon/2) M^{j_w \epsilon} \\
&\leq 2(1+\beta) Q_0^{|L(G_w)|} |v|_1^{|V(G_w)|} \lambda_{n_w}(j_{\pi(w)}, \epsilon/2) M^{j_{\pi(w)} \epsilon}.
\end{aligned}$$

- If w belongs to a $r-$fork, then in the sup-norm:

$$\begin{aligned}
|F_w|_0 &= |(1 - l^{(L)}) \sum_{j_w > j_{\pi(w)}} \sum_{J \in \mathfrak{J}(t_w, j_w)} \mathrm{Val}\,(G_w^J)|_0 \\
&\leq \sum_{j_w > j_{\pi(w)}} \alpha \sup_{(q_0, \mathbf{q}) \in\, \mathrm{supp} C_{j_{\pi(w)}}} |i q_0 - e(\mathbf{q})| \sum_{J \in \mathfrak{J}(t_w, j_w)} |\,\mathrm{Val}_L\,(G_w^J)|_1 \\
&\leq \alpha Q_0^{|L(G_w)|} |v|_1^{|V(G_w)|} M^{j_{\pi(w)}} \sum_{j_w > j_{\pi(w)}} \lambda_{n_w}(j_w, \epsilon/2) M^{j_w \epsilon} \\
&\leq 2\alpha Q_0^{|L(G_w)|} |v|_1^{|V(G_w)|} \lambda_{n_w}(j_{\pi(w)}, \epsilon/2) M^{j_{\pi(w)}}.
\end{aligned}$$

And for the derivative,

$$\begin{aligned}
|F_w|_1 &\leq (1+\beta) \sum_{j_w > j_{\pi(w)}} \sum_{J \in \mathfrak{J}(t_w, j_w)} |\,\mathrm{Val}_L\,(G_w^J)|_1 \\
&\leq (1+\beta) Q_0^{|L(G_w)|} |v|_1^{|V(G_w)|} \sum_{j_w > j_{\pi(w)}} \lambda_{n_w}(j_w, \epsilon/2) M^{j_w \epsilon} \\
&\leq 2(1+\beta) Q_0^{|L(G_w)|} |v|_1^{|V(G_w)|} \lambda_{n_w}(j_{\pi(w)}, \epsilon/2).
\end{aligned}$$

The lemma 4.15 can be applied by the support properties of the propagator at scale $j_{\pi(w)}$ that accompanies F_w.

- If w corresponds to a same scale insertion, then there is no sum over the scale j_w. We get therefore

$$|F_w|_0 \leq Q_0^{|L(G_w)|}|v|_1^{|V(G_w)|}\lambda_{n_w}(j,\epsilon/2)M^j,$$

and

$$|F_w|_1 \leq Q_0^{|L(G_w)|}|v|_1^{|V(G_w)|}\lambda_{n_w}(j,\epsilon/2)M^{j\epsilon}.$$

- If F_w belongs to a four-legged vertex, and G_w is non-overlapping at scale j_w, then

$$\begin{aligned}
|F_w|_s &\leq Q_0^{|L(G_w)|}|v|_1^{|V(G_w)|}\sum_{j_w>j_{\pi(w)}}\lambda_{n_w}(j_w,\epsilon/2)M^{-sj_w}\\
&\leq Q_0^{|L(G_w)|}|v|_1^{|V(G_w)|}M^{-sj_{\pi(w)}}\sum_{j_w>j_{\pi(w)}}\lambda_{n_w}(j_w,\epsilon/2)\\
&\leq Q_0^{|L(G_w)|}|v|_1^{|V(G_w)|}M^{-sj_{\pi(w)}}\lambda_{n_w+1}(j_{\pi(w)},\epsilon/2).
\end{aligned}$$

- If F_w belongs to a four-legged fork, and G_w is overlapping at scale j_w, then

$$\begin{aligned}
|F_w|_s &\leq Q_0^{|L(G_w)|}|v|_1^{|V(G_w)|}\sum_{j_w>j_{\pi(w)}}\lambda_{n_w}(j_w,\epsilon/2)M^{j_w(\epsilon-s)}\\
&\leq Q_0^{|L(G_w)|}|v|_1^{|V(G_w)|}M^{-sj_{\pi(w)}}\sum_{j_w>j_{\pi(w)}}\lambda_{n_w}(j_w,\epsilon/2)M^{j_w\epsilon}\\
&\leq Q_0^{|L(G_w)|}|v|_1^{|V(G_w)|}M^{-sj_{\pi(w)}}\lambda_{n_w}(j_{\pi(w)},\epsilon/2).
\end{aligned}$$

We consider now the graph G' having vertex functions F_w. The two-legged subgraphs of G' are string of two-legged vertices, while the four-legged subgraphs of G' are four-legged vertices with string two-legged vertices attached to it. For simplicity, we denote with f (rather than f') the forks of the tree t', and $D_f := D_f(G')$. In the same way, $D_\phi = D_\phi(G')$. As for $P = 0$, we consider the following cases separately:

Case 4: $s = 0$.

We bound the value of the graph G' using the naive power counting of theorem 6.15(i).

$$\begin{aligned}
|\operatorname{Val}(G'^{J'})|_0 &\leq (K_1 M^{j\epsilon})^{1_{G'(\phi)\in\mathcal{O}}}(4K_0)^{|L(G')|}\prod_{v\in V_4(G')}|F_v|_0\prod_{w\in V_2(G')}|F_w|_0 \\
&\quad \cdot M^{D_\phi j}\prod_{f>\phi}M^{D_f(j_f-j_{\pi(f)})} \\
&\leq (2(\beta+1)\alpha)^{|V(G')|}(4K_0K_1)^{|L(G')|}Q_0^{\sum_{v\in V(G')}|L(G_v)|}|v|_1^{|V(G)|} \\
&\quad \cdot \prod_{v\in V_4(G')}M^{\epsilon j_{\pi(v)}1_{G_w\in\mathcal{O}}}\lambda_{n_v}(j_{\pi(v)},\epsilon/2))\prod_{w\in V_2(G')}\lambda_{n_w}(j_{\pi(w)},\epsilon/2)M^{j_{\pi(w)}} \\
&\quad \cdot M^{D_\phi j}\prod_{f>\phi}M^{D_f(j_f-j_{\pi(f)})}.
\end{aligned}$$

The product over the $\lambda-$functions is bounded by lemma 6.22:

$$\prod_{v\in V(G')}\lambda_{n_v}(j_{\pi(w)},\epsilon/2)\leq\lambda_{n_\phi}(j,\epsilon/2),$$

and the improvement factor for the four-legged vertices can be bounded by one. We use the telescope sum

$$j_{\pi(w)}=j+\sum_{\substack{w>f>\phi\\ f\ni w}}(j_f-j_{\pi(f)}),$$

where the sum runs over the forks in t', in order to get

$$\sum_{w\in V_2(G')}j_{\pi(w)}=j|V_2(G')|+\sum_{w>f>\phi}|V_2(G_f)|(j_f-j_{\pi(f)}).$$

It follows that the product

$$\prod_{v\in V_2(G')}M^{j_{\pi(v)}}=M^{j|V_2(G')|}\prod_{w>f>\phi}M^{|V_2(G_f)|(j_f-j_{\pi(f)})}.$$

Collecting both products over the forks of t', we get

$$\begin{aligned}
|\operatorname{Val}(G'^{J'})|_0 &\leq M^{j\epsilon 1_{G'\in\mathcal{O}}}(2(\beta+1)\alpha)^{|V(G')|}(4K_0K_1)^{|L(G')|}Q_0^{\sum_v|L(G_v)|}|v|_1^{|V(G)|} \\
&\qquad \cdot\lambda_{n_\phi}(j,\epsilon/2)M^{D_\phi^R j}\prod_{f>\phi}M^{D_f^R(j_f-j_{\pi(f)})},
\end{aligned}$$

where $D_f^R=D_f-|V_2(G')|=2-m_f$. The claim follows finally, summing over the scales as in *case 1*. We have to choose

$$Q_0\geq\frac{8(\beta+1)\alpha K_0K_1M(1+W)e^2}{1-M^{-1}}(M^\epsilon-1).$$

Case 5: $s = 1$, $E(G) > 2$ or $E(G) = 2$ and $\tilde{G}(\phi)$ is overlapping.

In this case, the derivative can apply on a line of $T(G')$, or on the vertices F_w of G'. We bound the number of term obtained by the product rule with $2|V(G')|$, and use $M^{-j_{\pi(w)}} \leq M^{-j}$, in order to obtain the desired bound. Follow the *case 4* in order to get the final result.

Case 6: $s = 1$ and $E(G) = 2$ and $\tilde{G}(\phi)$ is non-overlapping.

This case is similar to *case 3*. The graph G' is a ST-graph, with a unique external vertex v_1, which corresponds to a connected subgraph of G' at the overlapping scale or higher. Thus, as in *case 3*, the derivative applies only on v_1, and the effect of the derivative is bounded by

$$M^{(\epsilon-1)j^*} \leq M^{(\epsilon-1)j},$$

as in *case 3*. Up to this factor, the claim follows exactly in the same way, as in *case 4*. □

6.5 Removing the Cutoff

6.5.1 The Convergence of 1PI Graphs with j fixed

Lemma 6.24: Let $(f^{(L)})_{L\in\mathbb{N}}$ be a sequence of functions

$$f^{(L)} : (\mathbb{R} \times \mathbf{\Lambda}_L^\sharp)^n \to \mathbb{C}$$

that converges uniformly to a differentiable function $f : (\mathbb{R} \times \mathbb{T})^n \to \mathbb{C}$ in the limit $L \to \infty$. Assume further that all the functions $f^{(L)}$ and f have a compact support in the k_0-variables, independent of L. Then for each $m \leq n$,

$$L^{-dm} \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_m\in\mathbf{\Lambda}_L^\sharp} \int \frac{dk_{10}}{2\pi} \cdots \frac{dk_{m0}}{2\pi} f^{(L)}(k_1,\ldots,k_n) \overset{L\to\infty}{\to} \int \frac{d^{d+1}k_1}{(2\pi)^{d+1}} \cdots \frac{d^{d+1}k_m}{(2\pi)^{d+1}} f(k_1,\ldots,k_n),$$

in the supremum norm with respect to the $n-m$ remaining variables.

<u>Proof:</u> Since the support of the functions $f^{(L)}$ and f are compact, the k_0-integrals can

be bounded by a constant times the supremum over the k_0-variables. Hence, we bound

$$\begin{aligned}
&\left| L^{-dm} \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_m \in \mathbf{\Lambda}_L^\sharp} f^{(L)}(k_1,\ldots,k_n) - \int \frac{d^d\mathbf{k}_1}{(2\pi)^d} \cdots \frac{d^d\mathbf{k}_m}{(2\pi)^d} f(k_1,\ldots,k_n) \right| \\
&\leq L^{-dm} \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_m \in \mathbf{\Lambda}_L^\sharp} |f^{(L)}(k_1,\ldots,k_n) - f(k_1,\ldots,k_n)| \\
&+ \left| L^{-dm} \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_m \in \mathbf{\Lambda}_L^\sharp} f(k_1,\ldots,k_n) - \int \frac{d^d\mathbf{k}_1}{(2\pi)^d} \cdots \frac{d^d\mathbf{k}_m}{(2\pi)^d} f(k_1,\ldots,k_n) \right| .
\end{aligned}$$

By hypothesis, the first term of the right hand side tends to zero as $L \to \infty$. For the second term,

$$\begin{aligned}
&\left| L^{-dm} \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_m \in \mathbf{\Lambda}_L^\sharp} f(k_1,\ldots,k_n) - \int \frac{d^d\mathbf{k}_1}{(2\pi)^d} \cdots \frac{d^d\mathbf{k}_m}{(2\pi)^d} f(k_1,\ldots,k_n) \right| \\
&\leq \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_m \in \mathbf{\Lambda}_L^\sharp} \int_{V_{\mathbf{k}_1}} \frac{d^d\mathbf{p}_1}{(2\pi)^d} \cdots \int_{V_{\mathbf{k}_m}} \frac{d^d\mathbf{p}_m}{(2\pi)^d} |f(\mathbf{k}_1,\ldots,\mathbf{k}_m) - f(\mathbf{p}_1,\ldots,\mathbf{p}_m)| \\
&\leq md||f||_1 \frac{4\pi}{L}.
\end{aligned}$$

Since the derivative norm of f is independent of L, the claim follows. □

Definition 6.25:

(i) Let $\mathcal{C}_0$ be the space of the functions on $(\mathbb{R} \times \mathbb{T})^{2m-1}$ with finite $||\cdot||_0$-norm, and $\mathcal{C}_1$ be the space of the C^1-functions on $(\mathbb{R} \times \mathbb{T})^{2m-1}$ with finite norm $||\cdot||_1$. Further, let $\mathcal{L}$ be the space of the L_1-functions on $(\mathbb{R} \times \mathbb{T})^{2m-1}$.

Let $l_1(\mathbb{Z}_-, B)$ be the space of the absolute summable sequences in the Banach space B, where B is $\mathcal{C}_0$, $\mathcal{C}_1$ or $\mathcal{L}$.

(ii) Let G be a 1PI graph with $2m$ external legs, and t be a planar tree rooted at ϕ compatible with G, such that the pair (G,t) contributes to the renormalized Green function $G^{(L)}_{2m,r}$ or $G^I_{2m,r}$ at scale $j<0$. For $J \in \mathcal{J}(t,j)$, let G^J be the labeled graph with $t(G^J) = t$. For $L > 0$, define the sequence $\gamma^{(L)}$ as

$$\gamma_j^{(L)} := \begin{cases} \sum_{J \in \mathcal{J}(j,t)} \mathrm{Val}_L(G^J), & j \geq I \\ 0, & j < I, \end{cases}$$

where we consider $\gamma_j^{(L)}$ as a step function on $\mathbb{R} \times \mathbb{T}$, with constant value on the fundamental zone of the lattice $\mathbf{\Lambda}_L^\sharp$.

(iii) Let $0 > I > -\infty$. Define the sequence

$$\gamma_j^I := \begin{cases} \sum_{J \in \mathfrak{J}_I(j,t)} \mathrm{Val}_\infty(G^J), & j \geq I \\ 0, & j < I. \end{cases}$$

Remark 6.26:

(i) $\gamma_j^{(L)} \in \mathcal{C}_0$ and $\gamma_j^{(L)} \in \mathcal{L}$.

(ii) $\gamma_j^I \in \mathcal{C}_0$, $\gamma_j^I \in \mathcal{C}_1$, and $\gamma_j^I \in \mathcal{L}$.

Proof: The bounds of 6.15 imply that the $||\cdot||_0-$norm of γ_j^I and its $||\cdot||'-$norm are finite, such that $\gamma_j^I \in \mathcal{C}_0$ and $\gamma_j^I \in \mathcal{L}$. The bound for the derivative and integral norms follows in the same way. □

Lemma 6.27: There is a sequence $(\gamma_j)_{j\in\mathbb{Z}_-}$ in $\mathcal{C}_1$, such that for each fixed $j \in \mathbb{Z}_-$,

$$\gamma_j^I \overset{I\to-\infty}{\to} \gamma_j \text{ in the } ||\cdot||_1 - \text{norm},$$

and

$$\gamma_j^{(L)} \overset{L\to\infty}{\to} \gamma_j \text{ in the } ||\cdot||_0 - \text{norm}.$$

Further,

$$\gamma_j^I \overset{I\to-\infty}{\to} \gamma_j \text{ and } \gamma_j^{(L)} \overset{L\to\infty}{\to} \gamma_j \text{ in the } ||\cdot||' - \text{norm}.$$

Proof: The proof of this theorem follows the proof of the theorem 6.23, by induction on P. Pick up $L > 0$ such that $j > j_L$.

$P = 0$

In the case $P = 0$, the graph G^J contains no 1PI two- or four-legged subgraphs. All the propagators are therefore at scale j or higher, and since $j > j_L$, $e^{(L)}(\mathbf{k}) = e(\mathbf{k})$. Hence $\gamma_j^I = \gamma_j$, and $\gamma_j^{(L)}$ is a Riemann sum that converges to γ_j by lemma 6.24 in the supremum as well as in the integral norm.

$P \geq 1$

As in the proof of theorem 6.23, we consider the graph G' defined in 6.19. Since all the scales of G' are bigger than $j > j_L$, $e^{(L)}(\mathbf{k}) = e(\mathbf{k})$. We first apply the induction hypothesis (IH) on the vertex functions $F_w^{(L)}$ or F_w^I corresponding to the vertex w of G',

and in a second step prove the claim for G' itself. Let w be a vertex of G', given by the value of the graph G_w. Let $\bar{\gamma}_{j_w}^{(L)}$ and $\bar{\gamma}_{j_w}^{I}$ be the corresponding of $\gamma_j^{(L)}$ and γ_j^I for the graph G_w.

By induction hypothesis, there is a sequence $\bar{\gamma}_{j_w}$, such that $\bar{\gamma}_{j_w}^{I} \to \bar{\gamma}_{j_w}$ for $I \to -\infty$, and $\bar{\gamma}_{j_w}^{(L)} \to \bar{\gamma}_{j_w}$ for $L \to \infty$ in the supremum and derivative norms. We consider the following cases:

- <u>w corresponds to a $c-$fork:</u> Let us consider the sequence

$$g_h^I := \begin{cases} l\bar{\gamma}_h^I, & h \in \{I, \dots, j_{\pi(w)}\} \\ 0 & \text{otherwise.} \end{cases}$$

By theorem 6.23, g_h^I is bounded by

$$||g_h^I||_1 \leq \text{Const } \lambda_{n_w}(h, \epsilon) M^{h\epsilon},$$

such by dominated convergence in $l_1(\mathbb{Z}_-, \mathcal{C}_1)$,

$$F_w^I \overset{I \to -\infty}{\to} F_w := \sum_{j_w = -\infty}^{j_{\pi(w)}} l\bar{\gamma}_w$$

in the $||\cdot||_1-$norm. We turn to the thermodynamic limit of

$$F_w^{(L)} = \sum_{j_w = I(L)}^{j_{\pi(w)}} l^{(L)} \bar{\gamma}_w^{(L)}.$$

Observe that

$$||l^{(L)}\bar{\gamma}_{j_w}^{(L)} - l\bar{\gamma}_{j_w}||_0 \leq ||l^{(L)}(\bar{\gamma}_{j_w}^{(L)} - \bar{\gamma}_{j_w})||_0 + ||(l^{(L)} - l)\bar{\gamma}_{j_w}||_0$$

The first term on the right hand side vanishes for $L \to \infty$ by IH. The lemma 4.18 implies

$$||(l^{(L)} - l)\bar{\gamma}_{j_w}||_0 \leq ||\bar{\gamma}_{j_w}||_1 \frac{4\pi}{L},$$

hence

$$l^{(L)}\bar{\gamma}_{j_w}^{(L)} \overset{L \to \infty}{\to} l\bar{\gamma}_{j_w}$$

in the sup-norm. Using the theorem 6.23 on $\bar{\gamma}_{j_w}^{(L)}$, we apply once again the dominated convergence in $l_1(\mathbb{Z}_-, \mathcal{C}_1)$ and prove

$$F_w^{(L)} \overset{L \to \infty}{\to} F_w$$

in the $||\cdot||_0-$norm.

- w is a $r-$fork

  By the same argument as for the $c-$forks, we see that in the sup-norm,

$$(1-l^{(L)})\bar{\gamma}_{j_w}^{(L)} \overset{L\to\infty}{\to} (1-l)\bar{\gamma}_{j_w}.$$

  Since the sum over the scales j_w contains only $|j_{\pi(w)}| \leq |j| < \infty$ terms, the function F_w^I as well as the function $F_w^{(L)}$ converge to

$$F_w = \sum_{j_w=j+1}^{0} \bar{\gamma}_{j_w}.$$

- w corresponds to a same scale insertion

  Since they are no sum, and no projection operator $l^{(L)}$, the convergence is given by the I.H.

- w corresponds to a four-legged forks

  Since the sum over the root scale j_w has at most $|j|$ terms, and that $F_w^{(L)}$ contains no projection operator $l^{(L)}$, the convergence follows from the I.H.

We turn now to the proof of the claim for G', using the fact that G' is a graph with vertex-functions F_w^I or $F_w^{(L)}$ that converge to F_w in the limit $I \to -\infty$ resp. $L \to \infty$.

The convergence of γ^I and $\gamma^{(L)}$ in $\mathcal{C}_1$ and $\mathcal{C}_0$ follows from the lemma 6.24, since the value of G' is given by the integral of a bounded function with compact supports. The convergence in $\mathcal{L}$ follows in the same way. □

6.5.2 Bounds on $\Sigma_r^{(L)}$ and $G_{2,r}^{(L)}$

Lemma 6.28:

(i) Let G be a 1PI, two-legged graph, that contributes to the self-energy at order r in λ, and t a tree compatible with G rooted at ϕ. Then

$$|\sum_{j=I}^{0} \sum_{J\in\mathcal{J}(j,t)} \mathrm{Val}_L(G^J)|_0 \leq \tilde{Q}_0^r |v|_1^r,$$

where $\tilde{Q}_0$ is a constant independent of r. Further,

$$|\sum_{j=I}^{0} \sum_{J\in\mathcal{J}(j,t)} \mathrm{Val}_L(G^J)|_1 \leq \tilde{Q}_1^r |v|_1^r.$$

(ii) Let G be a two-legged, connected graph that contributes to the two-points Green function $G^{(L)}_{2,r}$ at order r in λ. Then

$$|\sum_{j=I}^{0}\sum_{J\in\mathcal{J}(j,t)} \mathrm{Val}_L(G^J)|_0 \leq \bar{Q}_0^r |v|_1^r.$$

<u>Proof:</u> The claim (i) for 1PI graphs follows directly from theorem 6.23:

$$\begin{aligned}
|\sum_{j=I}^{0}\sum_{J\in\mathcal{J}(j,t)} \mathrm{Val}_L(G^J)|_0 &\leq \sum_{j=-\infty}^{0}\sum_{J\in\mathcal{J}(j,t)} |\mathrm{Val}_L(G^J)|_0 \\
&\leq Q_0^{|L(G)|}|v|_1^{|V(G)|}\sum_{j=-\infty}^{0}\lambda_n(j,\epsilon)M^j \\
&\leq 2Q_0^{|L(G)|}|v|_1^{|V(G)|}\lambda_n(0,\epsilon).
\end{aligned}$$

Since $|L(G)| = 2|V_4(G)| + |V_2(G)| - m$, with in our case, $|V_4(G)| = r$, $|V_2(G)| = 0$ and $m = 1$, and $n \leq |L(G)|$, we get the claim. The proof of the bound for the derivative norm is similar.

The bound (ii) for the two-legged graphs is obtained applying the power counting 6.15(i) on the graph G' corresponding to G:

$$|\sum_{j=I}^{0}\sum_{J\in\mathcal{J}(j,t)} \mathrm{Val}_L(G^J)|_0 \leq \sum_{j=-\infty}^{0}\sum_{J\in\mathcal{J}(j,t')} |\mathrm{Val}_L(G'^J)|_0,$$

where t' is the subtree of t, corresponding to the graph G'. Further, by 6.15(i),

$$|\mathrm{Val}_L(G'^J)|_0 \leq (4K_0)^{|L(G')|}\prod_{v\in V_4(G')}|U_v|_0\prod_{v\in V_2(G')}|\Theta_v|_0 M^{D_\phi j}\prod_{f>\phi}M^{D_f(j_f-j_{\pi(f)})},$$

where the product runs over the forks $f\in t'$, and $D_f = D_f(G')$. We bound the sup-norms of the vertices of G' with the help of theorem 6.23, and for all the two-legged vertices v of G', we use the telescope sum

$$j_{\pi(v)} = \sum_{v>f>\phi} f \ni v(j_f - j_{\pi(f)}) + j,$$

such that

$$\prod_{v\in V_2(G')} M^{j_{\pi(v)}} = M^{j|V_2(G')|}\prod_{f>\phi}M^{|V_2(G_f)|(j_f-j_{\pi(f)})}.$$

Hence

$$\begin{aligned}
|\mathrm{Val}_L(G'^J)|_0 \leq\ & (4K_0)^{|L(G')|}\prod_{v\in V_4(G')}|U_v|_0\prod_{v\in V_2(G')}M^{-j_{\pi(v)}}|\Theta_v|_0 \\
& \cdot M^{D^R_\phi j}\prod_{f>\phi}M^{D^R_f(j_f-j_{\pi(f)})},
\end{aligned}$$

where $D_f^R = D_f + |V_2(G_f)| = 2 - m_f < 0$, as in the proof of theorem 6.23. By lemma 6.22 and theorem 6.23,

$$|\operatorname{Val}_L(G'^J)|_0 \leq (4K_0)^{|L(G')|} Q_0^{\sum_{v\in V(G')}|L(G_v)|} |v|_1^{|V(G)|} \lambda_n(j,\epsilon) M^{D_\phi^R j} \prod_{f>\phi} M^{D_f^R(j_f - j_{\pi(f)})}.$$

We can therefore perform the sum over $J \in \mathcal{J}(j,t')$ in the same way as in the proof of 6.23, and get

$$\sum_{J\in\mathcal{J}(j,t')} |\operatorname{Val}_L(G'^J)|_0 \leq Q_0^{|L(G)|} |v|_1^{|V(G)|} \lambda_n(j,\epsilon) M^j$$

since $D_\phi^R = 1$. The sum over the root scale is therefore convergent. □

Proof of 5.4(i): By remarks 6.6 and 6.7,

$$|\Sigma_r|_s \leq \sum_{j=I(L)}^{0} \sum_t \prod_{f\in t} \frac{1}{u_f!} \sum_G \sum_{J\in\mathcal{J}(j,t)} |\operatorname{Val}_L G^J|_s.$$

Using the previous lemma,

$$|\Sigma_r|_s \leq \tilde{Q}_s^r |v|_1^r \sum_t \prod_{f\in t} \frac{1}{u_f!} \sum_G 1.$$

The lasts term corresponds to the number of connected graphs, with r four-legged vertices, which is bounded by $const^r (r!)^2$. Hence,

$$|\Sigma_r|_s \leq \operatorname{Const}^r (r!)^2.$$

The supremum norm of the two-points Green function is bounded in the same way. □

6.5.3 The Convergence of the Green Functions

Lemma 6.29: Let G be a 1PI, two-legged graph, contributing to the self-energy Σ_r, and t a planar tree with r leaves, compatible with G. Let γ be the sequence defined in 6.27. Then

(i) For $I \to -\infty$, γ^I converges to γ in $l_1(\mathbb{Z}_-, \mathcal{C}_1)$.

(ii) For $L \to \infty$, $\gamma^{(L)}$ converges to γ in $l_1(\mathbb{Z}_-, \mathcal{C}_0)$.

Further, for a two-legged connected graph G that contributes to $G_{2,r}^{(L)}$, $\gamma^{(L)} \to \gamma$ in $l_1(\mathbb{Z}_-, \mathcal{C}_0)$.

Proof: If G is a 1PI, two-legged graph, we have

$$|\gamma_j^{(L)}|_0 \leq \text{Const } \lambda_n(j,\epsilon)M^j \text{ and } ||\gamma_j^I||_0 \leq \text{Const } \lambda_n(j,\epsilon)M^j,$$

by 6.23. Hence, by dominated convergence in $l_1(\mathbb{Z}_-, \mathcal{C}_0)$,

$$\sum_{j=I}^{0} \gamma_j^{(L)} \to \sum_{j=-\infty}^{0} \gamma_j.$$

The convergence of γ^I in $l_1(\mathbb{Z}_-, \mathcal{C}_1)$ follows in the same way.

If $\gamma_j^{(L)}$ corresponds to a graph contributing to the two-points Green function $G_{2,r}^{(L)}$, we apply the bound obtained in 6.28 and get

$$\gamma_j^{(L)} \leq Q_0^{|L(G)|}|v|_1^{|V(G)|}\lambda_n(j,\epsilon)M^j,$$

such that the dominated convergence in $l_1(\mathbb{Z}_-, \mathcal{C})$ proves the claim. □

Remark 6.30: Point (ii) of theorem 5.4 follows directly from lemma 6.29, since the sum over the graphs that contribute to $\Sigma_r^{(L)}$ or $G_{2,r}^{(L)}$ is finite.

Proof of 5.4(iii):

We prove the convergence of the counterterm $K_r^{(L)}$ order by order in r. Set $K_r = -l\Sigma_r$. Then

$$\begin{aligned} ||K_r^{(L)} - K_r||_0 &= ||l^{(L)}\Sigma_r^{(L)} - l\Sigma_r||_0 \\ &\leq ||l^{(L)}(\Sigma_r^{(L)} - \Sigma_r)||_0 + ||(l^{(L)} - l)\Sigma_r||_0. \end{aligned}$$

The first term of the right hand side of last line tends to zero as $L \to \infty$ by 5.4(ii), and the second term vanishes by 4.18, and the fact that $||\Sigma_r||_1 \leq Const.$ □

6.5.4 The Convergence of $G_{2m,r}^{(L)}$ in the L_1-norm

Lemma 6.31: Let t be a planar tree with r leaves, and G be a $2m-$legged graphs, compatible with t. For $j < 0$, let $J \in \mathcal{J}(t,j)$ be such that $t(G^J) = t$. Then

$$\sum_{J\in\mathcal{J}(t,j)} |\operatorname{Val}_L(G^J)|' \leq Q_0^{|L(G)|}|v|_1^{|V(G)|}\, \lambda_n(j,\epsilon/2)M^{j/4}.$$

<u>Proof:</u> Let consider the graph G' defined in 6.19. Then by theorem 6.15(ii),

$$\begin{aligned} |\operatorname{Val}_L(G^J)|' &\leq (4K_0)^{|L(G'^J)|} \prod_{v\in V_4(G')} |U_v|_0 \prod_{w\in V_2(G')} M^{-j_{\pi(w)}} |\Theta_w|_0 \\ &\cdot \prod_{f,\,int} M^{D_f^R(j_f - j_{\pi(f)})} \prod_{f,\,ext} M^{\Delta_f(j_f - j_{\pi(f)})} \prod_{v,\,ext} M^{\Delta_v(0 - j_{\pi(v)})}. \end{aligned}$$

Bounding the vertex function with 6.23, and using the lemma 6.22 in order to bound the product over the $\lambda-$functions, we get

$$\begin{aligned} |\operatorname{Val}_L(G^J)|' &\leq (4K_0)^{|L(G'^J)|} \prod_{v\in V(G')} (Q_0^{L(G_v)|} |v|_1^{|V(G_v)|})\, \lambda_n(j, \epsilon/2) \\ &\cdot \prod_{f,\,int} M^{D_f^R(j_f - j_{\pi(f)})} \prod_{f,\,ext} M^{\Delta_f(j_f - j_{\pi(f)})} \prod_{v,\,ext} M^{\Delta_v(0 - j_{\pi(v)})}, \end{aligned}$$

where $n = \sum_v n_v$. Let $\bar{v}$ be one of the external vertex of G'^J. Observe that

$$-j = \sum_{f\ni\bar{v}} (j_f - j_{\pi(f)}) - j_{\pi(\bar{v})},$$

such that

$$\begin{aligned} |\operatorname{Val}_L(G'^J)|' &\leq (4K_0)^{|L(G'^J)|} \prod_{v\in V(G')} (Q_0^{L(G_v)|} |v|_1^{|V(G_v)|})\, \lambda_n(j, \epsilon/2) M^{j/4} \\ &\cdot \prod_{f,\,int} M^{D_f(j_f - j_{\pi(f)})} \prod_{f,\,ext\, \bar{v}\notin f} M^{\Delta_f(j_f - j_{\pi(f)})} \\ &\cdot \prod_{f\ni\bar{v}} M^{(\Delta_f + \frac{1}{4})(j_f - j_{\pi(f)})} \prod_{v,\,ext,\, v\neq\bar{v}} M^{-\Delta_v j_{\pi(v)}} M^{-(\Delta_{\bar{v}} + \frac{1}{4}) j_{\pi(\bar{v})}}. \end{aligned}$$

The sum over the scale of the four products over the forks is bounded by

$$\sum_{J\in\mathcal{J}(j,t)} \prod_{f>\phi} M^{-\frac{1}{4}(j_f - j_{\pi(f)})} \leq (1 - M^{-1/4})^{-|L(G'^J)|},$$

such that

$$\sum_{J\in\mathcal{J}(j,t')} |\operatorname{Val}_L(G'^J)|' \leq Q_0^{|L(G)|} |v|_1^{|V(G)|} \lambda_n(j, \epsilon) M^{j/4}.$$

□

The convergence of $G_{2m,r}^{(L)}$ in the $||\cdot||'-$norm is proved in the same way as the convergence of the two-points function $G_{2,r}^{(L)}$ in the $||\cdot||_0-$norm.

Appendix A

The Fourier Transform

The discrete Fourier transform of a function f on the lattice $\mathbf{\Lambda}$ satisfying

$$\sum_{\mathbf{x}\in\mathbf{\Lambda}} |\mathbf{x}|^r |f(\mathbf{x})| < \infty,$$

for a $r \geq 1$ is defined by

$$\hat{f}(\mathbf{k}) = \sum_{\mathbf{x}\in\mathbf{\Lambda}} f(\mathbf{x}) e^{-i\mathbf{k}\mathbf{x}}.$$

$\hat{f}$ is a $r-$time differentiable function on the torus $\mathcal{B} = \mathbb{R}^d/2\pi\mathbb{Z}^d$. The inverse Fourier transform is given by

$$f(\mathbf{x}) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}} \hat{f}(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}} d^d k.$$

Consider now a function f on the finite lattice $\mathbf{\Lambda}_L^\sharp$ satisfying

$$\sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} |f(\mathbf{k})| < \infty.$$

The discrete Fourier transform of f is defined by

$$\check{f}(\mathbf{x}) = L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} f(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}},$$

and its inverse is given by

$$f(\mathbf{k}) = \sum_{\mathbf{x}\in\mathbf{\Lambda}_L} \check{f}(\mathbf{x}) e^{-i\mathbf{k}\mathbf{x}}.$$

Note that $\check{f}$ satisfies periodic boundary conditions on $\mathbf{\Lambda}_L$, and is given by a Riemanian sum that converge to the Fourier coefficient of the function f on the torus $\mathbb{T}$. In particular,

$$\sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} e^{i\mathbf{k}\mathbf{x}} = L^d \delta_{\mathbf{x}\mathbf{0}}.$$

Let $x = (x^0, \mathbf{x}) \in \mathbb{R} \times \mathbf{\Lambda}_L$. For $k = (k_0, \mathbf{k})$, the Fourier transform of an integrable function f is given by

$$\hat{f}(k) = \sum_{\mathbf{x} \in \mathbf{\Lambda}_L^\sharp} \int dx^0 f(x) e^{i<k,x>},$$

where $< k, x >= -k_0 x^0 + \mathbf{kx}$. Remark that by the periodic boundary conditions,

$$\sum_{\mathbf{x} \in \mathbf{\Lambda}_L} \int dx^0 f(x-y) = \sum_{\mathbf{x} \in \mathbf{\Lambda}_L} \int dx^0 f(x).$$

Appendix B

Grassman Integrals

Let A be the Grassman algebra on $\mathbb{C}$ generated by the fields $\psi(\xi)$, with $\xi = (x, \sigma, a) \in \mathcal{B}$, where $\mathcal{B} = \mathbb{R} \times \mathbf{\Lambda}_L \times \{\uparrow, \downarrow\} \times \{0, 1\}$. A Grassman function $f(\psi)$ is a function on A of the form

$$f(\psi) = \sum_{n \geq 0} \int d\xi_1 \cdots \int d\xi_n \, f_n(\xi_1, \ldots, \xi_n) \, \psi(\xi_1) \cdots \psi(\xi_n),$$

where the $f_n(\xi_1, \ldots, \xi_n)$ are totally antisymmetric complex functions of the variables ξ_i, and

$$\int d\xi \, \cdot = \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{a \in \{0,1\}} \sum_{\mathbf{x} \in \mathbf{\Lambda}_L} \int_{-\infty}^{\infty} dx^0 \; \cdot \, .$$

A Grassman function with Grassman coefficients is a function g of the form

$$g(\phi, \psi) = \sum_{n \geq 0} \int d\xi_1 \cdots \int d\xi_n \, g_n(\phi; \xi_1, \ldots, \xi_n) \, \psi(\xi_1) \cdots \psi(\xi_n),$$

where the g_n's are Grassman functions in ϕ.

For Grassman fields ϕ and ψ, define the antisymmetric bilinear form

$$(\phi, \psi) = \int d\xi \int d\eta \, \phi(\xi) \, \mathcal{J}(\xi, \eta) \, \psi(\eta),$$

where

$$\mathcal{J}(\xi, \eta) = \delta(x^0 - y^0) \delta_{\mathbf{xy}} \delta_{\sigma\tau} J_{ab},$$

with

$$J_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In other words,

$$(\phi, \psi) = \sum_{\sigma \in \{\uparrow, \downarrow\}} \int dx^0 \sum_{\mathbf{x} \in \mathbf{\Lambda}_L} \left(\bar{\phi}_\sigma(x) \psi_\sigma(x) + \bar{\psi}_\sigma(x) \phi_\sigma(x) \right).$$

The exponential function is defined by its Taylor expansion,

$$e^{(\phi,\psi)} = \sum_{n\geq 0} \frac{1}{n!}(\phi,\psi)^n.$$

Definition B.1: Let C be an antisymmetric bilinear form on A, defined by

$$C(\psi,\phi) = \int d\xi d\eta\, \psi(\xi)\, C(\xi,\eta)\, \phi(\eta),$$

where

$$C(x,\sigma,a;y,\tau,b) = \begin{pmatrix} 0 & -C_{\tau\sigma}(y,x) \\ C_{\sigma\tau}(x,y) & 0 \end{pmatrix}.$$

The Grassman Gaussian integral with covariance C is then the unique linear map on the space of the Grassman functions

$$\int \cdot\, d\mu_C(\psi) : f(\psi) \mapsto \int f(\psi)\, d\mu_C(\psi)$$

such that

$$\int e^{(\psi,\phi)}\, d\mu_C(\psi) = e^{-\frac{1}{2}C(\phi,\phi)}.$$

In particular, if $a_i = 1$ for $i = 1,\ldots,n/2$ and $a_i = 0$ for $i = n/2+1,\ldots,n$,

$$\int \psi(\xi_1)\cdots\psi(\xi_n)\, d\mu_C(\psi) = \det\left(C_{\sigma_i\sigma_j}(x_i,x_j)\right)_{i,j=1}^n,$$

and the integral vanishes if $\sum_{i=1}^n a_i \neq \frac{n}{2}$.

B.1 Integral Bound

Lemma B.2: Let $\mathcal{F}_m(n)$ be the set of the antisymmetric functions on $\mathcal{B}^m \times \mathcal{B}^n$, defined in 3.3. For $f \in \mathcal{F}_m(n)$, define $f' \in \mathcal{F}_m(n-n')$ by

$$\begin{aligned} &f'(\eta_1,\ldots,\eta_m;\xi_{n'+1},\ldots,\xi_n) = \\ &\quad = \int d\xi_1\cdots d\xi_{n'}\, f(\eta_1,\ldots,\eta_m;\xi_1,\ldots,\xi_n) \int \psi(\xi_1)\cdots\psi(\xi_{n'})\, d\mu_C(\psi). \end{aligned}$$

Then

$$||f'||_{1,\infty} \leq S^{n'}(C)||f||_{1,\infty},$$

where

$$S(C) = \sup_{m\in\mathbb{N}} \sup_{\xi_1,\ldots,\xi_{2m}\in\mathcal{B}} \left(\int \psi(\xi_1)\cdots\psi(\xi_{2m})\, d\mu_C(\psi) \right)^{1/2m}.$$

<u>Proof:</u> Consider first the case $m > 0$. Then

$$\begin{aligned}
&\textstyle\int |f'(\eta_1,\ldots,\eta_m;\xi_{n'+1},\ldots,\xi_n)|\, d\xi_{n'+1}\cdots d\xi_n \\
&\leq \sup_{\xi_1,\ldots,\xi_{n'}\in\mathcal{B}} \left| \int \psi(\xi_1)\cdots\psi(\xi_{n'}) d\mu_C(\psi) \right| \int |f(\eta_1,\ldots,\eta_m;\xi_1,\ldots,\xi_n)| d\xi_1\cdots d\xi_n \\
&\leq S^{n'}(C)||f||_{1,\infty}.
\end{aligned}$$

If $m = 0$, then for $i \in \{n'+1,\ldots,n\}$,

$$\int |f'(\xi_{n'+1},\ldots,\xi_n)| \prod_{\substack{j=n'+1\\ i\neq j}}^{n} d\xi_j \leq S^{n'}(C) \int |f(\xi_1,\ldots,\xi_n)| \prod_{\substack{j=1\\ i\neq j}}^{n} d\xi_j.$$

such that

$$||f||_{1,\infty} \leq S^{n'}(C)||f||_{1,\infty}.$$

□

Lemma B.3: Let C and C' be two antisymmetric bilinear forms on the Grassman algebra A. Then

$$S(C + C') \leq S(C) + S(C').$$

<u>Proof:</u> For $\xi_1,\ldots,\xi_m \in \mathcal{B}$,

$$\int \psi(\xi_1)\cdots\psi(\xi_m)\, d\mu_{C+C'}(\psi) = \int (\psi(\xi_1)+\psi'(\xi_1))\cdots(\psi(\xi_m)+\psi'(\xi_m))\, d\mu_C(\psi) d\mu_{C'}(\psi').$$

Since

$$(\psi(\xi_1)+\psi'(\xi_1))\cdots(\psi(\xi_m)+\psi'(\xi_m)) = \sum_{\substack{I\cup J=\{1,\ldots,m\}\\ I\cap J=\emptyset}} (\pm 1) \prod_{i\in I}\psi(\xi_i) \prod_{j\in J}\psi'(\xi_j),$$

we get

$$\begin{aligned}
\left| \int \psi(\xi_1)\cdots\psi(\xi_m)\, d\mu_{C+C'}(\psi) \right| &\leq \sum_{\substack{I\cup J=\{1,\ldots,m\}\\ I\cap J=\emptyset}} \left| \int \prod_{i\in I}\psi(\xi_i)\, d\mu_C(\psi) \right| \cdot \left| \int \prod_{j\in J}\psi'(\xi_j)\, d\mu_C(\psi') \right| \\
&\leq \sum_{\substack{I\cup J=\{1,\ldots,m\}\\ I\cap J=\emptyset}} S^{|I|}(C) S^{|J|}(C') \\
&= (S(C)+S(C'))^m.
\end{aligned}$$

□

B.2 Gram Bound

Theorem B.4: Let $\mathcal{H}$ be an Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, and let $f_1,\ldots,f_n \in \mathcal{H}$ and $g_1,\ldots,g_n \in \mathcal{H}$. Then

$$|\det\left(\langle f_i, g_j\rangle_{\mathcal{H}}\right)_{i,j=1}^n| \leq \prod_{i=1}^n ||f_i||_{\mathcal{H}} \prod_{j=1}^n ||g_j||_{\mathcal{H}}.$$

Proof: Without restriction of the generality, we can assume that the vectors f_i are linear independent. (In the other case, the determinant would vanish.) In the same way, the vectors g_j are also assumed to be linear independent. Further, let P be the orthogonal projection on the span of the vectors f_i. Then

$$\left|\det\left(\langle f_i, g_j\rangle_{\mathcal{H}}\right)_{i,j=1}^n\right| = \left|\det\left(\langle f_i, Pg_j\rangle_{\mathcal{H}}\right)_{i,j=1}^n\right|,$$

and $||Pg_j||_{\mathcal{H}} \leq ||g_j||_{\mathcal{H}}$. Thus, we assume that the vectors g_j are in the span of the f_i's.

Let $\{\tilde{f}_1,\ldots,\tilde{f}_n\}$ be the set of orthogonal vectors, obtained applying the Gram-Schmidt orthogonalization procedure on the f_i's. Define the vectors $\tilde{g}_j$ in the same way. In particular,

$$\left|\det\left(\langle f_i, g_j\rangle_{\mathcal{H}}\right)_{i,j=1}^n\right| = \left|\det\left(\langle \tilde{f}_i, \tilde{g}_j\rangle_{\mathcal{H}}\right)_{i,j=1}^n\right|.$$

Further, since the span of the $\tilde{f}_i$'s and of the $\tilde{g}_j$'s are the same, the matrix $\left(\langle \tilde{f}_i, \tilde{g}_j\rangle_{\mathcal{H}}\right)_{i,j=1}^n$ is, up to reordering of the columns, diagonal. Thus,

$$\left|\det\left(\langle f_i, g_j\rangle_{\mathcal{H}}\right)_{i,j=1}^n\right| = \prod_{i=1}^n ||\tilde{f}_i||_{\mathcal{H}} \prod_{j=1}^n ||\tilde{g}_j||_{\mathcal{H}}.$$

Finally, by construction, $||\tilde{f}_i||_{\mathcal{H}} \leq ||f_i||_{\mathcal{H}}$ and $||\tilde{g}_j||_{\mathcal{H}} \leq ||g_j||_{\mathcal{H}}$, and the claim follows. □

Corollary B.5: Suppose that the covariance of definition B.1 can be written in the form

$$C_{\sigma\sigma'}(x,x') = \langle f_{x,\sigma}, g_{x',\sigma'}\rangle_{\mathcal{H}},$$

where $f_{x,\sigma}$ and $g_{x,\sigma}$ are vectors in an Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Suppose further that for all $x \in \mathbb{R}\times\mathbf{\Lambda}_L$ and $\sigma \in \{\uparrow,\downarrow\}$, $||f_{x,\sigma}||_{\mathcal{H}} \leq K_1^2$ and $||g_{x,\sigma}||_{\mathcal{H}} \leq K_2^2$. Then

$$S(C) = \sup_{m\in\mathbb{N}} \sup_{\xi_1,\ldots,\xi_{2m}\in\mathcal{B}} \left(\int \psi(\xi_1)\cdots\psi(\xi_{2m})\, d\mu_C(\psi)\right)^{1/2m} \leq K_1K_2.$$

B.3 Symmetries

Definition B.6: For a function $S : \mathcal{B} \times \mathcal{B} \to \mathbb{C}$, consider the following linear transformation of the fields

$$\psi^S(\xi) := \int d\zeta\, S(\xi,\zeta)\psi(\zeta).$$

(i) The transformation S is invertible if there is a function $S^{-1} : \mathcal{B} \times \mathcal{B} \to \mathbb{C}$ such that

$$\int d\zeta\, S(\xi,\zeta)S^{-1}(\zeta,\eta) = \int d\zeta\, S^{-1}(\xi,\zeta)S(\zeta,\eta) = \delta(\xi,\eta),$$

where $\delta(\xi,\eta) = \delta_{ab}\delta_{\sigma\tau}\delta_{\mathbf{xy}}\delta(x^0 - y^0)$.

(ii) A Grassman function

$$f(\psi) = \sum_{n\geq 0} \int d\xi_1 \cdots d\xi_n\, f_n(\xi_1,\ldots,\xi_n)\, \psi(\xi_1)\cdots\psi(\xi_n)$$

is invariant under the transformation S if $f(\psi^S) = f(\psi)$, or equivalently

$$f_n^S(\xi_1,\ldots,\xi_n) = f_n(\xi_1,\ldots,\xi_n), \text{ for all } n \geq 0,$$

where[1]

$$f_n^S(\xi_1,\ldots,\xi_n) = \int d\xi_1' \cdots d\xi_n'\, S(\xi_1',\xi_1)\cdots S(\xi_n',\xi_n)\, f_n(\xi_1',\ldots,\xi_n').$$

Remark B.7: The generating functional for the Green functions contain the Grassman function (ϕ,ψ). All the physical symmetries should therefor leave this function invariant. Hence, we consider only symmetries satisfying

$$\mathcal{J}^S = \mathcal{J},$$

or

$$\int d\xi' d\eta'\, S(\xi',\xi)\, S(\eta',\eta)\, \mathcal{J}(\xi',\eta') = \mathcal{J}(\xi,\eta).$$

Suppose that the symmetry does not mix the particle and the hole states,

$$S(\xi,\eta) = s_{\sigma\tau}(x,y;a)\delta_{ab},$$

where s is a transformation of the spacial and spin coordinates only, that depends on the particle species. For such a transformation, the condition $\mathcal{J}^S = \mathcal{J}$ reads

$$\sum_{\delta\in\{\uparrow,\downarrow\}} \int dz^0 \sum_{\mathbf{z}\in\mathbf{\Lambda}_L} \bar{s}_{\delta\sigma}(z,x)s_{\delta\tau}(z,y) = \delta_{\sigma\tau}\delta(x^0 - y^0)\delta_{\mathbf{xy}},$$

[1]Observe that the coefficients f_n transform as contravariant tensors.

with $\bar{s}_{\delta\sigma}(z,x) = s_{\delta\sigma}(z,x,0)$ and $s_{\delta\sigma}(z,x) = s_{\delta\sigma}(z,x,1)$. Thus

$$\bar{s}_{\delta\sigma}(z,x) = (s^{-1})_{\sigma\delta}(x,z)$$

which corresponds to a "unitarity" condition.

Example B.8: Let $R \in SO(3)$ be a rotation. Then

$$s_{\sigma\tau}(x^0,\mathbf{x};y^0,\mathbf{y}) := \delta(x^0-y^0)\delta_{R^T\mathbf{x},\mathbf{y}}U_{\sigma\tau}(R),$$

and

$$\bar{s}_{\sigma\tau}(x^0,\mathbf{x};y^0,\mathbf{y}) := \delta(x^0-y^0)\delta_{R^T\mathbf{x},\mathbf{y}}U^*_{\sigma\tau}(R),$$

where $U(R)$ is the two dimensional spinor representation of the rotation R, and $U^*_{\sigma\tau}$ is the complex conjugate of $U_{\sigma\tau}$. Thus,

$$\psi^R_\sigma(x^0,\mathbf{x}) = \sum_{\tau\in\{\uparrow,\downarrow\}} U_{\sigma\tau}(R)\psi_\tau(x^0,R^T\mathbf{x}),$$

and

$$\bar{\psi}^R_\sigma(x^0,\mathbf{x}) = \sum_{\tau\in\{\uparrow,\downarrow\}} U^*_{\sigma\tau}(R)\bar{\psi}_\tau(x^0,R^T\mathbf{x}).$$

Remark B.9: Suppose that the covariance $C(\xi,\eta)$ defined in B.1 is invariant under the transformation $S(\xi,\eta) = s_{\sigma\tau}(x,y;a)\delta_{ab}$. Then

$$\int d\xi' d\eta'\, S(\xi,\xi')\, S(\eta,\eta')\, C(\xi',\eta') = C(\xi,\eta).$$

<u>Proof:</u> First observe that[2]

$$\int d\xi' d\eta'\, S(\xi',\xi)\, S(\eta',\eta)\, C(\xi',\eta') =$$
$$= \sum_{\sigma',\tau'} \int dx' \int dy' \begin{pmatrix} 0 & -s_{\sigma'\sigma}(x',x)\bar{s}_{\tau'\tau}(y',y)C_{\tau'\sigma'}(y',x') \\ \bar{s}_{\sigma'\sigma}(x',x)s_{\tau'\tau}(y',y)C_{\sigma'\tau'}(x',y') & 0 \end{pmatrix}.$$

Since C is invariant under the transformation S,

$$\sum_{\sigma',\tau'} \int dx' \int dy' s_{\sigma'\sigma}(x',x)\bar{s}_{\tau'\tau}(y',y)C_{\tau'\sigma'}(y',x') = C_{\tau\sigma}(y,x).$$

[2]In order to simplify the notation, we write $\int dx$ instead of $\sum_{\mathbf{x}\in\mathbf{\Lambda}_L} \int dx^0$.

Further,

$$\int d\xi' d\eta'\, S(\xi,\xi')\, S(\eta,\eta')\, C(\xi',\eta') =$$
$$= \sum_{\sigma',\tau'} \int dx' \int dy' \begin{pmatrix} 0 & -s_{\sigma\sigma'}(x,x')\bar{s}_{\tau\tau'}(y,y')C_{\tau'\sigma'}(y',x') \\ \bar{s}_{\sigma\sigma'}(x,x')s_{\tau\tau'}(y,y')C_{\sigma'\tau'}(x',y') & 0 \end{pmatrix}.$$

Using the "unitarity" condition, one gets

$$\sum_{\sigma',\tau'} \int dx' \int dy'\, s_{\sigma\sigma'}(x,x')\bar{s}_{\tau\tau'}(y,y')C_{\tau'\sigma'}(y',x') =$$
$$= \sum_{\sigma',\tau'} \int dx' \int dy'\, (\bar{s}^{-1})_{\sigma'\sigma}(x',x)(s^{-1})_{\tau'\tau}(y',y)C_{\tau'\sigma'}(y',x').$$

The invariance of C under the transformation S, and consequently under its inverse S^{-1} leads us to

$$\sum_{\sigma',\tau'} \int dx' \int dy'\, s_{\sigma\sigma'}(x,x')\bar{s}_{\tau\tau'}(y,y')C_{\tau'\sigma'}(y',x') = C_{\tau\sigma}(y,x).$$

□

Lemma B.10: Suppose that the covariance is invariant under the transformation S. Then

$$\int \psi^S(\xi_1)\cdots\psi^S(\xi_n)\, d\mu_C(\psi) = \int \psi(\xi_1)\cdots\psi(\xi_n)\, d\mu_C(\psi).$$

<u>Proof:</u> By definitions of the transformation S and of the Grassman Gaussian integral, for n even,

$$\int \psi^S(\xi_1)\cdots\psi^S(\xi_n) d\mu_C(\psi) =$$
$$= \sum_{\pi\in S_{n/2}} (-1)^\pi \int d\xi'_1\cdots d\xi'_n\, S(\xi_1,\xi'_1)\cdots S(\xi_n,\xi'_n) \prod_{i=1}^{n/2} C(\xi'_i,\xi'_{\pi(i)}).$$

If the covariance C is invariant under the transformation S, the claim follows directly form remark B.8. For odd n, the Grassman integral vanishes, and the claim is verified. □

Lemma B.11: Suppose that the covariance is invariant under the transformation S. Then, if $f \in \mathcal{F}_m(n)$ is a Grassman function which is invariant under the same transformation S,

$$f'^S = f',$$

where

$$f'(\eta_1,\ldots,\eta_m;\xi_{n'+1},\ldots,\xi_n) = \int d\xi_1\cdots d\xi_{n'}\, f(\eta_1,\ldots,\eta_m;\xi_1,\ldots,\xi_n)\int \psi(\xi_1)\cdots\psi(\xi_{n'})d\mu_C(\psi).$$

Proof:

$$\begin{aligned}
&f'^S(\eta_1,\ldots,\eta_m;\xi_{n'+1},\ldots,\xi_n)\\
&= \int d\eta_1'\cdots d\eta_m'\, S(\eta_1',\eta_1)\cdots S(\eta_m',\eta_m)\int d\xi_{n'+1}'\cdots d\xi_n'\, S(\xi_{n'+1}',\xi_{n'+1})\cdots\\
&\qquad\qquad \cdots S(\xi_n',\xi_n) f'(\eta_1',\ldots,\eta_m';\xi_{n'+1}',\ldots,\xi_n')\\
&= \int d\eta_1'\cdots d\eta_m'\, S(\eta_1',\eta_1)\cdots S(\eta_m',\eta_m)\int d\xi_{n'+1}'\cdots d\xi_n'\, S(\xi_{n'+1}',\xi_{n'+1})\cdots S(\xi_n',\xi_n)\\
&\qquad\cdot\int d\xi_1'\cdots d\xi_{n'}'\, f(\eta_1',\ldots,\eta_m';\xi_1',\ldots,\xi_n')\int \psi(\xi_1')\cdots\psi(\xi_{n'}')\, d\mu_C(\psi)\\
&= \int d\xi_1\cdots d\xi_{n'}\int d\eta_1'\cdots d\eta_m'\, S(\eta_1',\eta_1)\cdots S(\eta_m',\eta_m)\int d\xi_1'\cdots d\xi_n'\, S(\xi_1',\xi_1)\cdots\\
&\qquad \cdots S(\xi_n',\xi_n) f(\eta_1',\ldots,\eta_m';\xi_1',\ldots,\xi_n')\int d\xi_1''\cdots d\xi_{n'}''\, S^{-1}(\xi_1,\xi_1'')\cdots S^{-1}(\xi_{n'},\xi_{n'}'')\\
&\qquad\qquad\cdot\int \psi(\xi_1'')\cdots\psi(\xi_{n'}'')\, d\mu_C(\psi)\\
&= \int d\xi_1\cdots d\xi_{n'}\, f^S(\eta_1,\ldots,\eta_m;\xi_1,\ldots,\xi_n)\int \psi^{S^{-1}}(\xi_1)\cdots\psi^{S^{-1}}(\xi_{n'})\, d\mu_C(\psi)
\end{aligned}$$

By the previous lemma and the invariance of f, the claim follows. □

Corollary B.12: The generating functional for the Green functions is invariant under the symmetries of the Hamiltonian. Thus, the Green functions have the same symmetries as the Hamiltonian.

B.3.1 Spin Symmetry

Lemma B.13: Suppose that the covariance is diagonal in the spin index, $C_{\sigma\tau}(x,y) = \delta_{\sigma\tau}C^{(\sigma)}(x,y)$, and that the interaction is given by the potential

$$V(\xi_1,\xi_2,\xi_3,\xi_4) = \delta_{a_1a_2}\delta_{a_3a_4}\delta_{a_11}\delta_{a_30}\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4}\delta(x_1-x_3)\delta(x_2-x_4)V_{\sigma_1\sigma_2}(x_1-x_2).$$

Then the two-point Green function is diagonal in the spin index.

Proof: Consider the following transformation, involving only the spin structure of the fields:

$$S(x,\sigma,a;y,\tau,b) = \delta_{ab}\delta(x-y)s^{(a)}_{\sigma\tau},$$

where $s^{(a)} \in SU(2)$. Then

$$C^S(x,\sigma,1;y,\tau,0) = \sum_{\sigma'\in\{\uparrow,\downarrow\}} s^*_{\sigma\sigma'} s_{\tau\sigma'} C^{(\sigma')}(x,y)$$

and

$$V^S(\xi_1,\xi_2,\xi_3,\xi_4) = \delta_{a_10}\delta_{a_20}\delta_{a_31}\delta_{a_41}\delta(x_1-x_3)\delta(x_2-x_4) \sum_{\sigma_1',\sigma_2'\in\{\uparrow,\downarrow\}} V_{\sigma_1'\sigma_2'}(x_1-x_2)|s_{\sigma_1\sigma_1'}|^2|s_{\sigma_2\sigma_2'}|^2.$$

Let s be the spinor representation of a rotation along the vertical axis,

$$s_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$

then $C^{S_\theta} = C$ and $V^{S_\theta} = V$, and all Green functions are invariant under the spin transformations S_θ defined above. In particular,

$$G_2^{S_\theta}(x,\sigma;y,\tau) = G_2(x,\sigma;y,\tau).$$

Hence for all $\theta \in [0,2\pi)$

$$G_2(x,\uparrow;y,\downarrow) = G_2^{S_\theta}(x,\uparrow;y,\downarrow) = e^{2i\theta}G_2(x,\uparrow;y,\downarrow),$$

and we deduce

$$G_2(x,\uparrow;y,\downarrow) = 0.$$

□

Appendix C

Spin-Dependent Hamiltonian

In this chapter, we consider a system of spin $\frac{1}{2}$ fermions in a weak, constant magnetic field, parallel to the vertical axis. Fermions with different spin orientations have a different energy, such that the Fermi surface is actually split in two folds, corresponding to the two different spin orientations. With the argument that proved Luttinger's theorem, we derive the results obtained by Luttinger in [1].

C.1 Free Electrons in a Weak Magnetic Field

Let $T(\mathbf{x}-\mathbf{y})$ be the hopping amplitude between sites of the lattice $\mathbf{\Lambda}$ in a weak magnetic field h, parallel to the vertical axis.[1] The (non-interacting) Hamiltonian of the system is then given by

$$H_0 = \sum_{\sigma\in\{\uparrow,\downarrow\}} \sum_{\mathbf{x},\mathbf{y}\in\mathbf{\Lambda}} T(\mathbf{x}-\mathbf{y}) c_\sigma^+(\mathbf{x}) c_\sigma(\mathbf{x}) + \mu_B h \sum_{\mathbf{x}\in\mathbf{\Lambda}} \left(c_\downarrow^+(\mathbf{x}) c_\downarrow(\mathbf{x}) - c_\uparrow^+(\mathbf{x}) c_\uparrow(\mathbf{x}) \right).$$

In momentum space,

$$H_0 = \sum_{\sigma\in\{\uparrow,\downarrow\}} \int \frac{d\mathbf{k}^d}{(2\pi)^d} \varepsilon_\sigma(\mathbf{k}) c_\sigma^+(\mathbf{k}) c_\sigma(\mathbf{k}),$$

where

$$\varepsilon_\sigma(\mathbf{k}) = \begin{cases} \varepsilon(\mathbf{k}) - \mu_B h, & \sigma = \uparrow \\ \varepsilon(\mathbf{k}) + \mu_B h, & \sigma = \downarrow \end{cases}$$

and

$$T(\mathbf{x}) = \int \frac{d\mathbf{k}^d}{(2\pi)^d} \varepsilon(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}}.$$

[1] In the tight-binding approximation, the electrons are trapped in bounded states by the positive ions. The hopping amplitude determines the probability for an electron to jump from one ions to an other. A weak magnetic field modifies slightly the bounded states, and the hopping amplitude is modified as well.

Let consider a many-particle state containing n electrons, obtained filling all the energy levels of the spin up states up to the energy $E_\uparrow$, and all the levels with spin down, up to the energy $E_\downarrow$. The total energy of such a many-particle state is given by

$$\begin{aligned} E_{tot} &= \int_{\varepsilon(\mathbf{k})<E_\uparrow+\mu_B H} \frac{d\mathbf{k}^d}{(2\pi)^d}(\varepsilon(\mathbf{k})-\mu_B h) + \int_{\varepsilon(\mathbf{k})<E_\downarrow-\mu_B h} \frac{d\mathbf{k}^d}{(2\pi)^d}(\varepsilon(\mathbf{k})+\mu_B h) \\ &= \int_0^{E_\uparrow+\mu_B h} d\epsilon\, \rho(\epsilon)(\epsilon-\mu_B h) + \int_0^{E_\downarrow-\mu_B h} d\epsilon\, \rho(\epsilon)(\epsilon+\mu_B h), \end{aligned}$$

where $\rho(\epsilon)$ is the spectral density of states. Minimizing the total energy with respect to $E_\uparrow$ and $E_\downarrow$, fixing the total density of particles

$$n = \int_0^{E_\downarrow-\mu_B h} d\epsilon \rho(\epsilon) + \int_0^{E_\uparrow+\mu_B h} d\epsilon \rho(\epsilon),$$

one gets $E_\uparrow = E_\downarrow = E_F$.[2] Thus, the ground state of a system of n independent electrons in a weak magnetic field is obtained filling all the one-particle states with energy less than the Fermi energy E_F, which is defined by the condition

$$n = \int_0^{E_F-\mu_B h} d\epsilon\, \rho(\epsilon) + \int_0^{E_F+\mu_B h} d\epsilon\, \rho(\epsilon).$$

The ground state energy is then given by

$$E_0 = \int_0^{E_F+\mu_B h} d\epsilon\, \rho(\epsilon)(\epsilon-\mu_B h) + \int_0^{E_F-\mu_B h} d\epsilon\, (\epsilon+\mu_B h),$$

and the Pauli magnetization is

$$m = \mu_B \int_{E_F-\mu_B h}^{E_F+\mu_B h} d\epsilon\, \rho(\epsilon).$$

The surfaces of Fermi are defined by

$$S_\uparrow = \{\mathbf{k}\in\mathbb{T} : \varepsilon(\mathbf{k}) = E_F+\mu_B h\} \text{ and } S_\downarrow = \{\mathbf{k}\in\mathbb{T} : \varepsilon(\mathbf{k}) = E_F-\mu_B h\},$$

such that

$$m = \mu_B(2\pi)^{-d}(\mathrm{Vol}(S_\uparrow) - \mathrm{Vol}(S_\downarrow)),$$

while

$$n = (2\pi)^{-d}(\mathrm{Vol}(S_\uparrow) + \mathrm{Vol}(S_\downarrow)).$$

[2] In order to avoid the confusion with μ_B, we use E_F for the Fermi energy, instead of the usual notation μ for the chemical potential.

C.2 Interacting Electrons in a Weak Magnetic Field

We turn now to the study of a system of interacting fermions in a weak magnetic field h, parallel to the vertical axis. In order to avoid divergence problems, we work on the finite lattice $\mathbf{\Lambda}_L$. We refer to chapter two for the basic definitions. Let $e(\mathbf{k})$, defined as in 2.3, be a band function describing electrons on a lattice with a weak magnetic field parallel to the vertical axis, and satisfying the assumptions **A2** and **A3**. Define

$$e_\sigma(\mathbf{k}) = e(\mathbf{k}) - (-1)^\sigma \mu_B h,$$

where $(-1)^\uparrow = -(-1)^\downarrow = 1$. If the magnetic field h is small enough, the functions $e_\sigma(\mathbf{k})$ still satisfies the assumptions **A2** and **A3**. Let the Hamiltonian of the system on the finite lattice be[3]

$$\overline{H}_u^{(L)} = L^{-d} \sum_{\substack{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp \\ \sigma\in\{\uparrow,\downarrow\}}} e_\sigma^{(L)}(\mathbf{k}) + \lambda V^{(L)} + K^{(L)}(\lambda),$$

where

$$e_\sigma^{(L)}(\mathbf{k}) = \begin{cases} e_\sigma(\mathbf{k}), & \text{if } |e_\sigma(\mathbf{k})| \geq \frac{A}{L} \\ \operatorname{sgn}(e_\sigma(\mathbf{k}))\frac{A}{L}, & \text{if } |e_\sigma(\mathbf{k})| < \frac{A}{L} \end{cases}$$

is defined as in 2.10. The interaction is given by

$$V^{(L)} = \frac{1}{2L^{3d}} \sum_{\substack{\mathbf{k}_1,\cdots,\mathbf{k}_4\in\mathbf{\Lambda}_L^\sharp \\ \sigma,\tau\in\{\uparrow,\downarrow\}}} \delta_{\mathbf{k}_1+\mathbf{k}_3,\mathbf{k}_2+\mathbf{k}_4} v_{\sigma\tau}(\mathbf{k}_1-\mathbf{k}_2)\cdot \\ \cdot c_\sigma^+(\mathbf{k}_1)c_\tau^+(\mathbf{k}_3)c_\sigma(\mathbf{k}_2)c_\tau(\mathbf{k}_4),$$

and the counterterm is now spin dependent:

$$K^{(L)}(\lambda) = L^{-d} \sum_{\substack{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp \\ \sigma\in\{\uparrow,\downarrow\}}} u(\mathbf{k},\sigma,\lambda) c_\sigma^+(\mathbf{k}) c_\sigma(\mathbf{k}),$$

with the function u in the set of the possible counterterms

$$\mathcal{K} = \Big\{ u : \mathbf{\Lambda}_L^\sharp \times \mathbb{C} \times \{\uparrow,\downarrow\} \to \mathbb{C} \,|\, u \text{ is analytic in } \lambda\in\mathbb{C} \text{ with } u(\mathbf{k},\sigma,0) = 0, \\ \text{and } \sup_\lambda |u|_0 \leq \frac{A}{2L} \Big\}.$$

Remark C.1: The magnetization operator

$$M^{(L)} = L^{-d} \sum_{\mathbf{k}\in\mathbf{\Lambda}_L^\sharp} \big(c_\uparrow^+(\mathbf{k})c_\uparrow(\mathbf{k}) - c_\downarrow^+(\mathbf{k})c_\downarrow(\mathbf{k}) \big)$$

[3]We assume that L is such that $\frac{A}{L}$ is much smaller than $\mu_B h$.

commutes with the Hamiltonian $\overline{H}^{(L)}$.

Proof: We proved in 2.12 that[4]

$$[N_\tau^{(L)}, V] = 0,$$

and thus $[M^{(L)}, V] = 0$. One easily verify that

$$[N_\tau^{(L)}, c_\sigma^+(\mathbf{k})c_\sigma(\mathbf{k})] = 0,$$

and the magnetization operator commutes with the Hamiltonian. □

Remark C.2: The spectrum of the magnetization operator $M^{(L)}$ is a subset of $\mathbb{Z}$.

We can repeat the construction of the Green functions presented in the second chapter, using now the spin dependent propagator

$$C_\sigma^{(L)}(k) = \frac{1}{ik_0 - e_\sigma^{(L)}(\mathbf{k})}.$$

The theorem 2.18 of chapter two can easily be extend to the spin dependent case, in order to obtain

Theorem C.3: For all $L \in \mathbb{N}$, there is a $\lambda_0^{(L)} > 0$ that depends on L such that

(i) For all $u \in \mathcal{K}$, the connected Green functions $\hat{G}_{2m}^{(L)}(k_1, \sigma_1 \ldots, k_{2m}, \sigma_{2m}, \lambda; u)$ are analytic in λ with analyticity radius $\lambda_0^{(L)}$. In particular, the two-point Green function $S_\sigma^{(L)}(k; \lambda)$ is analytic in λ.

(ii) For all $u \in \mathcal{K}$, the density of fermions $\rho^{(L)}(\lambda, u)$ is analytic in λ with analyticity radius $\lambda_0^{(L)}$.

(iii) For all $u \in \mathcal{K}$, the magnetization density of fermions, defined by

$$\begin{aligned} m^{(L)}(\lambda) &= L^{-d} \sum_{\mathbf{k} \in \mathbf{\Lambda}_L^\sharp} \langle c_\uparrow^+(\mathbf{k})c_\uparrow(\mathbf{k}) - c_\downarrow^+(\mathbf{k})c_\downarrow(\mathbf{k})\rangle_L \\ &= \lim_{x^0 \to 0_+} L^{-d} \sum_{\mathbf{k} \in \mathbf{\Lambda}_L^\sharp} \int \frac{dk^0}{2\pi} \left(S_\uparrow^{(L)}(k) - S_\downarrow^{(L)}(k) \right) e^{ik_0 x^0} \end{aligned}$$

is analytic in λ with analyticity radius $\lambda_0^{(L)}$.

[4]Observe that the determining condition was that the interaction was of the form $V = \sum_{\sigma,\tau \in \{\uparrow,\downarrow\}} \sum_{\mathbf{x},\mathbf{y} \in \mathbf{\Lambda}} V_{\sigma\tau}(\mathbf{x} - \mathbf{y}) \bar{\psi}_\sigma(\mathbf{x}) \bar{\psi}_\tau(\mathbf{y}) \psi_\tau(\mathbf{y}) \psi_\sigma(\mathbf{x})$.

Further, one easily verify that with the same argument as in 2.20,

Theorem C.4: For each $L < \infty$, and $u \in \mathcal{K}$, the density and the spin magnetization are independent of the coupling constant:

$$\rho^{(L)}(\lambda) = \rho^{(L)}(0) \text{ and } m^{(L)}(\lambda) = m^{(L)}(0).$$

Finally, one can extend the proof of the convergence of the density in the thermodynamic limit to the magnetization density:

Theorem C.5: Assume **A1-3**. Then there is a sequence of counterterms $\left(K^{(L)}(\mathbf{k}, \sigma, \lambda)\right)_{L\in\mathbb{N}}$ in $\mathcal{K}$ that converges uniformly in $(\mathbf{k}, \sigma) \in \mathbb{T} \times \{\uparrow, \downarrow\}$ to the formal power series $K(\mathbf{k}, \sigma, \lambda)$ of theorem 2.22, such that

(i) For $L \to \infty$, the two-point Green function $\hat{G}_\sigma^{(L)}$ converges uniformly in $k = (k_0, \mathbf{k}) \in \mathbb{R} \times \mathbb{T}$ to the (formal) Green function $\hat{G}_\sigma$ of the model in the infinite volume with dispersion relation $e_\sigma(\mathbf{k})$ and counterterm $K(\mathbf{k}, \sigma, \lambda)$.

(ii) For each $m \geq 1$, the $2m-$point Green function $\hat{G}_{2m}^{(L)}$ converges in the limit $L \to \infty$ to the formal Green function of the model in the infinite volume in the L_1-norm.

(iii) The density of fermions $\rho^{(L)}(\lambda)$ converges in the sense of formal power series to the density of fermions in the infinite volume:

$$\rho^{(L)}(\lambda) \overset{L\to\infty}{\to} \rho(\lambda)$$

(iv) The magnetization density $m^{(L)}(\lambda)$ converges in the sense of formal power series to the magnetization density in the infinite volume:

$$m^{(L)}(\lambda) \overset{L\to\infty}{\to} m(\lambda)$$

where $m(\lambda)$ is defined by the formal power series

$$m(\lambda) = \lim_{x^0\to 0_+} \int \frac{dk^{d+1}}{(2\pi)^{d+1}} \left(S_\uparrow(k) - S_\downarrow(k)\right) e^{ik_0x^0}.$$

We deduce as in the case of the density,

Corollary C.6: The spin magnetization $m(\lambda)$ is independent of the coupling constant.

C.3 Luttinger's Theorem

Definition C.7: The physical (or interacting) Fermi surfaces S_σ of the system are the surfaces of discontinuity of the occupation numbers $n_\sigma(\mathbf{p}, \lambda)$.

Luttinger's Theorem: Let $\varepsilon_\sigma(\mathbf{k}) - \mu$ be a dispersion relation of a system of fermions in a constant weak magnetic field h parallel to the vertical axis, satisfying the assumptions **A2** and **A3**. Assume further that the interaction $v_{\sigma\tau}$ between the fermions satisfies **A1**, and let $\bar{\rho}$ be the (given) density of fermions of the system.

Then the sum of the volumes enclosed by the Fermi surfaces is independent of the interaction strength.

<u>Proof:</u> In the free-fermion approximation, the Fermi surfaces are

$$S_\sigma^{(0)} = \{\mathbf{k} \in \mathbb{T} \,:\, \varepsilon_\sigma(\mathbf{k}) = E_F\}.$$

Since by assumption $\varepsilon_\sigma(\mathbf{k}) - E_F$ has convex level sets, there is a chemical potential E_0 such that

$$\bar{\rho} = \mathrm{Vol}(S_\uparrow^{(0)}) + \mathrm{Vol}(S_\downarrow^{(0)}).$$

Once the chemical potential is fixed, the spin magnetization is determined by

$$m^{(0)} = \mathrm{Vol}(S_\uparrow^{(0)}) - \mathrm{Vol}(S_\downarrow^{(0)}).$$

We turn now to the interacting system. For each E in a neighborhood of E_0, there is an interacting dispersion relation

$$e_{int}(\mathbf{k}, \sigma, E; \lambda) = R_\lambda^{-1}(\varepsilon(\cdot) - E)(\mathbf{k})$$

defined as a formal power series $e_{int}(\mathbf{k}, \sigma, E; \lambda) = \sum_{r \geq 0} \lambda^r e_r(\mathbf{k}, \sigma, E)$. For each $R > 0$, there is a $\lambda_0 > 0$ such that for λ with $|\lambda| \leq \lambda_0$, the dispersion relation

$$e^R(\mathbf{k}, \sigma, E; \lambda) = \sum_{r=0}^{R} \lambda^r e_r(\mathbf{k}, \sigma, E)$$

satisfies the assumptions **A2** and **A3**. For λ' with $|\lambda'| \leq |\lambda|$, consider the model defined by the generating functional

$$\mathcal{G}(\bar{\phi}, \phi; \lambda', e^R(\cdot; \lambda)) = \int e^{\lambda' \mathcal{V}(\bar{\psi}, \psi) + \mathcal{E}(\bar{\psi}, \psi; \lambda', e^R(\cdot; \lambda)) + (\bar{\phi}, \bar{\psi}) + (\phi, \psi)} d\mu_{C_{e^R(\cdot;\lambda)}}(\bar{\psi}, \psi),$$

which is well-defined by theorem 1.1. By corollary 1.2, we know that the occupation number corresponding to this model has a jump on the surfaces

$$S_\sigma^{(R)} = \{\mathbf{k} \in \mathbb{T} \,:\, e^R(\mathbf{k}, \sigma, E; \lambda) = 0\},$$

which are therefore the interacting Fermi surfaces up to the order R in perturbation theory. The theorem C.4 implies that the density of the model is

$$\rho(\lambda'; e^R(\cdot;\lambda)) = \rho(0; e^R(\cdot;\lambda)) = \mathrm{Vol}(S_\uparrow^{(R)}) + \mathrm{Vol}(S_\downarrow^{(R)}).$$

Further, we obtain for the magnetization

$$m(\lambda'; e^R(\cdot;\lambda)) = m(0; e^R(\cdot;\lambda)) = \mathrm{Vol}(S_\uparrow^{(R)}) - \mathrm{Vol}(S_\downarrow^{(R)}).$$

In order to achieve the right physical model, the chemical potential E has to be adjust such that the density of the system is just $\bar{\rho}$. There is a E_F near E_0 (depending on R and λ) such that

$$\bar{\rho} = \rho(0; e^R(\cdot;\lambda)) = \mathrm{Vol}(S_\uparrow^{(R)}) + \mathrm{Vol}(S_\downarrow^{(R)}).$$

Hence,

$$\mathrm{Vol}(S_\uparrow^{(R)}) + \mathrm{Vol}(S_\downarrow^{(R)}) = \mathrm{Vol}(S_\uparrow^{(0)}) + \mathrm{Vol}(S_\downarrow^{(0)}),$$

and to each order R in perturbation theory, the volume enclosed in the Fermi surface is independent of the interaction strength. □

Remark C.8: Observe that the Pauli magnetization of the interacting system is still given by the difference between the volumes enclosed in the two Fermi surfaces:

$$m(0; e^R(\cdot;\lambda)) = \mathrm{Vol}(S_\uparrow^{(R)}) - \mathrm{Vol}(S_\downarrow^{(R)}).$$

Nothing allows to say that the difference between this two surfaces should be constant, and thus, in general, the volume of each Fermi surface is not conserved separately.

C.4 Example: Spherical Fermi Surfaces

Strictly speaking, the case of a spherical Fermi surface is incompatible with the lattice structure considered in our work[5]. We although present this example, in order to illustrate the construction presented above.

Let consider a system of fermions, described by the band function $\varepsilon_\sigma(\mathbf{k}) = \frac{\mathbf{k}^2}{2m_e} - (-1)^\sigma \mu_B h$, where m_e and μ_B are the mass and the magnetic moment of the fermions. Obviously, the assumptions **A2** and **A3** are satisfied for small h-fields. Let $\bar{\rho}$ be the density of Fermion in the system.

[5]The band function corresponding to the discrete Laplacian on the lattice would be $\sum_i(\cos k_i - 1)$ rather than $|\mathbf{k}|^2$.

C.4.1 The Non-Interacting Case

In the free-fermion approximation, the ground state is obtained filling all the one-particle states with energy less than the Fermi energy E_0. We have

$$\begin{aligned}\mathrm{Vol}(S_\sigma^{(0)}) &= \int d\mathbf{k}^d\,\Theta(E_0-\varepsilon_\sigma(\mathbf{k}))\\ &= \frac{|S_{d-1}|}{d}(2m)^{d/2}\left(E_0+(-1)^\sigma\mu_B h\right)^{d/2}.\end{aligned}$$

Thus, in the free-fermion approximation, the Fermi energy E_0 is determined by

$$\bar\rho=\frac{|S_{d-1}|}{d}(2m)^{d/2}\left((E_0+\mu_B h)^{d/2}+(E_0-\mu_B h)^{d/2}\right)$$

In particular, in two dimension, $d=2$,

$$\bar\rho=4\pi m E_0,$$

and $E_0=\bar\rho/(4\pi m)$. Observe that the density is independent of the magnetic field. In three dimension, $d=3$,

$$\begin{aligned}\bar\rho &= \frac{4\pi}{3}(2mE_0)^{3/2}\left((1+\frac{\mu_B h}{E_0})^{3/2}+(1-\frac{\mu_B h}{E_0})^{3/2}\right)\\ &\simeq \frac{8\pi}{3}(2mE_0)^{3/2}\left(1+\frac{3}{8}\left(\frac{\mu_B h}{E_0}\right)^2+O(4)\right).\end{aligned}$$

The Pauli magnetization is given by

$$m^{(0)}=\mu_B\frac{|S_{d-1}|}{d}(2m)^{d/2}\left((E_0+\mu_B h)^{d/2}-(E_0-\mu_B h)^{d/2}\right).$$

In two dimensions, $d=2$,

$$m^{(0)}=4\pi m\mu_B^2h^2.$$

In three dimensions, $d=3$,

$$\begin{aligned}m^{(0)} &= \mu_B\frac{4\pi}{3}(2mE_0)^{3/2}\left((1+\frac{\mu_B h}{E_0})^{3/2}-(1-\frac{\mu_B h}{E_0})^{3/2}\right)\\ &\simeq 4\pi(2m)^{3/2}E_0^{1/2}\mu_B^2h+O(3)\\ &\simeq \frac{3}{2}\bar\rho\mu_B\frac{\mu_B h}{E_0}+O(3).\end{aligned}$$

C.4.2 Interacting Fermions

We consider now the case of interacting fermions, and proceed as in the proof of Luttinger's theorem. In order to find the interacting dispersion relation, we invert the renormalization map

$$e_{int}(\mathbf{k}, \sigma, \lambda, E) = R_\lambda^{-1}(\varepsilon_\sigma(\cdot) - E)(\mathbf{k}).$$

By the rotational symmetry, we know that $e_{int}(\mathbf{k}, \sigma) = \frac{\mathbf{k}^2}{2m} - (-1)^\sigma \mu_B h - E_\lambda(E)$, where the term $(-1)^\sigma \mu_B h$ is single out for convenience. The Fermi surfaces are the surfaces on which the energy takes the value $E_\lambda(E)$.

In order to recover the initial model, one has to pick up E_F, such that the sum of the volumes enclosed in the Fermi surfaces is just $\bar{\rho}$, solving the equation

$$E_0 = E_\lambda(E_F),$$

since E_0 gives the radius of the Fermi surfaces for which the density is just $\bar{\rho}$. In order to compute the density and the Pauli magnetization, we don't need to solve explicitly this equation. Since $\mathrm{Vol}(S_\sigma)$ depends only on $E_\lambda(E_F) = E_0$, we see that

$$\mathrm{Vol}(S_\sigma) = \mathrm{Vol}(S_\sigma^{(0)}).$$

The case of a spherical Fermi surface is thus particular, in the sense that the volume of each Fermi surface is independent of the interaction strength. This conclusion cannot be proved in general.

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Curriculum Vitae

I was born in June 22, 1976 in the canton of Valais, Switzerland. I attend to primary and secondary schools in Saillon (VS). In the year 1995 I obtained my "Maturité" at the "Lycée Collège des Creusets" in Sion (VS), and began the same year my studies in physics at the ETHZ. I spend the academic year 1997-98 at the State University of St. Petersburg, Russia, as an exchange student. In the year 2000 I get my diploma in theoretical physics at the ETHZ, with the diploma thesis "On Gauge Invariance in Quantum Gravity" under the supervision Prof. Dr. G. Scharf. Since October 2003 I worked as teaching assistant at the Department of Mathematics of the ETHZ.

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