Prevalence of non-Lipschitz Anosov foliations

PREVALENCE OF NON-LIPSCHITZ ANOSOV FOLIATIONS

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Abstract. We give sharp regularity results for the invariant distributions of hyperbolic dynamical systems in terms of eigenvalue data at periodic points and prove optimality in a strong sense: we construct open dense set of codimension one systems where this regularity is not exceeded. We furthermore have open dense sets of symplectic, geodesic, and codimension one systems where the analogous regularity results of [?] are optimal. Most importantly, we produce open sets of symplectic Anosov diffeomorphisms and flows with low transverse Hölder regularity of the invariant foliations *almost everywhere*. An important ingredient is a result of independent interest: we establish a new connection between the transverse regularity of these foliations and their tangent distributions.

1. Introduction

A particular degree of control of expansion and contraction rates along periodic orbits of a hyperbolic system produces a corresponding degree of regularity of the stable or unstable distribution ("α-bunching implies C^{α}") and this regularity, as well as that asserted by [?] for the holonomies associated with the invariant foliations, is not exceeded for a generic symplectic system, geodesic flow, or system with 1-dimensional distribution (Theorem **??**).

Thus, the regularity of the invariant distribution is generically low at some periodic point. Anosov [?] found an Anosov diffeomorphism whose invariant foliations are at most 2/3-Hölder almost everywhere, but it is not clear that perturbations have the same property. Neither result gives low regularity on a large set for many Anosov systems. Since the foliations are absolutely continuous and Hölder continuous, one may ask whether they are generically quasiconformal, or at least Lipschitz almost everywhere. Furthermore, Lipschitz regularity is clearly relevant to dimension theory. Hence we address the main question: Do Anosov systems generically have invariant foliations that are Lipschitz continuous almost everywhere?

The answer is negative. There are open sets of Anosov diffeomorphisms whose invariant foliations are non-Lipschitz on a set of full (volume) measure. Indeed, for any given Hölder exponent we construct an open set of (symplectic) Anosov diffeomorphisms whose foliations fail almost everywhere to have this Hölder exponent. The key ingredient is the study of an obstruction that is very simple and locally defined and there is no technical machinery needed at all. The main result is Proposition **??** and its primary consequences are the next two results. We use the C^k-topology on Anosov systems, for any $k \in \mathbb{N} \cup \{\infty\}$ and denote by C^r the space of $C^{\lfloor r \rfloor}$ maps whose $\lfloor r \rfloor$th derivatives have modulus of continuity $O(x^{r-\lfloor r \rfloor})$.

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Theorem 1. *For $\alpha \in (0,1)$ there is a C^k-open set of symplectic Anosov flows and diffeomorphisms such that the holonomy maps defined by the weak or strong unstable foliation and those of the stable foliation are almost nowhere C^α.*

We call a set in the phase space of a diffeomorphism *negligible* it its complement is residual and of full measure for any ergodic invariant probability measure that is fully supported, i.e., positive on nonempty open sets.

Theorem 2. *For $\alpha \in (0,1)$ there is a C^k-open set of symplectic Anosov flows and diffeomorphisms such that the weak or strong stable and unstable distributions are C^α at most on a negligible set.*

Furthermore in these examples neither distributions nor foliations have C^α restrictions to any set of full measure (because then they would be C^α on that set by density and continuity). It will become apparent in the proof that there are many homotopy classes of Anosov systems in which we can find such open sets.

In the case of flows we prove low regularity of the weak unstable distribution, which implies low regularity of the strong unstable distribution because it is the product of strong unstable distribution and the smooth flow direction. Thus, throughout the following we always discuss the weak distributions and foliations.

John Franks suggested the following corollary to our construction:

Theorem 3. *For $\alpha \in (0,1)$ there is a linear symplectic Anosov diffeomorphism A and a C^k-neighborhood U of A of symplectic diffeomorphisms such that for an open dense set of $f \in U$ the conjugacy to A is almost nowhere bi-C^α.*

This uses that our main examples are perturbations of linear maps and that a conjugacy conjugates holonomy maps. Thus conjugacies between Anosov systems are typically of low regularity on a large set, even when close to the identity.

Theorem **??** follows from Theorem **??** via a separate result of independent interest, namely that Hölder regularity of holonomy maps implies (essentially) the same regularity for the tangent distributions. This is proved by an induction involving ever higher derivatives along the leaves (Proposition **??**) which allows the possibility of a genuine difference in regularity for the case of foliations with leaves of finite smoothness.

Definition 4. A map between metric spaces is $C^{\alpha-}$ if it is C^β for every $\beta < \alpha$.

Theorem 5. *Let $\mathcal{F}$ be a foliation of a Riemannian manifold M whose leaves are uniformly C^{n+1} (C^∞). Suppose the $\mathcal{F}$ holonomies are $C^{\alpha-}$ (almost everywhere). Then the tangent distribution $T\mathcal{F}$ is $C^{\alpha n/(n+1)-}$ ($C^{\alpha-}$) (almost everywhere).*

Here "almost everywhere" is with respect to the Riemannian volume element on M (in the case of the tangent distribution) and on smooth transversals to $\mathcal{F}$ (in the case of holonomy maps). Thus the holonomy maps of $\mathcal{F}$ are Hölder a.e. if for almost every pair of smooth transversals D_1, D_2, to a leaf of $\mathcal{F}$ the $\mathcal{F}$-holonomy map $h\colon D_1 \to D_2$, where defined, satisfies a Hölder condition, with fixed exponent, at almost every point of D_1 (a precise definition is in Section **??**).

Theorem **??** was outlined in the first paragraph of this introduction. The results about the distributions were proved by the first author, the others follow from Theorem **??**. If M is a compact manifold, $f\colon M \to M$ an Anosov diffeomorphism, p an n-periodic point, $\mu_f(p) < \mu_s(p) < 1 < \nu_s(p) < \nu_f(p)$ the minimal and maximal absolute values of the eigenvalues of Df_p^n in and outside the unit circle, let

$$B^u_{\text{per}}(f) := \inf_{p \text{ periodic}} \frac{\log \mu_s(p) - \log \nu_s(p)}{\log \mu_f(p)}.$$

$B^u_{\text{per}}(f)$ may be large but $\min(B^u_{\text{per}}(f), B^u_{\text{per}}(f^{-1})) \le 2$ [?]. In the symplectic case $\nu_s = 1/\mu_s$, so $B^u_{\text{per}}(f) = 2\inf_p \log \mu_s(p)/\log \mu_f(p)$ is close to 2 iff the contraction rates are close together. Let $TM = E^u \oplus E^s$ be the Anosov splitting (for a flow $E^{0u} := E^u \oplus \langle\dot\varphi\rangle$) and W^i the foliation integrating E^i for $i = s, 0s, u, 0u$.

Theorem 6. 1. *If f is transitive Anosov and $B^u_{per}(f) \notin \mathbb{N}$ then $E^u \in C^{B^u_{per}(f)}$.*
2. *If f is transitive Anosov and $B^u_{per}(f) \in \mathbb{N}$ then $E^u \in C^{B^u_{per}(f)-1, O(x|\log x|)}$.*
3. *??* and *??* *hold for the weak unstable distribution of flows.*
4. *For an open dense set of symplectic diffeomorphisms and flows the regularity of E^u, E^{0u}, W^u, or W^{0u} is at most that asserted in* ?? *and* ??.
5. *For an open dense set of diffeomorphisms and flows with* $\dim(E^u) = 1$ *the regularity of E^u, E^{0u}, W^u, W^{0u} is at most that asserted in* ?? *and* ??.
6. ??–?? *hold for hyperbolic sets.*
7. *Among the metrics on a compact manifold with sectional curvature $\le -k^2$ and injectivity radius $\ge \log 2/k$ there is an open dense set whose horospheric foliations (hence structure at infinity) have at most the regularity claimed in* ?? *and* ??.

For example, an area preserving diffeomorphism in dimension two has $C^{1,O(x|\log x|)}$ foliations, a volume preserving codimension one diffeomorphism (i.e., $\dim(E^u) = 1$) has both foliations $C^{1+\epsilon}$ [?, Corollary 1.9]. Also the geodesic flow g^t of a compact Riemannian manifold with 1/4-pinched negative sectional curvature has $B^u_{\text{per}}(g^t) \ge 1$ because along every orbit of the geodesic flow the largest and smallest expansion rates differ by at most a factor of 2. Transitivity in ?? and ?? can be replaced by a bunching condition on all orbits, see [?]. About horospheric foliations see [?].

Our techniques for proving Theorems ?? and ?? also apply to geodesic flows and codimension one Anosov diffeomorphisms and flows, see Proposition ??. To construct examples one needs widespread failure of bunching rather than failure at a periodic point, as in Theorem ??.. Accordingly, we consider systems where no orbit satisfies a given α-bunching condition (Definition ??). This is most easily achieved by small perturbations of a linear (or algebraic) system. The proof of Theorem ?? is completed in Section ?? and Section ?? proves Theorem ??.

2. Adapted coordinates

We define *adapted coordinates* as a family of smooth local coordinates depending continuously on the point and in which the filtration of the stable leaf is given by coordinate planes. These exist whenever this filtration is globally defined (see [?, Lemma A.3.16]). In the symplectic case they can be taken symplectic:

Lemma 7. *If (M, ω) is a symplectic manifold of dimension $2n$ and $f\colon M \to M$ a symplectic Anosov diffeomorphism whose stable and unstable foliations admit global filtrations $\{W^s_i \mid i \in I \subset \{1, \dots, n\}\}$ and $\{W^u_i \mid i \in I\}$ then there is a continuous family (h_x, U_x) of smooth symplectic coordinates with respect to which*

$W^s_i(x)$ is coordinatized by points $(0, \dots, 0, 0, \dots, q_{n-i+1}, \dots, q_n)$

$W^u_i(x)$ is coordinatized by points $(0, \dots, 0, p_{n-i+1}, \dots, p_n, 0, \dots, 0)$.

Proof. Fix $x \in M$ and a neighborhood U_x. Complete the point filtrations $\{W_i^s(x) \mid i \in I\}$ and $\{W_i^u(x) \mid i \in I\}$ to isotropic filtrations $\{W_i^s(x) \mid 1 \le i \le n\}$ and $\{W_i^u(x) \mid 1 \le i \le n\}$ (continuous in x), where $\dim W_i^s(x) = \dim W_i^u(x) = i$. Now take a $2(n-1)$-dimensional submanifold $M_{n-1} \supset W_{n-1}^s(x) \cup W_{n-1}^u(x)$ of U_x and a function $p_1 : U_x \to \mathbb{R}$ which vanishes on $W^s(x) \cup M_{n-1}$ and such that P_1 defined by $dp_1 = \omega(P_1, \cdot)$ is transverse to a hypersurface $N_n \supset W^u(x) \cup M_{n-1}$. Denote by P_1^t the Hamiltonian flow of P_1 and define q_1 on U_x by $z = P_1^{q_1(z)}(y)$ for a unique $y \in N_n$. These steps can be taken to depend continuously on x. We now have $\{q_1, p_1\} = 0$, $q_1 = 0$ on N_n, and $M_{n-1} = \{z \in U_x \mid p_1(z) = q_1(z) = 0\}$. If $n > 1$ perform a similar construction inside M_{n-1}, etc., and iteratively obtain coordinates $\{p_i, q_i\}_{i=1}^n$. These parametrize the filtration as needed, and are symplectic [**?**, 8.43E]. ☐

These coordinates can be varied continuously with the Anosov diffeomorphism. For flows adapted coordinates consist of smooth coordinate systems on transversals, with continuous dependence on the point. In codimension one these are straightforward, for symplectic flows (in odd dimension, with a preserved form that restricts to symplectic forms on transversals) they are constructed as above [**?**].

3. An obstruction to high regularity

Our main argument is based on [**?**] where a nongeneric condition was found at periodic points where the unstable distribution is smoother than in Theorem **??**. We show that if any orbit encounters a set of excessive Hölder regularity then the unstable distribution satisfies a nongeneric condition along its fast stable leaf.

Corresponding to unstable and stable directions adapted coordinates at a point x split into $\mathbb{R}^u \times \mathbb{R}^s$ and the differential of f at a point y on the stable leaf of x is

$$Df = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \tag{1}$$

because stable leaves are preserved. Represent $E^u(y)$ as the graph of a linear map $D\colon \mathbb{R}^u \to \mathbb{R}^s$ or the image of $\begin{pmatrix} I \\ D \end{pmatrix} : \mathbb{R}^u \to \mathbb{R}^u \times \mathbb{R}^s$. Then $Df(E^u)$ is the image of

$$Df \begin{pmatrix} I \\ D \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} I \\ D \end{pmatrix} = \begin{pmatrix} A \\ B + CD \end{pmatrix}.$$

The image of this map is unchanged when we reparametrize the preimage by A^{-1}:

$$f^* \begin{pmatrix} I \\ D \end{pmatrix} = \begin{pmatrix} A \\ B + CD \end{pmatrix} A^{-1} = \begin{pmatrix} I \\ (B + CD)A^{-1} \end{pmatrix}.$$

Thus the action of f on E^u is given in terms of D by $f^*D = (B + CD)A^{-1}$. By invariance this gives the unstable direction at $f(y)$, so if z_i are the coordinates of the point $f^i(y)$ in the coordinate system at $f^i(x)$ then this can be written as $D(z_1) = (B(z_0) + C(z_0)D(z_0))A(z_0)^{-1}$ or $D(z_{i+1}) = (B(z_i) + C(z_i)D(z_i))A(z_i)^{-1}$. To refine our analysis we now assume the following condition for some $\alpha \le 1$.

Definition 8. f is called *α-u-spread* if the stable distribution E^s has proper fast stable subdistributions $E^{fs} \subset E^{ms}$ with $\|Df^n_{\restriction_{E^{fs}}}\| < \text{cst.}\mu_f^n$ and $\|Df^{-n}(v)\| < \text{cst.}\mu_s^{-n}\|v\|$ for $v \in E^s \setminus E^{ms}$, and $\|Df^{-n}_{\restriction_{E^u(x)}}\| > \text{cst.}\nu^{-n}$ for all x, where μ_f, μ_s, and ν are constants with $\nu\mu_f^\alpha < \mu_s$.

This means that the Mather spectrum has rings in the regions $\{|z| < \mu_f\}$ and $\{\mu_s < |z| < 1\}$ and a ring overlapping $\{1 < |z| < \nu\}$, and is an open condition. Assume $\dim(E^u) = 1$ or f is symplectic (in which case we take $\nu = \mu_s^{-1}$).

E^{fs} is tangent to the fast stable foliation W^{fs}. For $y \in W^{fs}(x)$ the square matrix C in (**??**) is lower block triangular. Denote the upper left $k \times k$ block corresponding to the complement of E^{ms} by c. In the symplectic case A, B, and D are of the same size as C and we denote by a, b, and d the corresponding blocks. If $\dim(E^u) = 1$ then $A =: a$ is scalar and B and D are column vectors whose top k entries define column vectors b and d. In either case these blocks decouple from the others:

$$d(z_{i+1}) = \big(b(z_i) + c(z_i)d(z_i)\big)a(z_i)^{-1}. \tag{2}$$

(In the symplectic case we used that $A = C^{t^{-1}}$ is upper block triangular.) Let us pause to note the significance of this decoupling. Df stretches horizontally and compresses vertically, making E^u closer to horizontal (smoothing). On the other hand the base point of E^u moves closer to the reference orbit. Regularity of E^u along stable leaves results from a favorable balance between the smoothing and the rate at which the base point approaches the reference orbit. For high regularity it is best to smooth rapidly while the base point approaches the reference orbit slowly. But we consider base points y in the fast stable leaf, which approach the reference orbit at the fastest rate, and in (**??**) we have isolated the slowest smoothing action. So d represents those parts of E^u which are likely to have the lowest regularity. Let

$$\xi_{z_0}^n := \prod_{i=0}^{n-1} c(z_{n-i-1}),\ \eta_{z_0}^n := \prod_{i=0}^{n-1} a(z_i)^{-1},\ \text{and}\ \Delta_{z_0}^n := -\sum_{i=0}^{n-1} (\xi_{z_0}^{i+1})^{-1} b(z_i) (\eta_{z_0}^i)^{-1}.$$

Now $\nu\mu_f^\alpha < \mu_s$ and $\alpha < 1$, so $\mu_s^{-1}\mu_f\nu < 1$ and $\Delta_{z_0}^n \to \Delta_{z_0}^\infty$ as $n \to \infty$ since $\|(\xi_{z_0}^i)^{-1}\| \le \text{cst.}\mu_s^{-i}$, $\|b(z_i)\| \le \text{cst.}\|z_i\| \le \text{cst.}\mu_f^i$, and $\|(\eta_{z_0}^i)^{-1}\| \le \text{cst.}\nu^i$ (the latter is obvious for codimension one and follows from $a^{-1} = c$ in the symplectic case). As the uniform limit of continuous functions $\Delta_{z_0}^\infty$ is continuous in z_0 and f. Let

$$O(x) := \sup_{z \in W^{fs}(x)} \|d(z) - \Delta_z^\infty\|, \tag{3}$$

where $W^{fs}(x)$ is the local fast stable leaf of x defined by the adapted coordinate neighborhood and we use any coordinate norm. O depends on the choices made, but in a continuous way and is hence continuous in x and f.

Iterating (**??**) gives $d(z_n) = \xi_{z_0}^n \cdot (d(z_0) - \Delta_{z_0}^n) \cdot \eta_{z_0}^n$ and if $f^n(x) \in H_{C,d}^\alpha$, the set where E^u is α-Hölder with multiplicative constant C and up to a distance d, then

$$\|d(z_0) - \Delta_{z_0}^n\| = \|(\xi_{z_0}^n)^{-1} d(z_n) (\eta_{z_0}^n)^{-1}\| \le \text{cst.}\mu_s^{-n}\mu_f^{\alpha n}\nu^n = \text{cst.}(\mu_s^{-1}\mu_f^\alpha\nu)^n.$$

Since $\nu\mu_f^\alpha < \mu_s$ this shows

Lemma 9. *For α-u-spread symplectic or codimension one Anosov diffeomorphisms O is continuous on M and in f. $O(x) = 0$ if $f^n(x) \in H_{C,d}^\alpha$ for infinitely many $n \in \mathbb{N}$.*

Proposition 10. *If $O \ne 0$ for an α-u-spread symplectic or codimension one Anosov diffeomorphism then $H^\alpha := \bigcup_{C,d} H_{C,d}^\alpha$ is negligible.*

Proof. If H^α is not negligible then either $\mu(H^\alpha) > 0$ for some fully supported ergodic invariant probability measure, so $\mu(H_{C,d}^\alpha) > 0$ for some C, d and almost every orbit encounters $H^{\alpha}_{C,d}$ infinitely many times (in positive time), so $O=0$ a.e., i.e., on a dense set, hence identically. Or some $H^{\alpha}_{C,d}$ has nonempty interior and $O=0$ on any dense orbit (which exists by [**?**] in the codimension one case, by ergodicity of volume [**?**, Theorem 20.4.1] in the symplectic case), hence identically. ☐

For flows the same arguments work. In codimension one the weak-unstable foliation is 2-dimensional, but in adapted coordinates on transversals we have 1-dimensional unstable leaves and the calculations are identical. Consequently

Proposition 11. *Lemma* **??** *and Proposition* **??** *hold for flows.*

4. Construction of examples

Any α-u-spread example with $O\neq 0$ proves Theorem **??**, because this property persists under perturbation. We find one by a perturbation result for symplectic diffeomorphisms and flows and geodesic flows from [**?**] that makes $O\neq 0$. For codimension one diffeomorphisms, and to prove **??** of Theorem **??**, we now carry out this construction; for flows it works the same way.

For an α-u-spread periodic point x (in the context of proving Theorem **??** all orbits are α-u-spread) take a negatively nonrecurrent point y in the fast stable leaf of x (Proposition 4.1 of [**?**]), i.e., such that there is a neighborhood U of y with $U\cap\{f^{-n}(y)\mid n\in\mathbb{N}\}=\varnothing$. Taking U small enough we may assume that $f^n(y)\notin U$ for any $n\in\mathbb{N}$ and $x\notin U$. Consider now a perturbation J of the identity on M supported in U such that in the adapted coordinates for x the differential at y is $I+\epsilon e_{21}$, where e_{21} is the matrix whose only nonzero entry is a 1 in the (2,1)-slot. Then for the perturbation $J\circ f$ of f we have a periodic point x with negatively nonrecurrent y on the fast stable leaf with unstable distribution

$$J\begin{pmatrix}1\\D\end{pmatrix}=\begin{pmatrix}1\\D\end{pmatrix}+(0,\epsilon,0,\dots,0)^t,$$

so $d(y)\neq\Delta^{\infty}_y$ (note that Δ^{∞}_y is unchanged under this perturbation), proving the density assertion of **??** of Theorem **??**. Openness follows from the closing lemma, i.e., persistence of the periodic orbit under perturbations, and continuity of adapted coordinates and Δ^{∞}_y under perturbation. This proves **??** of Theorem **??**. Regarding Theorem **??** note that $O(x)\neq 0$ and hence

Proposition 12. *In a sufficiently small neighborhood of an α-u-spread symplectic or codimension one Anosov diffeomorphism or flow or hyperbolic set the systems whose unstable distribution is C^{α} on a negligible set are C^k-open dense.*

If the geodesic flow of a metric of negative curvature on a compact manifold is α-u-spread, then, in any neighborhood of this metric, there is a C^{k+2}-open dense set of metrics whose horospheric distributions are C^{α} on a negligible set only.

For geodesic flows one can drop the condition on the injectivity radius in [**?**]. There it yielded a "negatively nonreturning" point on the fast stable leaf of a periodic point at which to center the perturbation. We instead pick a point on whose fast stable manifold there is a point heteroclinic to different periodic points. It will then never return to a sufficiently small neighborhood on the base manifold. Thus the perturbation in [**?**] has the desired effect.

Proof of Theorem **??**. $A_\alpha := B \times B^{\lfloor 2/\alpha \rfloor + 1}$, where $B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, is α-u-spread. Proposition **??** gives Theorem **??** for E^u. Considering inverses makes E^s simultaneously C^α on a negligible set. Suspensions prove the result for flows. $\square$

Proof of Theorem **??**. This follows from Theorem **??**: Consider a symplectic Anosov system with almost nowhere $C^{\alpha-\epsilon}$ unstable distribution. By Theorem **??** the invariant set where the holonomies are C^α cannot have full measure, hence is a null set by ergodicity of volume. $\square$

Proof of Theorem **??**. Suppose f is symplectic Anosov near $A := A_{\alpha^2}$. If the conjugacy h is bi-C^α at x, i.e., there is a $C > 0$ such that for x' near x

$$(1/C)d(x, x')^{1/\alpha} \le d(h(x), h(x')) \le Cd(x, x')^\alpha,$$

then so is $A \circ h = h \circ f$. Thus if h is bi-C^α on a set U of positive measure then U is f-invariant, hence has full measure. For $x \in U$ and $y \in W^u_{\text{loc}}(x) \cap U$ the unstable holonomy $\pi^u\colon W^s_{\text{loc}}(x) \to W^s_{\text{loc}}(y)$ is C^{α^2} at x because

$$\begin{aligned} d(\pi^u(x), \pi^u(x')) &\le Cd(h(\pi^u(x)), h(\pi^u(x'))) \\ &= Cd(\rho^u(h(x)), \rho^u(h(x')))^\alpha \le C'd(h(x), h(x'))^\alpha \le C''d(x, x')^{\alpha^2}, \end{aligned}$$

where $\rho^u\colon W^s_{\text{loc}}(A, h(x)) \to W^s_{\text{loc}}(A, h(y))$ is the unstable holonomy for A. Let $U' := \{x \in U \mid W^s_{\text{loc}}(x) \text{ is essentially } U\text{-saturated}\}$. Then for $x \in U'' := \{x \in U' \mid W^u_{\text{loc}}(x) \text{ is essentially } U'\text{-saturated}\}$, a.e. $y \in W^u_{\text{loc}}(x)$, and a.e. $z \in W^s_{\text{loc}}(x)$ we have $z, \pi^u(z) \in U$, so the unstable holonomy for f is C^{α^2} a.e. Now use Proposition **??**. $\square$

Finding α-u-spread symplectic systems was easy but among geodesic flows there is none widely known because it is hard to control contraction and expansion rates through curvature information, except to the effect of deducing bunching information from curvature pinching (see, e.g., [**?**]), which is contrary to our objective of finding 1-u-spread examples. The closest are nonconstantly curved locally symmetric metrics, whose geodesic flow is $1+\epsilon$-u-spread for every $\epsilon > 0$. But it is not clear whether such metrics can be perturbed to give a 1-u-spread geodesic flow. By the way, our arguments include implicitly a construction corresponding to failure of $C^{1+\alpha}$ regularity of the foliations (see [**?**]). This implies

Proposition 13. *For any $\epsilon > 0$ and any nonconstantly curved locally symmetric metric (on a compact manifold) there is a C^∞ neighborhood in which metrics whose horospheric distributions are $C^{1+\epsilon}$ on a negligible set are C^{k+2}-open dense.*

Linear codimension one flows and diffeomorphisms preserve volume and thus, as remarked after Theorem **??** have C^1 distributions. This does not rule out examples, but there are none preserving volume or close to a linear one.

Proof of Theorem **??**. For all $p \in M$ there exist $\mu_f < \mu_s < 1-\epsilon < 1+\epsilon < \nu_s < \nu_f$ so that for $v \in E^s(p), u \in E^u(p)$ and $n \in \mathbb{N}$ we have

$$\mu_f^n \|v\|/C \le \|Df^n(v)\| \le C\mu_s^n\|v\| \text{ and } \nu_f^{-n}\|u\|/C \le \|Df^{-n}(u)\| \le C\nu_s^{-n}\|u\|.$$

Let $B^u(f) := \inf_{p\in M}(\log\mu_s - \log\nu_s)/\log\mu_f$. [?] defines f to be *α-u-bunched* if $\sup_{p\in M}(Q_n^1(p)+Q_n^2(p)-\alpha Q_n^3(p))<0$ for some $n\in\mathbb{N}$, where

$$Q_n^1(p) := \sup\{\log(|Df^n(v)|/|v|) \mid 0\neq v\in E^s(p)\}/n$$
$$Q_n^2(p) := \sup\{\log(|Df^{-n}(v)|/|v|) \mid 0\neq v\in E^u(p)\}/n$$
$$Q_n^3(p) := \inf\{\log(|Df^n(v)|/|v|) \mid 0\neq v\in E^s(p)\}/n,$$

and that the periodic orbits are *α-u-bunched* if for every n-periodic point p the absolute values of the eigenvalues of $Df^n\restriction_{E^s(p)}$ are contained in an interval $[q_p^3, q_p^1]$ and the absolute values of the eigenvalues of $Df^n\restriction_{E^u(p)}$ are bounded by q_p^2, where $\log q_p^1+\log q_p^2-\alpha\log q_p^3\leq 0$. [?] shows that a transitive Anosov diffeomorphism with α-u-bunched periodic orbits is $(\alpha-\epsilon)$–u-bunched. Thus $B^u(f)\geq B^u_{\mathrm{per}}(f)-\epsilon$ for any $\epsilon>0$ and $B^u(f)=B^u_{\mathrm{per}}(f)$, proving **??** and **??** of Theorem **??** by [?]. **??** is proved similarly. The statement of **??** for the distributions is in [?]. For the foliations, i.e., holonomies, it then follows from Theorem **??**. **??** follows similarly from [?] and Theorem **??**. **??** was proved above. □

5. Non-Lipschitzness of the holonomy maps

In [?] it is shown that if the local contraction and expansion rates of a diffeomorphism satisfy the pointwise bunching condition $\nu_s(p)\mu_f(p)^\alpha > \mu_s(p)$ everywhere, then the unstable holonomy maps are uniformly C^α. This complements the corresponding earlier result for the unstable tangent distribution in [?]. Theorem **??** is the corresponding non-C^α result for unstable holonomies.

A foliation tangent to a Hölder continuous distribution does not always have Hölder continuous holonomy maps, even when the leaves of the foliation are uniformly smooth ([?]). This is closely related to the fact that a Hölder vector field is not always uniquely integrable; near non-unique trajectories, points with nearby initial values can move apart arbitrarily rapidly. Surprisingly, perhaps, the converse implication holds. This is Theorem **??**, which we now prove.

Since the question is a local one, invariant under smooth coordinate changes, we may formulate the problem so that the manifold is $\mathbb{R}^{u+s}=\mathbb{R}^u\times\mathbb{R}^s$ and the leaf of $\mathcal{F}$ through $(0,y)\in\{0\}\times\mathbb{R}^s$ is the graph of a smooth function g_y:

$$\mathcal{F}_{(0,y)} = \mathrm{graph}(g_y\colon \mathbb{R}^u\to\mathbb{R}^s).$$

The assertion that the leaves of $\mathcal{F}$ are uniformly smooth is equivalent to the one that $g(x,y) := g_y(x)\colon \mathbb{R}^u\times\mathbb{R}^s\to\mathbb{R}^s$ is C^k in the first u coordinates with all kth order derivatives $\partial^k g/\partial x^k$ uniformly continuous in the last s coordinates.

The statement that $\mathcal{F}$ has C^α holonomy implies (and is in fact equivalent to) the statement that the function $g(x,\cdot)$ is C^α, for every $x\in\mathbb{R}^u$. One statement holds uniformly, or almost everywhere, if and only if the other does, since the (x,y) coordinate system is smooth. Similarly, the tangent distribution $T\mathcal{F}$ is (a.e.) C^α if and only if for every vector $v\in\mathbb{R}^u$, the directional derivative $\partial g/\partial v$ is (a.e.) C^α. Thus Theorem **??** boils down to:

Proposition 14. *Suppose $g\colon \mathbb{R}^u\times\mathbb{R}^s\to\mathbb{R}^s$ is C^{α_0-} (a.e.), and there exists an $n\geq 0$ such that for every vector $v\in\mathbb{R}^u$ and for $k=1,\dots,n+1$, the kth directional derivatives in $\partial^k g/\partial v^k$ exist and are uniformly continuous (as a function of all $u+s$ variables and of v). Then for every k, with $0\leq k\leq n$, the kth directional derivatives $\partial^k g/\partial v^k$ are $C^{\alpha_0(n+1-k)/(n+1)-}$ (a.e.).*

Restricting to the coordinate functions of g and taking partial derivatives allows us to further assume that $u = s = 1$. The uniform version of Proposition **??** in the case $u = s = 1$ is Proposition **??**; after proving it, we turn to the almost everywhere version, Proposition **??**, below. Proposition **??** can be viewed as an extension of the results in [**?**] and [**?**] to the case where the function is less than differentiable in one of the variables.. To simplify notation write $F_x^{(k)}$ for the partial derivative $\partial^k F/\partial x^k$ with respect to the first coordinate.

Proposition 15. *Suppose $F\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is C^{α_0-}, and there is an $n \geq 0$ such that $F_x^{(k)}$ is uniformly continuous in both variables for $k = 1, \dots, n+1$. Then $F_x^{(k)}$ is $C^{\alpha_0(n+1-k)/(n+1)-}$ for $0 \leq k \leq n$. If $n = \infty$ then $F_x^{(k)}$ is C^{α_0-} for all k.*

Proof. We use induction on n. For $n = 0$ there is nothing to prove. For $n = 1$ we illustrate the method of the argument. Let us show that $\partial F/\partial x$ is $C^{\alpha_0/2-}$ at the origin $(0,0)$. By subtracting the smooth function $h(x,y) = F(x,0) + xF_x^{(1)}(x,0)$ from F we may assume that $F(x,0) = F_x^{(1)}(x,0) = 0$ for all x. Expand $F(\,\cdot\,,y)$ about $(0,0)$ to obtain

$$F(\epsilon,y) - F(0,y) = \epsilon F_x^{(1)}(0,y) + \epsilon^2 M(\epsilon,y)$$

for ϵ near 0, where $M(\epsilon,y)$ is uniformly continuous in both variables. Fix $\alpha < \alpha_0$ and let $\epsilon = \epsilon(y) = |y|^{\alpha/2}$. Divide by $|y|^\alpha$ to get

$$\frac{F(\epsilon,y)}{|y|^\alpha} - \frac{F(0,y)}{|y|^\alpha} = \frac{F_x^{(1)}(0,y)}{|y|^{\alpha/2}} + M(\epsilon,y).$$

The left-hand side is uniformly bounded for $|y|$ near 0 because F is C^α. $M(\epsilon,y)$ is bounded in any bounded region of $\mathbb{R}^2$, and so $F_x^{(1)}(0,y)/|y|^{\alpha/2}$ is bounded for all y in a bounded region of $\mathbb{R}^2$. Since $\alpha < \alpha_0$ is arbitrary, this implies that $\partial F/\partial x$ is $C^{\alpha_0/2-}$ at $(0,0)$. The origin was arbitrary so $\partial F/\partial x$ is $C^{\alpha_0/2-}$ everywhere.

Suppose then that the assertion holds for some value of n; that is, for any $\beta_0 > 0$ and for any G of class C^{β_0-} with $G_x^{(k)}$ continuous for $k = 1, \dots n+1$, $G_x^{(k)}$ is $C^{\beta_0(n+1-k)/(n+1)-}$. We show this implies the assertion for $n+1$.

Let $F\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be C^{α_0-} with $F_x^{(k)}$ continuous for $1 \leq k \leq n+2$. We aim to show that for $\alpha < \alpha_0$, and for every x_0, y_0, the function $F_x^{(k)}(x_0,\,\cdot\,)$ is Hölder continuous at the point (x_0,y_0), with exponent $\alpha(n+2-k)/(n+2)$. We may assume $x_0 = y_0 = 0$ and $F(x,0) = F_x^{(1)}(x,0) = \cdots = F_x^{(n+1)}(x,0) = 0$ for all x.

For $0 \leq d \leq \dfrac{n+1}{n+2}$ let $g(d) := \dfrac{1}{2}\left(1 + \dfrac{nd}{n+1}\right)$ and $h(d) := 1 - g(d)$. Then

$$1 - kh(d) \leq \frac{d(n+2-k)}{n+1} \text{ for } k \geq 2 \tag{4}$$

because both sides are linear in k with equality for $k = 2$ and reversed inequality for $k = 0$. Now $F(\,\cdot\,,y)$ is C^{n+2}, so $F_x^{(1)}$ is $C^{\alpha_0 n/(n+1)-}$ by the induction hypothesis.

Claim 1. *If $F_x^{(1)}$ is $C^{d\alpha_0-}$, with $0 \leq d \leq (n+1)/(n+2)$, then $F_x^{(1)}$ is $C^{g(d)\alpha_0-}$.*

Proof. $F(\,\cdot\,,y)$ is C^{n+2}, so for all ϵ near 0

$$F(\epsilon,y) - F(0,y) = \epsilon F_x^{(1)}(0,y) + \cdots + \frac{\epsilon^{n+1} F_x^{(n+1)}(0,y)}{(n+1)!} + \epsilon^{n+2} M(\epsilon,y), \tag{5}$$

where $M(\epsilon,y)$ is bounded uniformly in ϵ and y. Now pick $\alpha<\alpha_0$. Dividing by $|y|^\alpha$, we have

$$\frac{F(\epsilon,y)}{|y|^\alpha}-\frac{F(0,y)}{|y|^\alpha}=\frac{\epsilon F_x^{(1)}(0,y)}{|y|^\alpha}+\cdots+\frac{\epsilon^{n+1}F_x^{(n+1)}(0,y)}{(n+1)!|y|^\alpha}+\frac{\epsilon^{n+2}M(\epsilon,y)}{|y|^\alpha}. \tag{6}$$

The left hand side is bounded for all ϵ and y because F is C^α at $(\epsilon,0)$ and $F(\epsilon,0)=0$ for all ϵ. If $F_x^{(1)}$ is $C^{d\alpha_0-}$ then the induction hypothesis (applied to $F_x(\,\cdot\,,y)$, which is C^{n+1}) implies that $F_x^{(k)}$ is $C^{\beta-}$, for $k\geq 2$, where

$$\beta=(d\alpha_0)\left(\frac{n+2-k}{n+1}\right)\geq\alpha(1-kh(d))$$

by (**??**). Let $\epsilon=\epsilon(y):=|y|^{h(d)\alpha}$. The kth term on the right-hand side of (**??**) is on the order (as y approaches 0) of:

$$\frac{F_x^{(k)}(0,y)}{|y|^{\alpha(1-kh(d))}},$$

hence bounded for $2\leq k\leq n+1$, since $F_x^{(k)}(\epsilon,0)=0$ for all ϵ. The last term

$$\frac{M(\epsilon,y)}{|y|^{\alpha(1-(n+2)h(d))}}$$

is bounded because $1-(n+2)h(d)\leq 0$ by (**??**). Since all other terms in (**??**) are bounded as $|y|\to 0$, so is the term

$$\frac{F_x^{(1)}(0,y)}{|y|^{\alpha(1-h(d))}}=\frac{F_x^{(1)}(0,y)}{|y|^{\alpha g(d)}}.$$

Hence $F_x^{(1)}$ is $C^{\alpha g(d)}$ at $(0,0)$, for all $\alpha<\alpha_0$. The point $(0,0)$ was arbitrary, so $F_x^{(1)}$ is $C^{\alpha_0 g(d)-}$. This proves Claim **??**. □

Since $F_x^{(1)}$ is $C^{\alpha_0 n/(n+1)-}$ Claim **??** iteratively shows that $F_x^{(1)}$ is $C^{\alpha_0 g^m(n/(n+1))-}$ for all $m>0$. The contraction g has fixed point $g(d_0)=d_0=(n+1)/(n+2)$, so $F_x^{(1)}$ is $C^{\alpha_0 d_0-}$ and $F_x^{(k)}$ is $C^{\alpha_0 d_0(n+2-k)/(n+1)-}=C^{\alpha_0(n+2-k)/(n+2)-}$, proving Proposition **??**. □

Remark. To see that Proposition **??** is sharp consider the function defined by $F(\cdot,0)=0$ and $F(x,y)=|y|^{\alpha_0}\sin(x\cdot|y|^{-\alpha_0/(n+1)})$ for $y\neq 0$.

We now turn to the case where F is C^{α_0-} almost everywhere, i.e., for almost every $p\in\mathbb{R}^2$ and for every $\alpha<\alpha_0$, there exists $C(p,\alpha)>0$ such that

$$|F(p)-F(q)|\leq C(p,\alpha)d(p,q)^\alpha$$

for all $q\in\mathbb{R}^2$ satisfying $d(p,q)\leq 1$. (C is measurable if chosen optimally.) For $m>0$ and $\alpha<\alpha_0$ let

$$B_m^\alpha:=\{p\in\mathbb{R}^2\mid d(p,q)\leq 1\implies|F(p)-F(q)|\leq md(p,q)^\alpha\}.$$

Then $\mathbb{R}^2=\bigcup_{m=1}^\infty B_m^\alpha$ and so $\lambda(B_N^\alpha)>0$ for N sufficiently large, where λ denotes Lebesgue measure. For $A\in\mathbb{R}^2$ let

$$\langle A\rangle:=\{(x,y)\in A\mid\lim_{r\to 0}\lambda_y\big(A\cap([x-r,x+r]\times\{y\})\big)/2r=1\}$$

be the set of *horizontal density points* of A in A, where λ_y is Lebesgue measure on the line $\mathbb{R}\times\{y\}$. For a given α almost every (x,y) is in some $\langle B_N^\alpha\rangle$.

At the heart of the proof of Proposition **??** is the choice of $\epsilon = \epsilon(y) = |y|^{\alpha h(d)}$ in Claim **??**. On the one hand, $\epsilon(y)$ is chosen small relative to $|y|$ so that the terms $\dfrac{F_x^{(k)}(0,y)}{|y|^{\alpha(1-kh(d))}}$ and $\dfrac{M(\epsilon,y)}{|y|^{\alpha(1-(n+2)h(d))}}$ are uniformly bounded in y. On the other hand, we use that the term $F(\epsilon,y)/|y|^\alpha$ is uniformly bounded. When F is uniformly C^α, this is automatic. When F is not uniformly C^α, the value of $\epsilon(y)$ must be chosen more carefully. If, for example, the Hölder constant of F at $(\epsilon(y),0)$ is unbounded as $y \to 0$, then so is $F(\epsilon,y)/|y|^\alpha$. We avoid this possibility by assuming that the origin $(0,0)$ is in some $\langle B_N^\zeta\rangle$, and then making a more delicate choice of $\epsilon(y)$ to bound the other terms. This allows us to prove

Proposition 16. *Suppose $F\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is C^{α_0-} a.e., and there is an $n \geq 0$ such that $F_x^{(k)}$ is uniformly continuous for $k = 1,\dots,n+1$. Then $F_x^{(k)}$ is $C^{\alpha_0(n+1-k)/(n+1)-}$ a.e. for $k \leq n$. If $n = \infty$ then $F_x^{(k)}$ is C^{α_0-} a.e. for all k.*

Proof. We first prove the assertion for $n = 1$. Fix $\alpha < \alpha_0$ and $\delta < (\alpha_0-\alpha)/2$. Suppose the origin $(0,0)$ is in some $\langle B_N^{\alpha+\delta}\rangle$. As in the previous proof divide $F(\epsilon,y)-F(0,y) = \epsilon F_x^{(1)}(0,y) + \epsilon^2 M(\epsilon,y)$ by $|y|^{\alpha+\delta}$ to get

$$\frac{F(\epsilon,y)}{|y|^{\alpha+\delta}} - \frac{F(0,y)}{|y|^{\alpha+\delta}} - \frac{\epsilon^2}{|y|^{\alpha+\delta}}M(\epsilon,y) = \frac{\epsilon F_x^{(1)}(0,y)}{|y|^{\alpha+\delta}}.$$

The density of $[-|y|^{(\alpha+2\delta)/2}, |y|^{(\alpha+2\delta)/2}]\times\{0\}$ in $[-|y|^{(\alpha+\delta)/2}, |y|^{(\alpha+\delta)/2}]\times\{0\}$ is $2|y|^\delta \to 0$ as $|y|\to 0$ and $(0,0)\in\langle B_N^{\alpha+\delta}\rangle$, so we can choose $(\epsilon,0)\in B_N^{\alpha+\delta}$ such that $|y|^{(\alpha+2\delta)/2} < |\epsilon| < |y|^{(\alpha+\delta)/2}$. Then the left hand side is bounded for $|y|\neq 0$, and hence so is the right hand side. But the right hand side bounds $F_x^{(1)}(0,y)/|y|^{\alpha/2}$, so $F_x^{(1)}$ is $C^{\alpha/2}$ at $(0,0)$.

Since $(0,0)$ was essentially arbitrary, this implies that $F_x^{(1)}$ is $C^{\alpha/2}$ almost everywhere for every $\alpha < \alpha_0$. But then, taking countable intersections of full-measure sets, this implies that $F_x^{(1)}$ is $C^{\alpha_0/2-}$ almost everywhere.

Suppose that the statement holds for every C^{α_0-} a.e. function $n+1$ times differentiable in x. Assume now that F is $n+2$ times differentiable in x and that $F_x^{(k)}$ is $C^{\alpha_0(n+1-k)/(n+1)-}$ almost everywhere for $k = 0,\dots,n+1$. We show that $F_x^{(k)}$ is $C^{\alpha_0(n+2-k)/(n+2)-}$ almost everywhere for $k = 0,\dots,n+2$.

The proof proceeds almost exactly as that of Proposition **??** using

Claim 2. *If $\dfrac{n}{n+1} \leq d < \dfrac{n+1}{n+2}$ and $F_x^{(1)}$ is $C^{d\alpha_0-}$ a.e. then $F_x^{(1)}$ is $C^{g(d)\alpha_0-}$ a.e.*

Proof. Fix $\alpha < \beta < \alpha_0$ with $\beta - \alpha < \alpha/(n+1)$. Let G_N be the set of (x,y) such that $F_x^{(k)}$ is $C^{d\beta(n+1-k)/(n+1)-}$ with constant N for every $0 \leq k \leq n+1$. By the inductive hypothesis almost every point (x,y) is in some $\langle G_N\rangle$. Assume $(0,0)$ is in $\langle G_N\rangle$. Then (**??**) implies that $F_x^{(k)}$ is $C^{\beta(1-kh(d))}$ at $(0,0)$ (with constant $\leq N$) for $k = 2,\dots,n+1$.

In this step $\dfrac{n}{n+1} \leq d < \dfrac{n+1}{n+2}$, so $\dfrac{1}{n+2} < h(d) \leq \dfrac{2n+1}{2(n+1)^2}$ and

$$\alpha - \frac{\beta-\alpha}{n+1} < \gamma := \alpha - (\beta-\alpha)\left(\frac{1}{(n+1)h(d)} - 1\right) \leq \alpha - \frac{\beta-\alpha}{2n+1}.$$

Our choice of β ensures $0 < \gamma < \alpha$ and

$$\alpha - (n+2)\gamma h(d) = \alpha(1-(n+2)h(d)) + (\alpha-\gamma)(n+2)h(d) < 0.$$

The definition of γ implies $\alpha - k\gamma h(d) \leq \beta - k\beta h(d)$ for $k = 2, \dots, n+1$.

For y sufficiently close to 0

$$\lambda_0 \left(([-|y|^{\gamma h(d)}, -|y|^{\alpha h(d)}] \cup [|y|^{\alpha h(d)}, |y|^{\gamma h(d)}]) \cap G_N \right) > 0$$

because the density of $[-|y|^{\alpha h(d)}, |y|^{\alpha h(d)}] \times \{0\}$ in $[-|y|^{\gamma h(d)}, |y|^{\gamma h(d)}] \times \{0\}$ is $2|y|^{(\alpha-\gamma)h(d)}$, which approaches 0 as $|y| \to 0$, while the density of G_N in the same interval approaches 1 as $|y| \to 0$ (since $(0,0) \in \langle G_N \rangle$). Thus for small $|y|$ we can take $(\epsilon, 0) \in G_N$ with $|y|^{\alpha h(d)} < |\epsilon| < |y|^{\gamma h(d)}$. Then the terms on the order

$$\frac{\epsilon^k F_x^{(k)}(0,y)}{|y|^\alpha} \leq \frac{F_x^{(k)}(0,y)}{|y|^{\alpha - k\gamma h(d)}} \leq \frac{F_x^{(k)}(0,y)}{|y|^{\beta(1-kh(d))}} \quad \text{and} \quad \frac{\epsilon^{n+2} M(\epsilon, y)}{y^\alpha} \leq \frac{M(\epsilon, y)}{y^{\alpha-(n+2)\gamma h(d)}}$$

are bounded in y. Since $(\epsilon, 0) \in G_N \subset B_N^\beta \subset B_N^\alpha$, the left-hand term $F(\epsilon, y)/|y|^\alpha$ of (**??**) is bounded. Hence the remaining term

$$\frac{\epsilon F_x^{(1)}(0,y)}{|y|^\alpha} > \frac{F_x^{(1)}(0,y)}{|y|^{\alpha - \alpha h(d)}},$$

is bounded, and so $F_x^{(1)}$ is of class $C^{g(d)\alpha}$ at $(0,0)$. Since $(0,0)$ is essentially arbitrary and $\alpha < \alpha_0$ is arbitrary, this proves Claim **??**. □

To prove Proposition **??** apply Claim **??** iteratively; each time a new full-measure set is generated where $F_x^{(1)}$ is $C^{\alpha_0 g^m(n/(n+1))-}$. Intersecting these sets over $m \in \mathbb{N}$ gives a full-measure set where $F_x^{(1)}$ is $C^{\alpha_0(n+1)/(n+2)-}$. □

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