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Default and Recovery Risk Valuation in Incomplete Markets

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presented by
MARCEL B. RÜEGG
Dipl. Math. ETH
born November 26, 1970
citizen of Lumino TI and St.Gallenkappel SG

accepted on the recommendation of
Prof. Dr. F. Delbaen, examiner
Prof. Dr. P. Embrechts, co-examiner

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"L'oeuvre on la termine pas, on l'abandonne."

Unknown French autor.

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Contents

1 Abstract

Credit risk related securities are typically illiquid, have high transaction costs, the premiums are not transparent, the prices are not just risk based but also show some institutional arbitrage. Moreover there are only few and big transactions as well as few and big market players. This list of credit market imperfections shows that these products are typically dealt in incomplete markets. The assumptions of a perfect market are too strong. We are convinced that weaker assumptions like no-arbitrage are adequate for modelling prices of default prone securities. For valuing default prone securities one has to address not only default prone securities, but also one has to study their relations to default-free and securities of already defaulted companies. For this reason we split up the default and recovery valuation into three parts: i) building a multi company default model (at what time which company defaults), which allows for "default caused dependencies"; ii) modelling the recoveries in case of default events; iii) embedding and merging the multi-default and the recovery model into a martingale pricing frame, where security prices can be represented as risk neutral expectations of conditional to default discounted cash flows.

So assuming the existence of an equivalent martingale measure $\mathbb{P}$, we have first constructed a multi company default time model. A simple model with dependencies, which are caused by defaults, is presented. In this model the dependencies between default event probabilities are driven by a natural system of probability equations.

After having observed some recovery features in the credit related markets, we have developed a recovery model, which gives some insights about the post bankruptcy procedure and the settlement of the defaulted company's liabilities. Using priority rules and remembering that after default the company's liability are just affected by the same recovery risk, we can extract from defaulted security's prices some information about recovery risks.

For obtaining recovery rates we compare the recoveries with several definitions of credit exposures.

Finally embedding the multi default and the recovery model components together in a martingale valuation model we obtain a credit securities pricing representation model. Several credit risk affected instruments are analyzed by this model. We observe that default feedbacks cause an additional jump term as in the usual default martingale representation theory.

We also present some preliminary ideas, how to invert the default term structure for calibrating the model with market prices.

In the last section we present an algorithm for simulating consistent multi company defaults with default feedbacks and recoveries. These two simulation procedures are merged for creating a credit portfolio, risk controlling or credit derivative evaluation engine.

Summarizing we have developed a model for estimating, valuing, managing and controlling credit risks.

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2 Zusammenfassung

Kreditrisiko ist das Risiko, ausgeliehenes Geld nicht vollständig zurückzubekommen. Schweizer Banken mussten im Jahr 1997 insgesamt 40 Milliarden CHF wegen faulen Inland Immobilienkrediten abschreiben. Das Jahr 1997 war somit negativ auffallend in der Erfolgsgeschichte der Schweizer Banken, welche lernen mussten, dieses Risiko nicht zu unterschätzen. Kreditrisiken sind nicht nur in Schweizer Banken vorhanden und relevant, sondern in der ganzen Finanzwelt. Die richtige Einschätzung dieses Risikos ist von zentraler Bedeutung für ein gutes Management dieser Risiken. Das Risk-Management optimiert Portfolios so, dass zu einer gegebenen Gewinnvorstellung die nicht ganz vermeidbaren Kreditausfälle zumindest minimal sind. Banken sichern zum Teil ihre Kreditportfolios ab, indem sie Versicherungsdeckungen von Versichern oder anderen Banken kaufen, welche gegen eine Prämie die zukünftigen Verlusten übernehmen. Natürliche Fragen tauchen bei solchen Transaktionen auf: i) Welches sind die Firmenkandidaten, die am ehesten Konkurs gehen? ii) Wie sieht die Verlustverteilung des Portfolio aus? iii) Was kostet eine Kreditabsicherung? iv) Kann man Kreditrisiken systematisch minimieren (hedgen)?

Mit der vorliegende Arbeit versuchen wir die oben aufgeführten Fragen zu beantworten.

Zu diesem Zweck haben wir im ersten Schritt ein ökonomisch-probabilistisches Firmenausfallmodell gebaut, das uns die Information liefert, zu welchem Zeitpunkt mit höchster Wahrscheinlichkeit welche Firma bankrott geht. Dieses Ausfallmodell hat die Eigenschaft, dass der Konkurs einer Firma die Ausfallwahrscheinlichkeit der anderen überlebenden Firmen beeinflusst. Diese Rückkoppelung im ökonomischen System ist eine Erweiterung von herkömmlichen Ausfallmodellen und sinnvoll, da im Moment des Ausfalles viele neue Informationen (Welche Firma?; Wo war sie tätig (Industrien, Länder)?; Warum ist sie bankrott?;...) zu den Investoren gelangen, welche Schlüsse über die noch überlebenden Firmen ziehen. Falls zum Beispiel Firma XY in einer ähnlichen (respektiv verwandten oder abhängigen) Branche tätig ist wie die Firma YZ, die eben ihre Zahlungsunfähigkeit ankündigte wegen einer Branchenschwäche, dann werden die Investoren, zum Beispiel, den Schluss ziehen, dass Firma YZ möglicherweise ähnliche Probleme haben könnte. Somit wird der Investor von Firma YZ eine höhere Verlustdeckungsprämie verlangen, um für das erhöhte Kreditrisiko kompensiert zu werden.

Im zweiten Teil der Arbeit beschreiben wir, wie man die Verluste eines Kreditausfalles modelliert und bewertet. In diesem Abschnitt brauchen wir erstens, dass alle Finanzierungs-Instrumente der bankrotten Firma am Zeitpunkt der Bankrottanmeldung gleichzeitig das Ausfallrisiko verlieren (da die Firma schon bankrott ist) und zweitens, dass dessen Finanzierungs-Instrumente trotzdem noch weiterhin im Markt gehandelt werden.

Wir zeigen, dass die Marktpreise dieser Instrumente der ausgefallenen Firma der Geldmenge, die man erwartet zurückzubekommen, entsprechen. Um diese Werte zu beschreiben, haben wir explizit den Firmenwert und den Kollokationsplan (wer, wann und wieviel aus der bankrotten Firma bekommt), modelliert. Somit können wir in einem einfachen Modell die Preisdynamik von bankrotten Anlagen beschreiben.

Indem wir die modellierten Verluste mit den Investitionen vergleichen, erhalten wir ein Modell, das Verlustraten von bankrotten Firmen beschreibt.

Das Ausfallmodell und das Verlustmodell haben wir schliesslich in ein Martingal-Bewertungsmodell eingebettet, welches uns erlaubt, Finanzierungs-Instrumenten von Firmen zu bewerten.

Die Preisdynamik, welches unser Modell beschreibt, ist sehr ähnlich zu deren der traditionellen Modellen. Neu ist, dass wir einen zusätzlichen Term in den Bewertungsgleichungen ausweisen, der die Ausfallrückkoppelung widerspiegelt. Mit diesem Modell haben wir ein paar kreditsensitive Finanzierungs-Instrumente bewertet.

In einem weiteren Teil erläutern wir eine mögliche Kalibrierung des Modells mit Marktpreisen.

Da die Preise wegen dieser Rückkoppelung analytisch ziemlich kompliziert zu berechnen sind, haben wir eine Simulationsprozedur für die Fimenausfälle und deren Verlustraten konstruiert. Zum Abschluss präzisieren wir, wie man das Modell in der Praxis implementieren kann.

Zusammengefasst haben wir ein Modell gebaut, welches erlaubt, Kreditrisiken zu bemessen, zu bewerten und zu managen.

3 Introduction

The credit risk models that have been studied can be divided into two classes: **structural** and **intensity based** models.

The **first class** considers the balance sheet structure of the company and defines the default time as the hitting time when the company's asset value falls under its liability value or under a critical liability threshold. When the asset are driven by a Brownian motion, the default time is a predictable event in the asset's filtration, which means that default time is not a surprise but announced. In these theoretical models, equities are considered as call options on the company's assets with strike equal to the liabilities and with infinite maturity. The Black-Scholes-Merton option pricing formula allows to evaluate the company's equity and thus with the put-call parity, to value the company's liability. These structural models with different extensions were mostly developed and studied by [Mr], [BC], [BS1], [Gr], [HW], [Lh], [LT], [Vo], [DGr] and [LS1].

The **second class,** a newer approach, does not explicitly consider the balance sheet of the company. Here the authors have generalized the idea that default of a company is actuarially modeled as a binomial random variable defined in such a way that with probability p the company defaults and with probability $(1-p)$ survives within a certain period ΔT. These models allow that the probability of default changes stochastically over time. [AD1] showed that these stochastic probabilities survive equivalent martingale measure transformations and that one can study prices of default prone securities in a martingale frame. Typically the default time is a surprise and unpredictable. These intensity based models, with several extensions, are studied in [AD1], [AD2], [AD3], [BR1], [Dd1], [DS2], [EJY], [JLT], [JT], [JR], [Ks], [Ld1], [Ld2], [MU], [Sp1], etc.

Both model classes underlie quite strong assumptions. For example in the structural models one has to assume that the company's assets are traded in a liquid market, but usually not all assets are actively traded as for example real estates, goodwill, know-how, etc.

Within primary and secondary markets several securities for financing the company's asset are traded: equities, bonds of different maturities and seniorities, convertibles, equity options, etc. In recent years one also observes several credit derivatives on the same company: default protections, total return swaps, default swaps, etc. These security's payoffs depend on the company's future cash flows, earnings, leverage, engaged risks, and on its credit quality. All these securities have as a consequence similar and common underlying risk factors but different payoff functions, depending on leverage, risk and interest of the investors. Since these securities have common underlying risk factors, some internal relationship between these prices must exist for avoiding **arbitrage.** For this we believe that the no-arbitrage concept makes sense for pricing credit sensitive securities.

One can synthetically construct or approximate several of these credit sensitive securities by combining others. For example credit risk can be partially hedged by shorting equities or equity call options or by simply buying equity put options.

For studying and valuing credit default sensitive securities one has also to consider default-free securities and such that are already defaulted, because default-free securities are used as references and defaulted ones for modelling recoveries. We principally identify four steps for valuing default prone securities:

1. Develop a market consistent multi company default time model, describing at what time which company defaults.
2. Model the losses in case of default, i.e. investments minus recoveries.
3. Embed default and recovery models into a risk neutral martingale pricing model, where prices are evaluated as expectations of discounted cash flows.
4. Gauge a risk neutral martingale measure by comparing observed market and model prices.

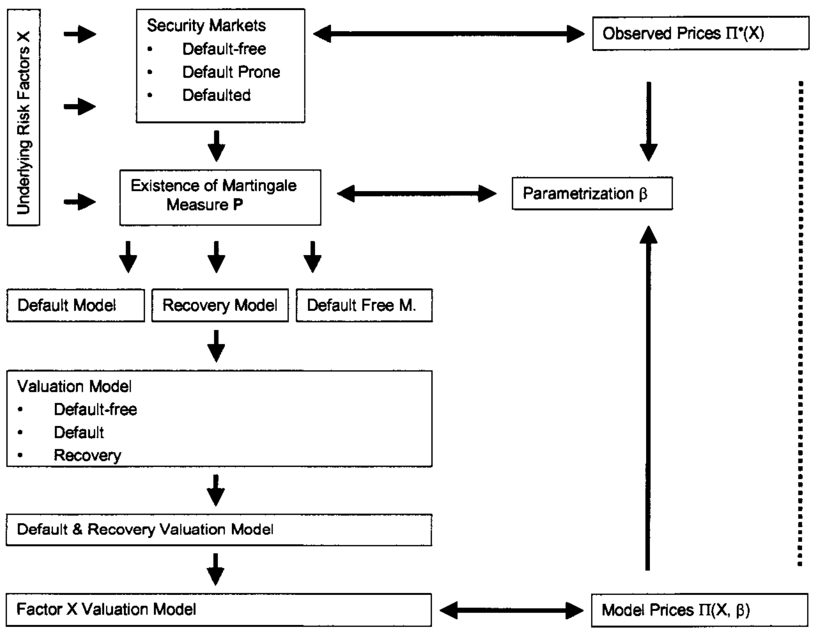


Figure 1: Default & Recovery Valuation Overview

According to our separation of the main default prone security's valuation issues and referring to Figure 1 we have organized this Ph.D. thesis in the following way.

In Section 4 we present some phenomena and properties, observed in primary and secondary credit risk markets. The presentation of these properties will become useful later when we have to motivate our model hypothesis and for giving some ideas about the economics and the dynamics of these credit markets. Section 5 recalls some definitions and some well known results about stand-alone intensity default risk models.

Our **first** part (Section 6) consists in extending under a risk neutral measure the intensity stand-alone model to a multi company default model, addressing default probability dependencies due to several important effects:

1. Economic cycles influence the company's credit quality. Thus default probabilities of companies operating in similar industries and/or countries behave probably similarly.

2. Default of a company may give new important information about the economic state of the industrial sectors and countries the defaulted company belonged to. As a consequence at default, investors may revalue the economic situations of the surviving companies and update their opinions about default probabilities.

3. Default of a company may cause financial troubles for its liability holders, which may also default due to the first default.

While we develop in this Section a rather general multi default model, in Section 11.3 a simplified model is studied.

In our **second** part (Section 7), after having listed recovery properties observed on the markets, we have developed a recovery model, which better reflects these market recovery properties. Using the defaulted company's liability structure and the priority rules, we recursively represent the recovery payoffs of the different liabilities. These recovery payoffs under some simplifying assumptions can be valued under an equivalent martingale measure. Finally different recovery rates are defined, depending whether we are comparing the recoveries to the security's face value, credit exposures or market value prior to default.

In the **third** part (Section 8) we embed the multi company default and the recovery model into the risk neutral martingale pricing model in order to give finally some martingale pricing representations of default prone securities. We already observe at this point that the default-feedbacks induce some additional terms in the pricing equation which are different from the usually expected ones. Several credit sensitive contracts are finally priced with this model.

In the **last** theoretical part (Section 8.6), gauging an equivalent martingale measure, is probably the most difficult task. The driving idea consists in applying all information delivered by the market prices of the various securities on the same company. Using the no-arbitrage concept, we filter out some information about the credit quality and the credit premiums of the underlying company and thus we extract some information about the market applied martingale measure. We will not completely solve the last problem in this thesis, but we give some ideas how one can "invert the default term structure".

Finally in Section 9 we present procedures for simulating the multi default, the recovery and the valuation model.

We observe that these sub-parts are presented in different sections because we believe that they also make sense as stand-alone components.

4 Credit Risk Characteristics

Before diving into the technical modelling of default and recovery risk pricing, we present several phenomena and properties observed in the real credit business markets. Some of these are reproduced by some theoretical credit models, while others have not yet been modeled. We will see that the credit risk market is such a complex market that including all features is extremely difficult. One should hence concentrate on the main influencing features that drive the most interesting phenomena. The following phenomena were observed in credit risk related markets:

1. **Default time is unpredictable**: Investors may observe that a company is not well performing and that its cash flow is negative. But the final default decision is generally taken by the management, the equity holders or the regulators. This is because debt and liability holder's actions are mostly constrained by the law of Chapter 11, which protects equity holder and management in order to give them a further chance for a turnaround. In the case where a turnaround cannot be expected within the chapter 11 protection period, the default time is then usually optimally chosen by the management and equity holders, such that equity value is maximized. They have thus insider information about the default time but for the market the default time still remains a surprise. Five events may trigger default: i) failure to meet payment obligations when due; ii) bankruptcy or moratorium; iii) repudiation; iv) material adverse restructuring of debt; v) obligation acceleration or obligation default. More generally speaking default may be caused either by a **liquidity** or a **solvency** problem. Unfortunately ex ante default cannot be predicted as a causal consequence of these presented triggers (i)-(v)), but ex post investors know what caused default because it is publicly declared by regulators with the respective trigger.

2. **Macroeconomic cycles** influence the company's credit qualities, since the credit quality depends on the company's finance, which heavily depends on the economic situation. Economic indicators like GDP, unemployment rates, housing costs, term structure, inflation rates influence credit loss cycles. An econometric analysis of the relations between the default rates and the economic indicators has been presented in the papers of [Wt] and [HK, Regensburg], which shows that the default rates have a time lag to these economic indicators. This time lag allows to predict credit crises because from the past state of the economy, one can estimate the actual one and estimate some future states of the credit cycle. Since these economic cycles differ among industries and/or countries, diversification is a saving recipe for minimizing default risk losses within credit portfolios.

3. **Default dependencies** can be observed in the clustering of default events: more defaults are observed in phases of low economic activities than while the economy is booming and expanding. We recall that the causes of these dependencies were presented in the introduction Section 3.

4. "**Flight-to-quality**": when the credit market is in downturn, manifested with several severe credit losses, investors become more risk averse and shift their portfolio exposures towards less credit risk sensitive assets. For example at the first signs of an

overheated emerging market, investors transfer their exposures from these emerging countries into safer assets (e.g. US government treasuries).

5. "**Aging effect**": each company passes trough different phases: it is founded, grows, expands, slows down its momentum, collapses and finally dies. One observes that the company's age actually influences its default likelihood. This phenomenon can be nicely seen in a family business company, where the grandfather founded the proper company with a great effort. His son inherited the company from his father and will probably manage it carefully since he still remembers and estimates his father's effort for building the company. The next generation, the grandson, will probably not remember anymore the family's effort invested in this company and will worry less about it. The default probability is for this reason not time homogeneous but may increase with aging. For a statistical analysis of this effect the interested reader is referred to [Dg].

6. **Taxes,** another market imperfection, influences how the company finances its assets, because interest expenses are tax deductible, while dividends are not. Because of these tax laws, companies would mostly finance their assets by debt, if there would not be a strong counter-force: default risk costs.

   In effect the more the company's asset are debt financed, the higher is the debt/equity leverage, the riskier the debt becomes and thus the higher the interest costs are. [Lh] and [Ej] showed that depending on the company's asset structure, there exists a natural optimal equilibrium between equity and debt financing. In a first view this seems to contradict the Modigliani-Miller [MM] Theorem: in perfect markets every financing structure is optimal, i.e. it does not matter, how a company is financed. But in a second view this is not a contradiction because taxes are a market imperfection and therefore the Modigliani-Miller Theorem does not apply. Concluding that taxes influence credit risks.

7. **Regulatory capital requirements**: Banks must reserve about 8% equities for their notional credit exposures, independently of their diversification and quality of the portfolio. Insurance companies have less severe restrictions for their credit risks. The difference of regulatory capital requirements between these two industries opened a new regulatory arbitrage business in transferring bank credit risks into insurance books. The prices of these deals are not only risk based, but also driven by the potential regulatory arbitrage.

8. "**Priority rules**": When a company is liquidated, the priority rules describe the settlement order of the liability securities. Senior liabilities are paid before junior ones. In case of default, these rules highly influence the recoveries payoffs. In fact it may happen that senior securities recover their investments, while junior ones lose almost everything.

9. **Restructuring or default cost**: restructuring, reorganizing or liquidating a company causes substantial extra costs because of social plans, early retirements, new management organizations, consultant's or lawyer's fees, write-offs (computer hardware, goodwill, etc.), transaction costs, etc.

10. **Credit spread dynamics behave similar to interest rate dynamics**, except that they might become **negative**. This might happen for example when off shore companies like Nestle Brasil have an international capital access and guarantee.

11. **Embedded options**: Frequently credit risk related securities include other embedded options:

    (i) Prepayment options that allow the borrower to exit the contract before maturity.

    (ii) Convertibles allow bondholders to transform their bonds into equity.

    All these options have an intrinsic value, which is reflected in the security's spread. For this reason before extracting any credit information from security's spreads, one has first to adjust the spreads for the embedded option's prices.

12. **Collaterals** reduce credit risks exposures. However it is not clear what the effects are, since most collaterals are marketed assets, which also bear some credit risks.

13. **Credit exposures** are very often **dynamic** and thus so are credit risks. For example a bank gives a credit limit of 100 millions CHF, but the company usually draws just 60 millions CHF. When the credit exposures are dynamic, one has to deal with notions like exposures potentials, percentile exposures, exposure at default, usage given default, etc.

14. **Credit risk is inhomogeneous in exposures**: lending 100 millions CHF to a company, whose assets are worth 200 millions CHF, may be more than twice as risky than lending just 50 millions CHF. This is because the default probability rises over-proportionally with the leverage and because the investor's portfolio has a larger concentration of this specific risk and hence is less diversified (default dependencies are positive).

15. **Netting agreements** reduce credit risk, but their effects are not well studied and depend heavily on local laws.

16. There are only few liability securities on a single company. You do not find a security for every maturity or a specific credit risk.

17. **Liquidity**: Credit risk affected securities are usually dealt in markets which are less liquid than equity markets. The typical credit transaction is normally of a quite significative amount with only few involved counterparts. For example few banks in collaboration lend a loan of 100 million CHF to a company. Other investors do not have access to this credit deal, except when the loan is later securitized. Since the non liquidity of these assets is viewed as a risk, investors want to be compensated for it. The spreads between the actual rate and the risk free rate is thus not just for compensating default risk but also for liquidity risk. The spread between bid and ask prices reflects the liquidity of the security: the smaller the spread the more liquid the security. Since these securities are not traded in liquid markets, the construction of credit risk hedging portfolios may become costly, if not impossible.

18. **Transaction costs** are relatively high because credit quality analysis is a quite labor consuming task.

19. The **rating** is based on the company's cash flows, its profitability, financial flexibility, industrial sector, country, market competitors, management, controls, financial reporting, legal structure, etc. The rating is a cardinal number which contains some default probability information but with a certain time lag since the ratings are updated only a few times a year. As a consequence the rating reflects more the expected average credit quality than the company's credit quality state at a specific moment.

With these credit market properties we conclude that **credit markets are probably not complete**. The assumption of complete markets is therefore too strong and we need weaker assumptions like no arbitrage.

5 Stand-alone Company Default Model

Before we start with the presentation of the multi company default model we want to present, as an introduction, some ideas and results about the single default model. We do not intend to give a complete overview of the credit and default model literature, since the interested reader can read it in several recent papers about intensity based default models: [Ld2], [JR], [BR1], [EJY] and [BR2]. Different definitions and known results presented in this section will be generalized later in the multi company default time model Section 6. Those readers, which are already familiar with these results can jump directly to the multi company default time Section 6.

5.1 Conventions and Notations

The basic models, we are going to study, are built on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, where $\mathbf{F} \stackrel{def}{=} (\mathcal{F}_t)_{\{t \geq 0\}}$ is the filtration, satisfying the "usual hypothesis":

(i) $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$;

(ii) $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$, all $t \geq 0$; that is, the filtration $\mathbf{F}$ is right continuous.

Heuristically $\mathcal{F}_t$ represents the **information** known to the investors at time t. For pricing purposes we will be working in an arbitrage-free setting and thus we will be considering the behavior of the involved processes directly under an equivalent martingale measure $\mathbb{P}$. We quickly recall few definitions and properties about stopping times and filtrations, which will become useful later and which can also be extensively read in [JS]:

1. An $\mathbb{R}^+ \cup \{\infty\}$-valued random variable τ is an **F-stopping time**, if $\{t \geq \tau\} \in \mathcal{F}_t$ for all $t \geq 0$.
2. The smallest filtration $\mathbf{F}$, where τ is an $\mathbf{F}$-stopping time, corresponds to the filtration generated by the sets $\{\tau \leq u\}$ and is usually denoted by $\sigma(\{\tau \leq u\} : u \leq t)$.
3. If $\mathbf{F}$ is a smaller filtration than $\mathbf{G}$ (this means $\mathcal{F}_t \subset \mathcal{G}_t$ for any $t \geq 0$), then any $\mathbf{F}$-stopping time τ is automatically a $\mathbf{G}$-stopping time.
4. A stopping time τ is **F-predictable**, if there exists an increasing sequence of $\mathbf{F}$-stopping times $\{\tau_i\}_{i\geq 0}$ such that for all $i \geq 0$, $\tau_i < \tau$ on $\{\tau > 0\}$ and $\lim_{i\to\infty} \tau_i = \tau$.
5. A stopping time τ is **F-totally inaccessible**, if for any $\mathbf{F}$-predictable stopping time α, $\mathbb{P}(\alpha = \tau < \infty) = 0$.
6. An example of an $\mathbf{F}$-totally inaccessible stopping time τ, when $\mathcal{F}_t = \sigma(\{\tau \leq u\} : u \leq t)$, is the first time when a Poisson process jumps.
7. For a completed with the null sets Brownian filtration $\mathbf{F}$, i.e. generated by a Brownian motion ($\mathcal{F}_t = \sigma(B_s,\ 0 \leq s \leq t)$ and completed, with B_s some d-dimensional Brownian motion on $\mathbb{R}^d$), any $\mathbf{F}$-stopping time τ is $\mathbf{F}$-predictable.

Continuing with the model description in the credit risk property's Section (4) we have seen that default might be triggered by different causes. However ex ante it is not clear what will trigger the default of a specific company. In fact some companies defaulted because their asset value was not sufficient to cover their liabilities whereas other companies defaulted because they had liquidity troubles while they had enough assets for covering their liabilities. Finally there also exist operating companies, whose asset values did not cover their liabilities and having had therefore some liquidity problems. For this an exact default time definition probably does not correspond to business reality. In practice default is officially announced by regulators because it has legal impacts, since it may trigger payoffs of securities. So we identify the default time with the default time announcement. It is important to stress that the default time is precept by the investors as a surprise.

Since we do not want to address further this "legal" issue we simply define the default time τ of a company to be an **unpredictable event in the future**. More precisely we assume

Assumption 1 *The* ***default time*** τ *of a company is an* $\mathbb{R}^+ \cup \{\infty\}$*-valued* $\mathbf{F}$***-totally inaccessible stopping time****, satisfying* $\mathbb{P}(\tau = 0) = 0$ *and* $\mathbb{P}(t < \tau) > 0$ *for all* $t \in \mathbb{R}^+$.

Notation 2 $\tau = \infty$ *stands for no default.*

Remark 3 *For the moment we allow an infinite time horizon* $T = \infty$ *for the default times. Later, when necessary, we will choose a finite time horizon* $0 < T < \infty$.

We associate to the default time τ in a natural way an increasing, càdlàg (right continuous paths with left limits), counting process

$$N_t \stackrel{def}{=} 1_{\{\tau \leq t\}} = \begin{cases} 0 & if \quad \tau > t, \\ 1 & if \quad \tau \leq t, \end{cases} \quad , \; t \geq 0,$$

which jumps from 0 to 1 exactly at the default time τ.

Definition 4 *The process* N_t *is called the* ***default indicator*** *process.*

The information structure, i.e. the filtration $\mathbf{F}$, is generated by two $(\mathbb{P}, \mathbf{F})$-completed filtrations $\mathbf{E}$ and $\mathbf{G}$, i.e.

$$\mathcal{F}_t = \mathcal{E}_t \vee \mathcal{G}_t, \; t \geq 0, \tag{1}$$

where

- $\mathcal{E}_t$ is generated by a Brownian motion and $(\mathbb{P}, \mathbf{F})$-completed with the $\mathcal{F}$ null sets, representing the information about the economy before time t; and
- $\mathcal{G}_t$ is the natural filtration of the default indicator process N_t: $\mathcal{G}_t = \sigma(N_u : u \leq t)$.

For the enlarged filtration $\mathbf{F}$ the following, as shown in the paper [Jt, Theorem 25, p. 16], is true

Proposition 5 $\mathcal{E}_t \subset \mathcal{F}_t = \mathcal{E}_t \vee \mathcal{G}_t \subset \mathcal{F}_t^*$, *for every* $t \geq 0$, *where*

$$\mathcal{F}_t^* \stackrel{def}{=} \{A \in \mathbf{F} \mid \exists B \in \mathcal{E}_t,\ A \cap \{\tau > t\} = B \cap \{\tau > t\}\}.$$

Further for any $t \geq 0$ *and for any event* $A \in \mathcal{E}_t \vee \mathcal{G}_\infty$ *we have* $A \cap \{\tau \leq t\} \in \mathcal{F}_t$.

Proof. See [Jt]. ■

Remark 6 *The natural filtration* $\mathbf{G} = \mathbf{F}^N$ *of the point process* N_t *is right continuous ([Pp, Theorem 25, p. 16]).* $\mathbf{E}$ *being a Brownian filtration is also right continuous, so is* $\mathbf{F} = \mathbf{E} \vee \mathbf{G}$ *as we will show later in the more general Proposition 36. Further being* $\mathbf{E}$ $(\mathbb{P}, \mathbf{F})$*-completed we have that the filtration* $\mathbf{F}$ *satisfies the usual hypothesis.*

Remark 7 *Heuristically the filtration* $\mathcal{F}_t$ *corresponds to knowing the evolution of the economy* $(\mathcal{E}_t)$ *up to time t and whether default has occurred or not before time t* $(\mathcal{G}_t)$.

Remark 8 *We assumed that the filtration* $\mathcal{E}_t$ *is generated by a Brownian motion because the economy is normally measured by economic indicators like GDP, unemployment rates, housing costs, interest rates, inflation rates, market and equity indices which are usually assumed to be driven by diffusion processes.*

Remark 9 *The default time* τ *is by construction a* $\mathbf{G}$*-stopping time and thus automatically becomes after the enlargement* $(\mathbf{G} \subset \mathbf{F})$ *an* $\mathbf{F}$*-stopping time. Furthermore the default time* τ *cannot be an* $\mathbf{E}$*-stopping time since by Assumption 1* τ *is* $\mathbf{F}$*-totally inaccessible and in a Brownian filtration there do not exist any totally inacessible stopping times. This implies that the default indicator process* N_t *is* $\mathbf{F}$*-adapted but not* $\mathbf{E}$*-adapted. Economically this means that the exact default time* τ *cannot be exactly predicted with economic indicators.*

5.2 Default Compensator and Intensity

The Doob-Meyer's decomposition result (see for example [JS, p. 32]) is formulated as

Theorem 10 *If* X *is a uniformly integrable sub-martingale, then there exists a unique increasing integrable predictable process* A *with* $A_0 = 0$ *and such that* $X - A$ *is a uniformly integrable martingale.*

This Theorem allows the following Definitions.

Definition 11 *The process* A *is called the* $\mathbf{F}$***-default compensator*** *of* N, *if there exists an increasing* $\mathbf{F}$*-predictable process* A *such that* $N - A$ *is a uniformly integrable* $\mathbf{F}$*-martingale.*

Remark 12 *The process* A *is also called* ***dual predictable projection*** *of* N.

Definition 13 *If the default compensator* A *further allows an integral representation like*

$$A_t = \int_0^{t \wedge \tau} \lambda_s \, ds = \int_0^t \left(1 - N_{s_-}\right) \lambda_s \, ds, \tag{2}$$

with λ *an* $\mathbf{E}$*-predictable bounded process, then* λ *is called the* $\mathbf{E}$***-default intensity of*** N.

Remark 14 *In [Jt] it is showed that for any* $\mathbf{F}$*-predictable bounded process* $\lambda^{\mathbf{F}}$*, there exists an* $\mathbf{E}$*-predictable bounded process* $\lambda^{\mathbf{E}}$*, such that* $\lambda_t^{\mathbf{F}}\ 1_{\{t\leq\tau\}} = \lambda_t^{\mathbf{E}}\ 1_{\{t\leq\tau\}}$.

Remark 15 *Since the default compensator process* A *is defined by a martingale property, where the filtration* $\mathbf{F}$ *matters, also the default compensator and the default intensity depend on the filtration* $\mathbf{F}$.

Remark 16 *The default intensity process* λ_s *has the nice heuristic interpretation of* ***infinitesimal default probability****, given the information of surviving until* $s > 0$*:* $\lambda_s\ ds = \mathbb{P}(dN_s = 1|\mathcal{F}_{s-})$.

5.3 Hazard Rate Process

For all $t \geq 0$ let F_t be the right continuous modification of the process $\mathbb{P}(\tau \leq t \mid \mathcal{E}_t)$, which is a bounded $0 \leq F_t \leq 1$ $(\mathbb{P}, \mathbf{E})$-sub-martingale. Moreover because the default event τ cannot be predicted nor seen with certainty by economic indicators $(\mathcal{E}_t)$ it holds $F_t < 1$. The strict inequality would not hold if we would have defined $F_t \stackrel{def}{=} \mathbb{P}(\tau \leq t \mid \mathcal{F}_t)$, since then the filtration $\mathcal{F}_t$ percepts the default time τ and the upper bound of $F_t = 1$ is attained after the default, i.e. on the set $\{\tau \leq t\}$. These bounds allow us to define

Definition 17 *The* $\mathbf{E}$*-**hazard** process* Γ *of* τ *is defined by*

$$\Gamma_t \stackrel{def}{=} -\ln(1 - F_t). \tag{3}$$

For the E-hazard rate process Γ we have the following result from [JR], which will become of practical use for the valuation of default prone securities.

Proposition 18 *For any* $\mathbf{F}$*-measurable random variable* Y *we have, for any* $t \in \mathbb{R}^+$

$$\mathbb{E}\left[1_{\{\tau>t\}}Y \mid \mathcal{F}_t\right] = 1_{\{\tau>t\}}\frac{\mathbb{E}\left[1_{\{\tau>t\}}Y \mid \mathcal{E}_t\right]}{\mathbb{P}\left[\tau > t \mid \mathcal{E}_t\right]} = 1_{\{\tau>t\}}\mathbb{E}\left[1_{\{\tau>t\}}e^{\Gamma_t}Y \mid \mathcal{E}_t\right], \tag{4}$$

any $0 \leq t < T$

$$\mathbb{P}\left[t < \tau \leq T \mid \mathcal{F}_t\right] = 1_{\{\tau>t\}}\frac{\mathbb{P}\left[t < \tau \leq T \mid \mathcal{E}_t\right]}{\mathbb{P}\left[\tau > t \mid \mathcal{E}_t\right]} = 1_{\{\tau>t\}}\mathbb{E}\left[1 - e^{\Gamma_t - \Gamma_T} \mid \mathcal{E}_t\right], \tag{5}$$

$$\mathbb{E}\left[1_{\{\tau>T\}}Y \mid \mathcal{F}_t\right] = 1_{\{\tau>t\}}\mathbb{E}\left[1_{\{\tau>T\}}e^{\Gamma_t}Y \mid \mathcal{E}_t\right]. \tag{6}$$

and for any $\mathcal{E}_T$*-measurable random variable* X *we have*

$$\mathbb{E}\left[1_{\{\tau>T\}}X \mid \mathcal{F}_t\right] = 1_{\{\tau>t\}}\mathbb{E}\left[e^{\Gamma_t - \Gamma_T}Y \mid \mathcal{E}_t\right]. \tag{7}$$

Proof. The proof follows from the fact that the restriction to the set $\{\tau > t\}$ of any $\mathcal{F}_t$-measurable random variable represents an $\mathcal{E}_t$-measurable random variable. So for example equation (4) follows from the fact that $\mathcal{F}_t$ is equivalent to knowing $\mathcal{E}_t$ and $\{\tau > t\}$ and hence conditioning on $\{\tau > t\}$, applying Bayes' rule we obtain

$$\mathbb{E}\left[1_{\{\tau>t\}}Y \mid \mathcal{F}_t\right] = 1_{\{\tau>t\}}\frac{\mathbb{E}\left[1_{\{\tau>t\}}Y \mid \mathcal{E}_t\right]}{\mathbb{P}\left[\tau > t \mid \mathcal{E}_t\right]}.$$

Further with (3) we have $\mathbb{P}[\tau > t \mid \mathcal{E}_t] = 1 - F_t = e^{-\Gamma_t}$, which is $\mathcal{E}_t$-measurable, thus (4) holds.

The same arguments can be applied for proving (5) and (6). Equality (7) follows by the $\mathcal{E}_T$-measurability of X

$$\begin{aligned}
\mathbb{E}\left[1_{\{\tau>T\}} e^{\Gamma_t} X \mid \mathcal{E}_t\right] &= \mathbb{E}\left[\, \mathbb{E}\left[1_{\{\tau>T\}} e^{\Gamma_t} X \mid \mathcal{E}_T\right] \mid \mathcal{E}_t\right] \\
&= \mathbb{E}\left[e^{\Gamma_t} X \, \mathbb{E}\left[1_{\{\tau>T\}} \mid \mathcal{E}_T\right] \mid \mathcal{E}_t\right] \\
&= \mathbb{E}\left[e^{\Gamma_t} X \, e^{-\Gamma_T} \mid \mathcal{E}_t\right].
\end{aligned}$$

■

Corollary 19 *If the* $\mathbf{E}$*-hazard process* Γ *is absolutely continuous, then there exists an* $\mathbf{F}$*-predictable bounded process* λ*, such that for any* $t \leq T$

$$\mathbb{P}[T < \tau \mid \mathcal{F}_t] = 1_{\{\tau>t\}} \mathbb{E}\left[e^{-\int_t^T \lambda_s \, ds} \mid \mathcal{E}_t\right], \tag{8}$$

and

$$\mathbb{P}[t < \tau \leq T \mid \mathcal{F}_t] = 1_{\{\tau>t\}} \mathbb{E}\left[1 - e^{-\int_t^T \lambda_s \, ds} \mid \mathcal{E}_t\right]. \tag{9}$$

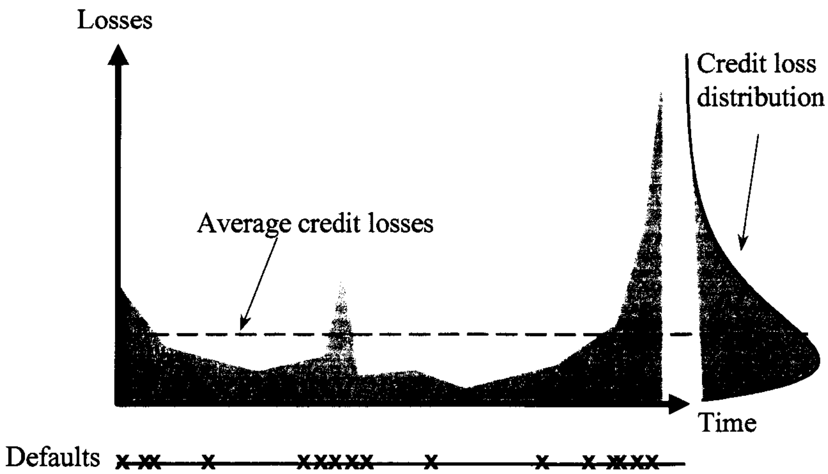
We see that the default indicator acts like a change in the discount process.

6 Multi Companies Default Model

6.1 Introduction

After the brief introduction to the stand-alone firm default model, we have to extend and to adapt those methods to a multi firm default model, considering default dependencies, which can be observed in credit risk affected markets:

1. **Clusters of default times**: there are periods with frequent and others with less frequent defaults, depending on the economic cycles as shown by [Wt] and [HK].



Historical Defaults and Credit Losses

2. **Similar credit spreads dynamics** for companies with similar rating and operating in similar countries and/or industry sectors.

These phenomena showing dependencies might be explained by the fact that each company i, being part of the economy influences with its financial activity the economy, which on the other side influences the company's activities

$$Company\ i \leftrightarrows\ Economy.$$

By transitivity each company in one way or the other influences other companies belonging to the economic system

$$Company\ i \leftrightarrows Company\ j.$$

Because of this economic dependence we conclude that the default of a company i may cause feedback effects to the rest of the economic system and to the activities of the other surviving companies k. We have identified three important effects that cause dependencies of default probabilities:

1. As shown by [Wt] and [HK] economic cycles influence the company's credit quality. Thus default probabilities of companies operating in similar industries and/or countries behave probably similarly.

2. Default of a company may give new important information about the economic state of specific industrial sectors and/or countries, where the defaulted company belonged to. As a consequence investors may revalue at default the economic situations of the surviving companies and update their opinions about their default probabilities. In fact if investors realize that the company defaulted because of a systemic weakness of a specific industry sector, then they might ask for a higher premium for the default risk of similar companies.

3. Default of a company may cause financial troubles for its liability holders, which may also default due to the first default.

We stress that in all three cases the **default of one company gives new insight information about the credit quality and thus about the default probability of the surviving companies**. After a specific default some companies might look riskier, others less. These changes of opinion are reflected in the market prices of default prone securities.

We present few examples of default probability dependencies.

Example 20 *The hedge fund company Long Term Capital Management (LTCM) has had serious financial troubles during autumn 1998. A default of LTCM might also have caused serious troubles for many other investors and institutions.*

Example 21 *A default of a company not only causes losses for the lending institutions but also for example to its delivering companies, that may have anticipated goods and not yet received their payments.*

Example 22 *The default of an airline may give some future market potentials for the surviving competitor airlines operating on similar flights.*

Before diving into the framework of our model we want to give an idea how practical business men, daily managing portfolios of default prone securities, deal with these default dependencies. They split intuitively their credit risks into two components of different natures: **systematic risk** S and **unsystematic risk component** U.

1. **Systematic risk component**

   (i) is driven by several underlying economic risk factors, e.g.: a) in the model KMV ([KMV1]): country industry factors, global, regional and industrial sector factors; b) in CreditMetrics ([JPM1]): country-industry factors; or c) CreditView ([Wt] and [HK]): country-industry factors, GDP, housing costs, employment rates, etc. We observe that these three systems of underlying risk factors represent "more or less" the same information but mathematical speaking, are represented in a different system of basis vectors. KMV for example constructed an orthonormal system of basis vectors, which facilitates the simulation.

   (ii) is not diversifying risk.

(iii) Since these economic factors like GDP or employment rates are not traded, this systematic risk cannot be easily hedged yet. This could change when these factors will be securitized and actively traded. Their securitization might have a high market potential, since banks are interested to hedge their risks.

(iv) Credit markets allow a risk premium for the systematic risk, since it cannot be diversified.

(v) Examples: risk free short rate, term structure, country-industry equity indices, GDP, housing costs, unemployment rates, order books, economic cycle indicators, etc.

2. **Unsystematic risk component**

   (i) Company specific factors are not depending on the systematic risk factors.

   (ii) This component can be diversified completely, when investing in different companies.

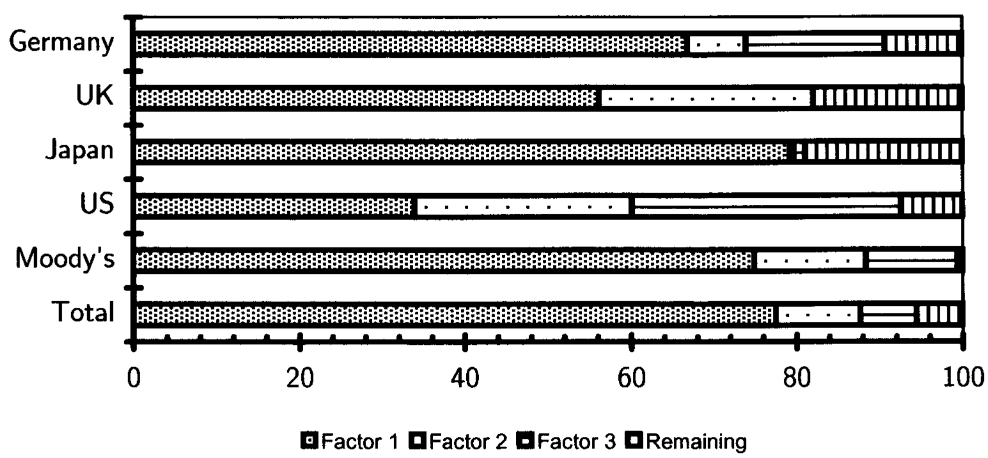
   (iii) Can be hedged by shorting other liability securities of the same company.

   (iv) Perfect credit markets do no pay a risk premium for unsystematic risks.

   (v) Examples: director's habitudes of buying nice cars, houses, etc.

The studies of [Wt] and [HK] confirm the presence of a substantial systematic credit risk component in default prone securities. The credit risk dependencies between firms can therefore be explained by a common set of underlying systematic risk factors. The more underlying risk factors two companies have in common, the more these credit risks are dependent.

Moreover [Wt] showed statistically that about 3 systematic factors (like GDP, housing costs, unemployment rates, etc.) are sufficient for representing the systematic default risk.



Systematic Default Risk by Systematic Factors

For this in the practical models like KMV, CreditMetrics or CreditView, the default probability dependencies are built in like a multi factor model similar to APT (Arbitrage

Pricing Theory). The underlying risk factors and indicators $X = (X^1, ..., X^k) \in \mathbb{R}^k$ are **assumed** to be driven by diffusion processes

$$dX_t^i = \mu_i(t, X_t)\, dt + \sum_{j=1}^{n} \sigma_{ij}(t, X_t)\, dW_t^j, \qquad i = 1, ..., k, \tag{10}$$

where

$\mu = (\mu_1, ..., \mu_k) \in \mathbb{R}^k$ is the **instantaneous mean drift** term;

$\sigma_i = (\sigma_{i1}, ..., \sigma_{in}) \in \mathbb{R}^n$ describes the **random dependence** of the i-th risk factor;

$\sigma = (\sigma_1, ..., \sigma_k) \in \mathbb{R}^{n \times k}$ is called the **instantaneous volatility matrix**;

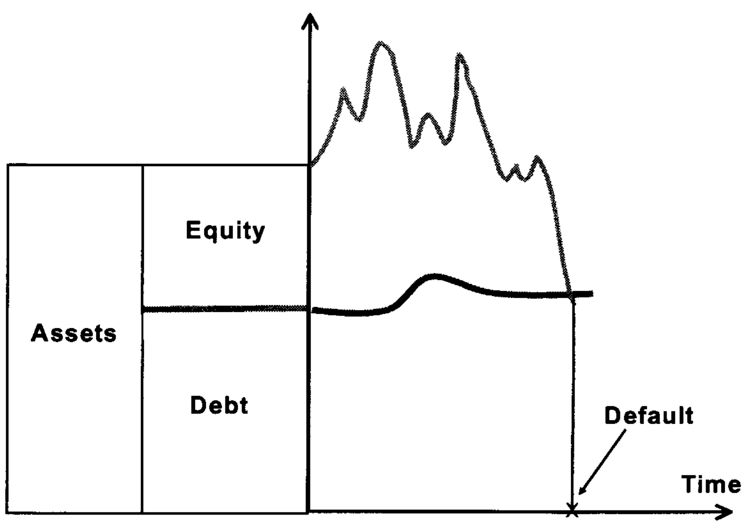
$W = (W^1, ..., W^n) \in \mathbb{R}^n$ is a standard n-dimensional $(\mathbb{P},\mathbf{F})$-Brownian motion.

For practical feasibility and estimation reasons practitioner assume time independent linear functions for $\mu_i(t, X(t)) = \mu_i X_t^i$ and $\sigma_{ij}(t, X(t)) = \sigma_{ij} X_t^i$. They model for each company i the market asset value A^i, corresponding to the company's project value, with APT

$$A_t^i = a^{i0} + \sum_{j=1}^{k} a^{ij}\, X_t^i + U_t^i, \tag{11}$$

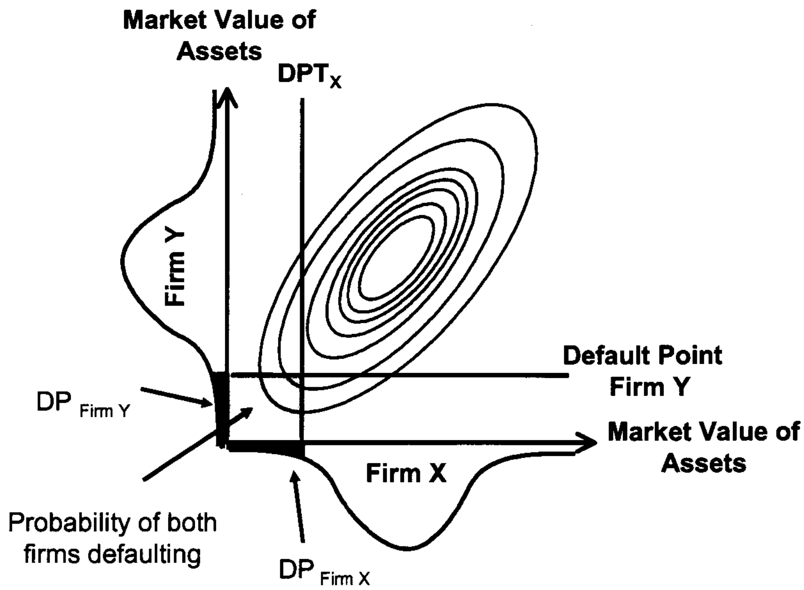
where a^{ij} are constants and U_t^i are the firm specific unsystematic risk component, not depending on the risk factors X_t^i and being also driven by a diffusion process like (10). The default time $\tilde{\tau}_i$ of company i is defined as the stopping time when the market asset value A_t^i falls under a critical "default threshold" L^i

$$\tilde{\tau}_i \stackrel{def}{=} \inf\left\{t \geq 0 : A_t^i \leq L^i\right\} \tag{12}$$



Structural Default Model

The default time dependences are due to similar underlying risk factors that drives the market asset value of the different companies: more common driving factors the higher the default dependence.



Correlated Defaults via Asset Correlation

Remark 23 *Equation (12) defines a predictable default time in the risk factor's filtration* $\sigma(X)$.

Remark 24 *In this kind of models we have dependencies implied by similar underlying economic drivers, but we do not have dependencies coming from any default feedback.*

Remark 25 *For a given basis of systematic risk factors* $(X^1, ..., X^k)$, *the* ***systematic default risk*** S^i *corresponds to the default risk component that is explainable in an APT model (like in (11)) with the systematic factors* $(X^1, ..., X^k)$, *while the* ***unsystematic default risk*** U^i *can be defined as the risk that is not explained by these systematic risk factors* $(X^1, ..., X^k)$

$$D^i(X^1, ..., X^k) = S^i(X^1, ..., X^k) + U^i. \tag{13}$$

6.2 Basic Ideas for Building a Multi Company Default Model

Having presented an overview about practical multi default models, we start presenting our model ideas, that are developed and precise in the following sections. For this let's study the following situation:

1. We are interested in modelling the default times of S companies. The today's market has an opinion, how the economic prospects of these companies look like and thus also has an opinion about their default probabilities and how these are going to evolve. This market opinion about default should be reflected in the market prices of securities affected by default risks.

2. At some time (say τ_1) a company defaults. At this time τ_1, investors receive new important information about which company (C_1) defaulted, of what region or which industrial sector it belonged to, the possible causes of default, the expected loss rates on the different defaulted securities, etc. At this point two important issues have to be modeled: first the recoveries or equivalent the losses given default (LGD), and second what happens to the survived companies. With the arrival of all this new information, investors may change their opinion on the different risks, prices and default rates of the remaining $S-1$ companies. Because of this new information, we might observe jumps in the prices of other default prone securities.

3. We are back in the starting situation (1.), interested in studying the default times, rates and prices of the remaining $S-1$ default prone companies.

The above presented steps (1.-3.) will drive the development of our multi company default time model.

6.3 Definitions and Notations

Most notations and concepts of the stand-alone default model section are extended and generalized to the multi company default model. Instead of having just one default prone company, we have S companies, {company 1, ..., company S}, denoted by $\{1, ..., S\}$. We introduce again a complete filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the "usual hypothesis". Later, for pricing purposes we will be working in an arbitrage-free setting and thus we will consider the behavior of the involved processes directly under an equivalent martingale measure $\mathbb{P}$. We introduce also the following definition for describing at what time which company defaults.

Definition 26 *$\{(\tau_i, C_i),\ i = 0, ..., S\}$ stands for the* ***default sequence*** *on the probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, where*

- $(\tau_0, C_0) \stackrel{def}{=} (0, 0)$.
- *The subsequence of $\mathbb{R}^+ \cup \{\infty\}$-valued* ***$\mathbf{F}$-stopping times*** *$\{\tau_i, i = 1, ..., S\}$ represents the* ***successive default times*** *of the S companies $\{1, ..., S\}$, satisfying for $i = 1, ..., S$*

  (i) *If $\tau_i < \infty \Rightarrow \tau_{i-1} < \tau_i$;*

  (ii) $\mathbb{P}(\tau_i = \infty) > 0$;

  (iii) *τ_i is $\mathbf{F}$-totally inaccessible.*

- *The subsequence $\{C_i, i = 1, ..., S\}$ of $\{1, ..., S\}$-valued random variable, where for $i = 1, ..., S$, C_i represents the* ***company that defaults at time*** *τ_i, satisfying for $i \neq j$, $\mathbb{P}(C_i{=}C_j) = 0$.*

Remark 27 *For an $i = 0, ..., S$ the random couple (τ_i, C_i) of the default sequence $\{(\tau_i, C_i),\ i = 0, ..., S\}$ gives the information that at the* ***default event*** *$\tau_i < \infty$,* ***company $C_i \in \{1, ..., S\}$ defaults.***

Remark 28 *At the default time τ_i, not necessarily company i defaults.*

Notation 29 *We write $\tau_i = \infty$ for an $i = 1, ..., S$, iff there are no more defaults after τ_{i-1}.*

Remark 30 *The imposed conditions on the default sequence are quite natural. In fact we want that*

1. *The default events are surprises;*
2. *At each default event just one company defaults;*
3. *A company defaults just once.*

Let's introduce some more definitions and notations, that will help later.

Definition 31 *For $t \geq 0$ the subset $\mathcal{D}_t = \{C_i : \tau_i \leq t\} \subset \{1, ..., S\}$ represents the* ***list of companies that have defaulted before and including time*** *t. Its complement in $\{1, ..., S\}$ is denoted by $\mathcal{S}_t \stackrel{def}{=} \{1, ..., S\} \setminus \mathcal{D}_t$ and corresponds to the* ***surviving companies until and including time*** *t.*

Definition 32 *We call the inter-arrival time between two defaults τ_i and τ_{i+1}, $i = 0, ..., S-1$, the* ***ith no-defaults period****, whose time length is denoted by*

$$\delta_{i+1} \stackrel{def}{=} \begin{cases} \tau_{i+1} - \tau_i & if\ \tau_{i+1} < \infty, \\ \infty & else. \end{cases}$$

Since in a no-defaults period there are no defaults, the subsets $\mathcal{D}_t$ and $\mathcal{S}_t$ remain constant within a no-defaults period, we introduce the notation.

Notation 33 *For all $i = 0, ..., S-1$ we denote by $\mathcal{D}^i \stackrel{def}{=} \{C_j : \tau_j \leq \tau_i\}$ (respective $\mathcal{S}^i \stackrel{def}{=} \{1, ..., S\} \setminus \mathcal{D}^i$) the set of defaulted (respective survived) companies before and including time τ_i.*

6.4 Multi Default Indicator Processes

In the stand-alone company default model we have seen that with a default time τ we can define the default indicator process N_t. Here in this subsection, we extend this definition to S companies. For this we denote the default time of company i by $\tilde{\tau}_i$. We recall that the default couple (τ_i, C_i), $i = 1, ..., S$ endorse the information that at time τ_i company C_i defaulted. Obviously between the two notations of default times we have that $\tau_i = \tilde{\tau}_{C_i}$.

To a given default sequence $\{(\tau_i, C_i),\ i = 0, ..., S\}$ we attribute an **S-variate $\{0,1\}^S$-valued point process** $N_t \stackrel{def}{=} (N_t^1, ..., N_t^S)$ on the probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, similarly as for the stand-alone default company model: to the default time $\tilde{\tau}_i$ of company i we associate a natural increasing, càdlàg, counting process

$$N_t^i \stackrel{def}{=} 1_{\{\tilde{\tau}_i \leq t\}} = \begin{cases} 0 & if \quad \tilde{\tau}_i > t, \\ 1 & if \quad \tilde{\tau}_i \leq t, \end{cases}$$

which jumps from 0 to 1 exactly at the default time $\tilde{\tau}_i$ of company i. The processes N_t^i are called again **default indicators**.

It is also clear that for a given S-variate default indicator process $N_t = (N_t^1, ..., N_t^S)$, we can construct our default sequence $\{(\tau_i, C_i),\ i = 0, ..., S\}$.

The information structure known to the investors at time $t \geq 0$, or more formally, the filtration, is composed as

$$\mathcal{F}_t = \mathcal{E}_t \vee \mathcal{G}_t, \tag{14}$$

where

- $\mathcal{E}_t$ is the right continuous, $(\mathbb{P}, \mathcal{F})$-completed Brownian filtration, representing the information deducted from the economy before time t;
- $\mathcal{G}_t$ stands for the natural filtration of the point process N_t or equivalently of the default time sequence $\sigma(\{(\tau_i, C_i), i : \tau_i \leq t\}) = \vee_{i=1}^S \mathcal{G}_t^i = \vee_{i=1}^S \mathcal{F}_t^{N^i}$, with $\mathcal{G}_t^i = \sigma(\{(\tau_i, C_i)\})$ and $\mathcal{F}_t^{N^i}$ the natural filtration of the default indicator point process N_t^i.

Heuristically the filtration $\mathcal{F}_t = \mathcal{E}_t \vee \mathcal{G}_t$ informs the investors about the economy until time t and at what time which company defaulted before time t. Again the economic filtration $\mathcal{E}_t$ is assumed to be a Brownian filtration since in our model $\mathcal{E}_t$ is generated by economic indicators $X = (X^1, ..., X^k) \in \mathbb{R}^k$ like GDP, unemployment rates, housing costs, interest rates, inflation rates or with market indices like equity indices, equities, or other asset prices, which usually are assumed to be driven by diffusion processes like (10). Some default sequence filtration $\mathcal{G}_t$ properties are presented in the next Proposition.

Proposition 34 *For $t \geq 0$, the filtration $\mathcal{G}_t$ is generated by the sets $\{C_i 1_{\{\tau_i \leq s\}},\ 0 \leq s \leq t,\ i = 0, ..., S\}$, i.e. $\mathcal{G}_t = \sigma(C_i 1_{\{\tau_i \leq s\}},\ 0 \leq s \leq t,\ i = 0, ..., S)$.*

Moreover the following equalities hold for all $i = 0, ..., S$

$$\begin{aligned} \mathcal{G}_{\tau_i} &= \sigma(\tau_1, C_1, ..., \tau_i, C_i), \\ \mathcal{G}_{\tau_i-} &= \sigma(\tau_1, C_1, ..., \tau_{i-1}, C_{i-1}, \tau_i), \\ \mathcal{G}_{\tau_i} &= \mathcal{G}_{\tau_i-} \vee \sigma(C_i). \end{aligned} \tag{15}$$

Proof. For the proof the reader is referred to [JS, p. 307, Theorem 30]. ∎

Moreover one knows that for point processes the following holds (see for example [BL, Theorem 2.2.4] or [Pp]).

Proposition 35 *Let $\mathcal{F}_t = \mathcal{G}_t \vee \sigma(Y)$ be the $(\mathbb{P}, \mathcal{F})$-completed filtration generated by a point process N and by a random variable Y. Then the filtration $\mathcal{F}_t$ is right continuous, i.e. $\mathcal{F}_t = \bigcap_{n \geq 1} \mathcal{F}_{t+\frac{1}{n}}$.*

Proof. Since the filtration $\mathcal{F}_t = \mathcal{G}_t \vee \sigma(Y)$ increases in time ($\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$), it is sufficient to show that

$$\bigcap_{n \geq 1} \mathcal{F}_{t_n} \subset \mathcal{F}_t,$$

where $t_n = t + \frac{1}{n}$. So we assume that the a random variable Z is $\mathcal{F}_{t_n}$ measurable for all n. By the $\mathcal{F}_{t_n}$ measurability it follows that

$$Z = f_n\left(Y, N^{t_n}\right),$$

for a $\mathcal{F}_{t_n}$measurable functions f_n, denoting $f_n\left(Y, N^{t_n}\right)$ that f_n depends on finite many points of the process N up to time t_n. Let's further introduce some subsets of Ω

$$A_n \stackrel{def}{=} \{w \in \Omega : N_t(w) = N_{t_n}(w)\},$$

and the function

$$f \stackrel{def}{=} \liminf_{n\to\infty} f_n.$$

Then since point processes are constant in a small time interval we have that by construction $A_1 \subset ... \subset A_n \to \Omega$ and hence we have that

$$\begin{aligned} Z &= \liminf_{n\to\infty} 1_{A_n}\, Z = \liminf_{n\to\infty} 1_{A_n}\, f_n\left(Y, N^{t_n}\right) \\ &= \liminf_{n\to\infty} 1_{A_n}\, f_n\left(Y, N^{t}\right) = f\left(Y, N^{t}\right), \end{aligned}$$

showing that Z is $\mathcal{F}_t$ measurable. ■

The right continuity of $\mathcal{G}_t$, the filtration generated by point process N_t, is proven in Proposition 35; the right continuity of the Brownian filtration $\mathcal{E}_t$ is assumed, hence it remains to show

Proposition 36 *The filtration $\mathcal{F}_t = \mathcal{E}_t \vee \mathcal{G}_t$ is right continuous.*

Proof. Since the filtration $\mathcal{F}_t = \mathcal{E}_t \vee \mathcal{G}_t$ increases in time ($\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$), it is sufficient to show that

$$\bigcap_{n\geq 1} \mathcal{F}_{t_n} \subset \mathcal{F}_t,$$

where $t_n = t + \frac{1}{n}$. So we assume that the process Z is $\mathcal{F}_{t_n}$ measurable for all $n \geq 1$ and it is sufficient to show that Z is $\mathcal{F}_t$ measurable. Similarly as in Proposition 35 we define on Ω the subsets

$$A_n \stackrel{def}{=} \{w \in \Omega : N_t(w) = N_{t_n}(w)\},$$

which are by construction $A_1 \subset ... \subset A_n \to \Omega$ since point processes are constant over a small time period (see Proposition 35). Again because the point process N_t is constant over a small time interval and thus no new information is added to $\mathcal{G}_{t_n}$, we **claim** that on the set A_n the following equality holds

$$\mathcal{E}_{t_n} \vee \mathcal{G}_t \,|_{A_n} = \mathcal{E}_{t_n} \vee \mathcal{G}_{t_n} \,|_{A_n}, \tag{16}$$

Now thanks to the equality (16) it follows that the $\mathcal{E}_{t_n} \vee \mathcal{G}_{t_n}$ measurable process Z can be written on the set A_n as

$$Z\, 1_{A_n} = F_n\, 1_{A_n}, \tag{17}$$

where F_n is $\mathcal{E}_{t_n} \vee \mathcal{G}_t$ measurable. Further knowing that $A_n \to \Omega$ we know that there exist for all $n \geq 1$ an $\varepsilon_n \downarrow 0$ (as $n \to \infty$) such that

$$\mathbb{E}[Z \mid \mathcal{E}_{t_n} \vee \mathcal{G}_t] = \mathbb{E}[Z \; 1_{A_n} \mid \mathcal{E}_{t_n} \vee \mathcal{G}_t] + \varepsilon_n. \tag{18}$$

Using relation (17) and using again the same arguments as used in (18) we obtain for another $\varepsilon_n \downarrow 0$ (as $n \to \infty$) that

$$\begin{aligned}
\mathbb{E}[Z \mid \mathcal{E}_{t_n} \vee \mathcal{G}_t] &= \mathbb{E}[Z \; 1_{A_n} \mid \mathcal{E}_{t_n} \vee \mathcal{G}_t] + \varepsilon_n \\
&= \mathbb{E}[F_n \; 1_{A_n} \mid \mathcal{E}_{t_n} \vee \mathcal{G}_t] + \varepsilon_n \\
&= \mathbb{E}[F_n \mid \mathcal{E}_{t_n} \vee \mathcal{G}_t] + 2 \; \varepsilon_n \\
&= F_n + 2 \; \varepsilon_n,
\end{aligned}$$

where the last equality follows since we know that F_n is $\mathcal{E}_{t_n} \vee \mathcal{G}_t$ measurable. Applying again relation (17), we get

$$\mathbb{E}[Z \mid \mathcal{E}_{t_n} \vee \mathcal{G}_t] = Z + 3 \; \varepsilon_n,$$

which converges against Z as $\varepsilon_n \downarrow 0$ and therefore Z is $\mathcal{E}_{t_n} \vee \mathcal{G}_t$ measurable. Now we claim that for $n \to \infty$

$$\mathcal{E}_{t_n} \vee \mathcal{G}_t \downarrow \mathcal{E}_t \vee \mathcal{G}_t, \tag{19}$$

that means

$$\bigcap_{n \geq 1} \mathcal{E}_{t_n} \vee \mathcal{G}_t = \mathcal{E}_t \vee \mathcal{G}_t,$$

because then we have that

$$\mathbb{E}[Z \mid \mathcal{E}_{t_n} \vee \mathcal{G}_t] \to \mathbb{E}[Z \mid \mathcal{E}_t \vee \mathcal{G}_t],$$

and hence summa summarum with the above calculation we obtain

$$\mathbb{E}[Z \mid \mathcal{E}_t \vee \mathcal{G}_t] = Z,$$

which shows that Z is $\mathcal{E}_t \vee \mathcal{G}_t$ measurable, completing the proof.

It remains therefore to show the two claims (16) and (19). Let's first prove that $\mathcal{E}_{t_n} \vee \mathcal{G}_t$ converges to $\mathcal{E}_t \vee \mathcal{G}_t$, which can be proven in a more general environment: let $\mathcal{A}$ and $\{\mathcal{B}_u\}_{u \leq 1}$ be two independent, $\mathbb{P}$-completed σ-algebras, such that for $u \to 0$, $\mathcal{B}_u \downarrow \{\emptyset, \Omega, N\}$, with N being the $\mathbb{P}$-nullsets. Under these assumptions we claim that for all $\mathcal{A} \vee \mathcal{B}_1$ measurable processes F we have as $u \to 0$ that

$$\mathbb{E}[F \mid \mathcal{A} \vee \mathcal{B}_u] \to \mathbb{E}[F \mid \mathcal{A}]. \tag{20}$$

Proof. The Proof follows almost the lines for proving equality (16). First, for $F = h \; k$, where h is $\mathcal{A}$ measurable and k is $\mathcal{B}_1$ measurable, we have that

$$\begin{aligned}
\mathbb{E}[F \mid \mathcal{A} \vee \mathcal{B}_u] &= \mathbb{E}[h \; k \mid \mathcal{A} \vee \mathcal{B}_u] \\
&= h \; \mathbb{E}[k \mid \mathcal{A} \vee \mathcal{B}_u] \\
&= h \; \mathbb{E}[k \mid \mathcal{B}_u],
\end{aligned}$$

using first the decomposition $F = h\ k$, second the $\mathcal{A}$ measurability of h and finally the $\mathcal{A}$ independence of k. At this point we have that $\mathbb{E}[k \mid \mathcal{B}_u] \to \mathbb{E}[k]$ for $u \to 0$ since $\mathcal{B}_u \downarrow \{\emptyset, \Omega, N\}$ and thus we have

$$h\ \mathbb{E}[\ k] = \mathbb{E}[h \mid \mathcal{A}]\ \mathbb{E}[k] = \mathbb{E}[h\ k \mid \mathcal{A}] = \mathbb{E}[h\ k \mid \mathcal{A}],$$

using again $\mathcal{A}$ measurability of h and the $\mathcal{A}$ independence of k. Summarizing we have shown that

$$\mathbb{E}[F \mid \mathcal{A} \vee \mathcal{B}_u] = h\ \mathbb{E}[k \mid \mathcal{B}_u] \to h\ \mathbb{E}[\ k] = \mathbb{E}[F \mid \mathcal{A}],$$

that means that we have proven (20) for all $F = h\ k$, where h is $\mathcal{A}$ measurable and k is $\mathcal{B}_1$ measurable. The proof for general F follows since first we can prove (20) for finite sums $F = \sum_{i=1}^n h_i\ k_i$, where h_i are $\mathcal{A}$ measurable and k_i are $\mathcal{B}_1$ measurable and second since all F can be represented as $F = \lim_{n\to\infty} \sum_{i=1}^n h_i\ k_i$ using the fact that these processes are dense in the space of predictable processes. ■

Finally for concluding the proof we show that claim (16) holds.

Proof. $\subset$: is clear.

$\supset$: Let Z be $\mathcal{E}_{t_n} \vee \mathcal{G}_{t_n}$ measurable such that $Z = h\ k$, with h is $\mathcal{E}_{t_n}$ and k is $\mathcal{G}_{t_n}$ measurable. Using first the decomposition of Z, then the $\mathcal{E}_{t_n}$ measurability of h and finally the $\mathcal{E}_{t_n}$ independence of k, we obtain

$$\begin{aligned} \mathbb{E}[Z\ 1_{A_n} \mid \mathcal{E}_{t_n} \vee \mathcal{G}_{t_n}] &= \mathbb{E}[h\ k\ 1_{A_n} \mid \mathcal{E}_{t_n} \vee \mathcal{G}_{t_n}] \\ &= h\ \mathbb{E}[k\ 1_{A_n} \mid \mathcal{E}_{t_n} \vee \mathcal{G}_{t_n}] \\ &= h\ \mathbb{E}[k\ 1_{A_n} \mid \mathcal{G}_{t_n}], \end{aligned}$$

At this point we have, because of Proposition 35, that $\mathbb{E}[k\ 1_{A_n} \mid \mathcal{G}_{t_n}] \to \mathbb{E}[k\ 1_{A_n} \mid \mathcal{G}_t]$ for $n \to \infty$ and thus we get

$$\begin{aligned} h\ \mathbb{E}[k\ 1_{A_n} \mid \mathcal{G}_t] &= h\ \mathbb{E}[k\ 1_{A_n} \mid \mathcal{E}_{t_n} \vee \mathcal{G}_t] \\ &= \mathbb{E}[h\ k\ 1_{A_n} \mid \mathcal{E}_{t_n} \vee \mathcal{G}_t] = \mathbb{E}[Z\ 1_{A_n} \mid \mathcal{E}_{t_n} \vee \mathcal{G}_t], \end{aligned}$$

using again the $\mathcal{E}_{t_n}$ independence of k and finally the $\mathcal{E}_{t_n}$ measurability of h. Summarizing we have shown that

$$\mathbb{E}[Z\ 1_{A_n} \mid \mathcal{E}_{t_n} \vee \mathcal{G}_{t_n}] = h\ \mathbb{E}[k\ 1_{A_n} \mid \mathcal{G}_{t_n}] \to h\ \mathbb{E}[k\ 1_{A_n} \mid \mathcal{G}_t] = \mathbb{E}[Z\ 1_{A_n} \mid \mathcal{E}_{t_n} \vee \mathcal{G}_t],$$

hence we have proven (16) for all $F = h\ k$, where h is $\mathcal{E}_{t_n}$ measurable and k is $\mathcal{G}_{t_n}$ measurable. The proof for general F follows since first we can prove (16) for finite sums $F = \sum_{i=1}^n h_i\ k_i$, where h_i are $\mathcal{E}_{t_n}$ measurable and k_i are $\mathcal{G}_{t_n}$ measurable and second since all F can be represented as $F = \lim_{n\to\infty} \sum_{i=1}^n h_i\ k_i$ using the fact that these processes are dense in the space of predictable processes. ■ ■

Being $\mathcal{F}_t = \mathcal{E}_t \vee \mathcal{G}_t$ right continuous and further being $\mathbf{E}$ $(\mathbb{P}, \mathbf{F})$-complete by construction, we have that the filtration $\mathbf{F}$ **satisfies the usual hypothesis.**

Remark 37 *We observe that all the default times τ_i, $i = 1, ..., S$ are $\mathbf{F}$-stopping times because $\mathbf{F}$ is the enlarged filtration of $\mathbf{G}$ ($\mathbf{G} \subset \mathbf{F}$), which is the natural filtration of the stopping times τ_i, $i = 1, ..., S$. Furthermore the* ***default times are assumed to be*** $\mathbf{F}$***-surprise*** *and thus they cannot be $\mathbf{F}$-predicted. Formally this concept corresponds that the default times τ_i are $\mathbf{F}$-totally inaccessible stopping times. But since in a Brownian filtration $\mathbf{E}$ all $\mathbf{E}$-stopping times τ are $\mathbf{E}$-predictable, we conclude that the stopping times τ_i, $i = 1, ..., S$ cannot be $\mathbf{E}$-stopping times.*

Remark 38 *This means that whenever the economic filtration is a Brownian filtration* $\mathbf{E}$, *we cannot construct any sure default predictors, i.e. from market information we cannot deduct that a specific company* i *must default exactly at a specific time* τ.

6.5 Dependent Default Probabilities

The default data are showing clusters of default times and the studies of [Wt] and [HK] and finally the practical model constructions ([KMV1], [JPM1] and [Wt]) show that default probabilities are most likely not independent. In what follows we will therefore develop a model with default probability dependencies. We **suppose** that investors, having the information $\mathcal{F}_t$, i.e. knowing the economic situation until time $t \geq 0$ ($\mathcal{E}_t$) and the companies that defaulted before t ($\mathcal{G}_t$), produce at time t a picture about the default likelihood before time $t + \Delta$ for the remaining surviving companies $i \in \mathcal{S}_t$

$$\mathbb{P}\left[\tau_{j+1} \leq t + \Delta, C_{j+1} = i \mid \mathcal{F}_t\right](w),$$

where we recall that C_{j+1} denotes the company that defaults at time τ_{j+1}.

More precisely, we further **suppose** that these default time distributions admit **densities** $d^{(j+1)}(.)$ for all $\mathbf{F}$-stopping times $t \in [\tau_j, \tau_{j+1})$ such that for all $\Delta > 0$, investors have an implicit opinion about the default time densities for each company $i \in \mathcal{S}^j$:

$$\mathbb{P}[\tau_{j+1} \leq t + \Delta, C_{j+1} = i \mid \mathcal{F}_t](w) = \int_0^{\Delta} d^{(j+1)}(w, x, t, i)\; dx \tag{21}$$

where the densities $d^{(j+1)}(.)$ are $(\Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{S}^j)$-measurable mappings $(w, x, t, i) \longmapsto d^{(j+1)}(w, x, t, i)$ from $\Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{S}^j$ to $\mathbb{R}^+$, with the arguments

- i denotes that this is the default time density of company $i \in \mathcal{S}^j$.
- $(j+1)$ denotes that j companies already defaulted.
- t marks that these default densities considers all the information $\mathcal{F}_t$ up to time t: they are updated not just at a default events τ_j but may also change at time $t \geq \tau_j$, at the arrival of new economic information $\mathcal{E}_t$. These economic changes are continuous because we supposed $\mathcal{E}_t$ to be a Brownian filtration.
- x stands for the time passed since the last information t, i.e. at time $t' \stackrel{def}{=} t + x$.

Further we will need a regularity **assumption**, that $d^{(j+1)}(w, x, t, i)$ is right continuous in x such that the limit $\lim_{x \downarrow 0} d^{(j+1)}(w, x, t, i)$ exists.

Example 39 $d^{(1)}(w, x, 0, i)$ *is the default time density of company* i *at time* x *in the probability state* w *having the information* $\mathcal{F}_0$.

Example 40 *Define the default time* τ *as the first jump of a homogeneous Poisson point process* N_t *with constant intensity* λ. *Then the probability of default before time* $T > 0$ *corresponds to* $\mathbb{P}[\tau < T] = \int_{[0,T]} \lambda e^{-\lambda t} dt = 1 - e^{-\lambda T}$, *and thus the density is* $d(w, t) = \lambda e^{-\lambda t}$. *Obviously in this simple model the density* $d(w, t)$ *is not depending on the arrival of new information.*

Example 41 *In several papers the above stand-alone default time model has been extended to a multi-default time model by simply choosing a* $\lambda_i > 0$ *for every default prone company* $i = 1, ..., S$, *such that* $\mathbb{P}[\tau_i < T] = 1 - e^{-\lambda_i T}$. *They have thus* ***assumed*** $d(w, t, i) = \lambda_i e^{-\lambda_i t}, t \leq \tau_i$.

Remark 42 *Since at time* $t \in [\tau_j, \tau_{j+1})$ *we have the information* $\mathcal{F}_t = \mathcal{E}_t \vee \mathcal{G}_t = \mathcal{E}_t \vee \mathcal{G}\tau_j \vee \sigma \{\delta_{j+1} \geq t - \tau_j\}$, *i.e. we know the economics* ($\mathcal{E}_t$), *the defaulted companies, their default times before* τ_j ($\mathcal{G}\tau_j$) *and finally that there were no further default events since the last default event* τ_j ($\sigma \{\delta_{j+1} \geq t - \tau_j\}$), *the probability of a default vanishes before* t, *i.e. for* $\Delta > 0$

$$\mathbb{P}[\tau_{j+1} \leq t + \Delta, C_{j+1} = i \mid \mathcal{F}_t](w) = 0.$$

Hence the default density $d^{(j+1)}(.)$ *of all companies vanishes before* t *as soon we have the information that the company has survived time* $t : d^{(j+1)}(w, x, t, i) = 0$ *for all* $x < 0 :$ *at time* t *there is no more uncertainty left about defaults before time* t.

Remark 43 *The today's default time density for time* $t + x$ ($d^{(j+1)}(w, x, t, i)$) *has to be in relation with the tomorrow's default time density for time* $t + x$ ($d^{(j+1)}(w, x, t + 1, i)$), *since opinions cannot abruptly change if there is no default event. Moreover default densities* $d^{(j+1)}(.)$ *are very similar to forward rate curves* $f(t, x)$, *which are usually modeled following the dynamics*

$$\begin{aligned} df(t,x) &= \alpha(t,x)\ dt + \sigma(t,x)\ dW_t, \\ f(0,T) &= f^*(0,T), \end{aligned} \tag{22}$$

where α *is the drift term,* σ *the volatility term and* $f^*(0, T)$ *the forward rate curve observed on the markets. The forward rate* $f(t, x)$ *as shown in the paper of [HJM] or [Bt2, Chapter 18] satisfy the Heath-Jarrow-Morton drift condition for excluding arbitrage, i.e. for all* $T \geq 0$ *and all* $t \leq T$

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s)'\ ds. \tag{23}$$

Since default densities can be seen as forward default rates, which are very similar to forward rates, we expect that also default time densities for consistency have to satisfy some internal relationship similar to (23).

The study of these kind of density relations was not subject of this thesis, however it seems quite interesting developing further this parallelism.

Summing the conditional default densities $d^{(j+1)}(w, x, t, i)$ over the not yet defaulted companies $i \in \mathcal{S}^j$,

$$d^{(j+1)}(w,x,t) \overset{def}{=} \sum_{i \in \mathcal{S}^j} d^{(j+1)}(w,x,t,i), \tag{24}$$

we obtain the **density of a default event** $d^{(j+1)}(w, x, t)$. We can thus express the **probability of a default event** within the time interval $\Delta \geq 0$ and for $t \in [\tau_j, \tau_{j+1})$ given the information $\mathcal{F}_t$ by

$$\mathbb{P}[\tau_{j+1} \leq t + \Delta \mid \mathcal{F}_t](w) = \int_0^{\Delta} d^{(j+1)}(w, x, t)\ dx. \tag{25}$$

In fact the following Proposition resumes formally, what was stated.

Proposition 44 *Given the information $\mathcal{F}_t$ for $t \in [\tau_j, \tau_{j+1})$ the probability of a next default event τ_{j+1} within Δ corresponds to*

$$\mathbb{P}[\tau_{j+1} \leq t + \Delta \mid \mathcal{F}_t](w) = \sum_{i \in \mathcal{S}^j} \mathbb{P}[\tau_{j+1} \leq t + \Delta, C_{j+1} = i \mid \mathcal{F}_t](w). \tag{26}$$

Proof. Knowing the information $\mathcal{F}_t$ we know all the economic history and which company when defaulted up to time $t \geq 0$. So we can assume without loss of generality that $t \in [\tau_j, \tau_{j+1})$ for some $j \geq 0$.

We recall some set notations on which we will calculate the conditional probability

- $\{\tau_{j+1} \leq t + \Delta\}$ denotes the set of a default event τ_{j+1} within the time interval Δ.
- $\{C_{j+1} = i\}$ represents the set that company $i \in \mathcal{S}^j$ defaults at the event τ_{j+1}.

These sets satisfy (see definition of default sequence), at each default event one and only one company defaults. This formally corresponds to

$$\bigcup_{i \in \mathcal{S}^j} \{C_{j+1} = i\} = \Omega, \tag{27}$$

and for all $i \neq k \in \mathcal{S}^j$

$$\{C_{j+1} = i\} \cap \{C_{j+1} = k\} = \varnothing. \tag{28}$$

With these conditions (27, 28) we have

$$\begin{aligned} \bigcup_{i \in \mathcal{S}^j} \{\tau_{j+1} \leq t + \Delta, C_{j+1} = i\} &= \bigcup_{i \in \mathcal{S}^j} \{\tau_{j+1} \leq t + \Delta\} \cap \{C_{j+1} = i\} \\ &= \{\tau_{j+1} \leq t + \Delta\} \cap \bigcup_{i \in \mathcal{S}^j} \{C_{j+1} = i\} \\ &= \{\tau_{j+1} \leq t + \Delta\}, \end{aligned} \tag{29}$$

Conditioning on $\mathcal{F}_t$ and applying the probability operator $\mathbb{E}$ on both sides of (29) we obtain the result (26) because for $i \neq k \in \mathcal{S}^j$,

$$\{\tau_{j+1} \leq t + \Delta, C_{j+1} = i\} \cap \{\tau_{j+1} \leq t + \Delta, C_{j+1} = k\} = \varnothing.$$

■

For clarification of the notations we list the different probabilities we will be interested in

1. $\mathbb{P}[\tau_{j+1} \leq t + \Delta \mid \mathcal{F}_t]$ denotes the probability of a default event τ_{j+1}, after t and within the period Δ, given the information $\mathcal{F}_t$.
2. $\mathbb{P}[\tau_{j+1} \leq t + \Delta, C_{j+1} = i \mid \mathcal{F}_t]$ denotes the probability that there is a default event after t and within Δ and that company i defaults at τ_{j+1}, given the information $\mathcal{F}_t$.

3. $\mathbb{P}[C_{j+1} = i \mid \tau_{j+1} \leq t+\Delta, \mathcal{F}_t]$ denotes the probability that company i defaults, given there is a default event after t and within Δ and given the information $\mathcal{F}_t$.

4. $\mathbb{P}[\tau_{j+1} \leq t + \Delta \mid C_{j+1} = i, \mathcal{F}_t]$ denotes the probability of a default event after t and within Δ, given that company i defaults and given the information $\mathcal{F}_t$.

5. $\mathbb{P}[C_{j+1} = i \mid \mathcal{F}_t]$ denotes the probability that company i defaults at the $j + 1$th default event, given the information $\mathcal{F}_t$.

With these definitions and notations we can prove with help of the classical equalities of probabilities the following results.

Corollary 45 *Given the information $\mathcal{F}_t$ for $t \in [\tau_j, \tau_{j+1})$ the probability that a next default event τ_{j+1} occurs after t and within Δ corresponds to*

$$\mathbb{P}[\tau_{j+1} \leq t + \Delta \mid \mathcal{F}_t] = \sum_{i \in \mathcal{S}^j} \mathbb{P}[\tau_{j+1} \leq t + \Delta \mid C_{j+1} = i, \mathcal{F}_t] \mathbb{P}[C_{j+1} = i \mid \mathcal{F}_t]. \tag{30}$$

Proof. Again knowing the information $\mathcal{F}_t$ we can assume without loss of generality that $t \in [\tau_j, \tau_{j+1})$. Since all the companies $i \in \mathcal{S}^j$, that have survived until t, have by assumption a positive probability of default: $\mathbb{P}[C_{j+1} = i \mid \mathcal{F}_t] > 0$, $i \in \mathcal{S}^j$. This allows to calculate the conditional probabilities by

$$\mathbb{P}[\tau_{j+1} \leq t + \Delta \mid C_{j+1} = i, \mathcal{F}_t] = \frac{\mathbb{P}[\tau_{j+1} \leq t + \Delta, C_{j+1} = i \mid \mathcal{F}_t]}{\mathbb{P}[C_{j+1} = i \mid \mathcal{F}_t]},$$

and replacing this expression into equation (30),

$$\mathbb{P}[\tau_{j+1} \leq t + \Delta \mid \mathcal{F}_t] = \sum_{i \in \mathcal{S}^j} \frac{\mathbb{P}[\tau_{j+1} \leq t + \Delta, C_{j+1} = i \mid \mathcal{F}_t]}{\mathbb{P}[C_{j+1} = i \mid \mathcal{F}_t]} \mathbb{P}[C_{j+1} = i \mid \mathcal{F}_t],$$

we obtain the equation (26) of Proposition 44. ■

Again the well known probability equalities allow to show.

Corollary 46 *Given the information $\mathcal{F}_t$ for $t \in [\tau_j, \tau_{j+1})$ we can calculate the probability that company $i \in \mathcal{S}^j$ defaults after t and within the period Δ by*

$$\mathbb{P}[\tau_{j+1} \leq t+\Delta, C_{j+1} = i \mid \mathcal{F}_t] = \mathbb{P}[C_{j+1} = i \mid \tau_{j+1} \leq t+\Delta, \mathcal{F}_t] \; \mathbb{P}[\tau_{j+1} \leq t+\Delta \mid \mathcal{F}_t]. \tag{31}$$

Proof. Since $\mathbb{P}[\tau_{j+1} \leq t + \Delta \mid \mathcal{F}_t]$ is positive for all $\Delta \geq 0$, we can represent

$$\mathbb{P}[C_{j+1} = i \mid \tau_{j+1} \leq t + \Delta, \; \mathcal{F}_t] = \frac{\mathbb{P}[C_{j+1} = i, \; \tau_{j+1} \leq t + \Delta \mid \mathcal{F}_t]}{\mathbb{P}[\tau_{j+1} \leq t + \Delta \mid \mathcal{F}_t]},$$

and thus similarly as in Corollary 45 we obtain the claimed result. ■

Corollary 47 *Given the information $\mathcal{F}_t$ for $t \in [\tau_j, \tau_{j+1})$ we can calculate the probability that company $i \in \mathcal{S}^j$ defaults after t and within Δ by*

$$\mathbb{P}[C_{j+1} = i \mid \tau_{j+1} \leq t + \Delta, \mathcal{F}_t] \; = \frac{\mathbb{P}[\tau_{j+1} \leq t + \Delta \mid C_{j+1} = i, \mathcal{F}_t] \mathbb{P}[C_{j+1} = i \mid \mathcal{F}_t]}{\mathbb{P}[\tau_{j+1} \leq t + \Delta \mid \mathcal{F}_t]}. \tag{32}$$

Proof. The procedure is similar as in the proof of Corollary 46, using elementary relations between probabilities. ■

6.6 Default Intensities

In the stand-alone company default model (Section 5) we have studied the default indicator process $N_t = 1_{\{\tau \leq t\}}$ and its associated **F**-default intensity process λ, which is defined such that the process $M_t \stackrel{def}{=} N_t - \int_{[0,t]} \lambda_s \, ds$ is an **F**-martingale. We have also seen that the default intensity characterizes the instantaneous default probability, i.e. the probability of default within the next instant dt, given the information of surviving until t. Because of this simple interpretation and also for future pricing purposes, we are interested in the associated **F**-default intensity processes $(\lambda_s^1, ..., \lambda_s^S)$ of the default indicator processes $N_t = (N_s^1, ..., N_s^S)$.

The one dimensional definitions of **F**-compensator and **F**-default intensity processes can easily be extended to the multidimensional case, when each component satisfies the requested martingale condition.

Definition 48 *The* $\mathbf{F}$*-predictable process* $A_t \stackrel{def}{=} (A_t^1, ..., A_t^S)$ *is called the* $\mathbf{F}$***-default compensator*** *for the jump process* N_t*, when all components* $N_t^i - A_t^i$ *are* $\mathbf{F}$*-martingales,* $i = 1, ..., S$*.*

Remark 49 *The Doob-Meyer decomposition result (see Theorem 10) guarantees the existence and uniqueness of the* $\mathbf{F}$*-default compensator* A*, which is filtration dependent, since it is defined by a martingale property.*

Definition 50 *If there exist further* $\mathbf{F}$*-predictable processes* $\lambda_t \stackrel{def}{=} (\lambda_t^1, ..., \lambda_t^S)$*, such that the* $\mathbf{F}$*-default compensator* A *allows for all* $i = 1, ..., S$ *an integral representation*

$$A_t^i = \int_{[0,t]} \lambda_s^i \, ds, \tag{33}$$

then λ_t *is called the* $\mathbf{F}$***-default intensity process****.*

Assumption 51 *We assume that the default intensity process* λ_t *exists for* N_t *and that it is right continuous.*

The next result is central for our multi company dependency default model.

Theorem 52 *Under the above assumption we define for all* $i = 1, ..., S$ *the processes* A_t^i *by (33), where the* λ_t^i *are defined by*

$$\lambda_t^i \stackrel{def}{=} \sum_{n=0}^{S-1} d^{(n+1)}(w, 0, t, i) 1_{\{\tau_n < t \leq \tau_{n+1}\}}, \tag{34}$$

Then $A_{t \wedge \tau_n}$ *is the* $\mathbf{F}$*-default compensator of the default indicator process* $N_{t \wedge \tau_n}$ *for all* $1 \leq n \leq S$*, i.e. for all* $i = 1, ..., S$ *and for all* $n = 1, ..., S$*,* $N_{t \wedge \tau_n}^i - A_{t \wedge \tau_n}^i$ *is an* $\mathbf{F}$*-martingale.*

Proof. For a fixed $t > 0$ and an $\varepsilon > 0$ we can make the following calculation

$$\begin{aligned}
\mathbb{P}\left[\widetilde{\tau}_i \leq t+\varepsilon \mid \mathcal{F}_t\right] \ 1_{\{N_t^i=0\}} &= \mathbb{E}\left[N_{t+\varepsilon}^i - N_t^i \mid \mathcal{F}_t\right] \ 1_{\{N_t^i=0\}} \\
&= \mathbb{E}\left[\sum_{n=0}^{S-1}\left(N_{t+\varepsilon}^i - N_t^i\right) \ 1_{\{\tau_n<t\leq\tau_{n+1}\}} \mid \mathcal{F}_t\right] \ 1_{\{N_t^i=0\}} \\
&= \sum_{n=0}^{S-1}\mathbb{E}\left[N_{t+\varepsilon}^i - N_t^i \mid \mathcal{F}_t\right] \ 1_{\{N_t^i=0\}}1_{\{\tau_n<t\leq\tau_{n+1}\}} \\
&= \sum_{n=0}^{S-1} \ 1_{\{N_t^i=0\}}1_{\{\tau_n<t\leq\tau_{n+1}\}}\int_0^{\varepsilon} d^{(n+1)}(w,u,t,i) \ du.
\end{aligned}$$

Dividing both sides by $\varepsilon > 0$ and letting $\varepsilon \downarrow 0$ we obtain thanks to the existence assumption of λ and the continuity assumption on the densities $d(.)$, on the left side

$$\lambda_t^i = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{P}\left[\widetilde{\tau}_i \leq t+\varepsilon \mid \mathcal{F}_t\right] \ 1_{\{N_t^i=0\}},$$

and on the right side

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\sum_{n=0}^{S-1} \ 1_{\{N_t^i=0\}}1_{\{\tau_n<t\leq\tau_{n+1}\}}\int_0^{\varepsilon} d^{(n+1)}(w,u,t,i) \ du &= \\
\sum_{n=0}^{S-1} 1_{\{N_t^i=0\}} \quad 1 \quad {}_{\{\tau_n<t\leq\tau_{n+1}\}} \ d^{(n+1)}(w,0,t,i).
\end{aligned}$$

Hence we have that

$$\lambda_t^i = \sum_{n=0}^{S-1} \ 1_{\{N_t^i=0\}}1_{\{\tau_n<t\leq\tau_{n+1}\}} \ d^{(n+1)}(w,0,t,i),$$

which concludes the proof. ■

With help of Theorem 52, we can further show the following Corollaries.

Corollary 53 *Under the same assumptions as Theorem 52, the process*

$$A_t = \int_{[0,t]} \lambda_s \ ds, \tag{35}$$

with

$$\lambda_t \overset{def}{=} \sum_{n=0}^{S-1} d^{(n+1)}(w,0,t) \ 1_{\{\tau_n<t\leq\tau_{n+1}\}}, \tag{36}$$

is the **F**-*default compensator of the default indicator process*

$$N_t = \sum_{i=1}^{S} N_t^i. \tag{37}$$

That is, $N_t - A_t$ is an $\mathbf{F}$-martingale. In particular, the default event intensity corresponds to the sum of the company's default intensities λ_t^i

$$\lambda_t = \sum_{i=1}^{S} \lambda_t^i \,. \tag{38}$$

Proof. Following the lines of Theorem 52 and considering that the default event density $d^{(n+1)}(w,x,t)$ corresponds to the sum of the individual company's default densities $d^{(n+1)}(w,x,t,i)$, equation (24).

Or we can prove it by remembering that the finite sum of $\mathbf{F}$-martingales is again an $\mathbf{F}$-martingale. ■

Corollary 54 *At the default time $\tilde{\tau}_i$ of company i, its default intensity λ_t^i vanishes, i.e. $\lambda_t^i = 0$ for $t \geq \tilde{\tau}_i$.*

Proof. Each company i can default just once (see default sequence properties), so at its default time $\tilde{\tau}_i$ its default densities $d^{(k+1)}(w,t,x,i)$ vanishes, thus its default intensity λ_t^i must also vanish. ■

Corollary 55 *If for an $i = 1,...,S$ and for an $n = 1,...,S$ the process $N^i_{t\wedge\tau_n} - A^i_{t\wedge\tau_n}$ is an $\mathbf{F}$-martingale, then*

$$\mathbb{E}\left[\int_{[0,\tau_n]} C_s \, dN_s^i\right] = \mathbb{E}\left[\int_{[0,\tau_n]} C_s \; \lambda_s^i \, ds\right], \tag{39}$$

for all nonnegative $\mathbf{F}$-predictable processes C_t .

Proof. When the process $M_t^i \stackrel{def}{=} N^i_{t\wedge\tau_n} - A^i_{t\wedge\tau_n}$ is an $\mathbf{F}$-martingale, then for all C_t $\mathbf{F}$-predictable processes the process $\int_{[0,\tau_n]} C_s \, dM_s$ is also an $\mathbf{F}$-martingale. Especially we have

$$0 = \mathbb{E}\left[\int_{[0,\tau_n]} C_s \, dM_s^i\right] = \mathbb{E}\left[\int_{[0,\tau_n]} C_s \; (dN_s^i - \lambda_s^i \, ds)\right],$$

and thus (39). ■

In the Appendix 11 we give few example of simplified default models, studying default sequences with limited information about the economy. Moreover in the Section 9.3 on Simulating Multi Default with Feedbacks, we explain, how one can simulate default sequences with default feedbacks.

7 Recoveries and Recovery Rates

7.1 Introduction

The goal of this section is to define and model **recoveries** R and **recovery rates** Z in the case when a company defaults. Recoveries are the second important key ingredient for valuing default prone securities. For this our recoveries R and recovery rates Z are modeled directly under an equivalent martingale measure $\mathbb{P}$. In the business world, investors mean for recovery rates Z the recovered values R over the invested values I

$$Z \stackrel{def}{=} \frac{R}{I}, \tag{40}$$

respectively they also define **losses given default** L, i.e. the losses conditional default by

$$L \stackrel{def}{=} I - R, \tag{41}$$

and their **loss rates** Q corresponding to

$$Q \stackrel{def}{=} \frac{L}{I} = 1 - Z. \tag{42}$$

We show some difficulties of defining recovery rates with an example.

Example 56 *We look at the recovery rates Z of a 3 years corporate loan with face value 100 millions CHF, 10% annual interest rate, which defaulted one month before maturity and which at the end of the transaction recovered 60 millions CHF. Some investors may argue that they have lend 100 millions CHF and that they recovered 60 millions CHF and thus their recovery rate Z corresponds to 60%. Other investors may argue that they have invested 100 millions CHF, which just before default had a value of about 109 millions CHF due to interest and thus their recovery rate Z is about 55%.*

From this loan recovery example we see that recovery rates Z are not well defined by (40), because

1. When have the recoveries R to be evaluated? At the default time τ, or at some future time $T' > \tau$ when the recovered amount is paid?
2. When have the investments I to be evaluated? Do the investments I correspond to the investment value, at the beginning of the contract, just before, at or after the default τ?

These several definition of recovery rates are due because there are different views: accounting, actuary and finance. Unfortunately this definition flexibility of recovery rates Z complicates the modelers life, but in the practical business world it is welcome because of the special recovery rates' nature:

1. Recoveries and recovery rates depend on the seniority, the defaulted company's asset value, local laws, tax rules and the chosen **post default procedure**: either the company is **reorganized** and debt holders receive equities or renegotiated debts;

or the company is **liquidated** and the remaining assets are shared between liability claimants. Unfortunately these two post default procedure causes costs. In fact reorganization costs because of regulator's intervention, consultant's fee, fluctuation in the management, social plans, new financial structuring, etc. Liquidation costs because of possible high write-offs of company/industry specific assets: machines, computers, special installations, etc. or because the company has to sell assets in unfavorable conditions. All these costs, which are more or less proportional to the asset value, are finally paid by the liability holders.

2. Over the world's economy there are different bankruptcy law designs: the US with Chapter 7 (liquidation) and Chapter 11 (reorganization); the British common law and the French, the German and the Scandinavian civil law, etc. For more details about these laws we refer the reader to [FM].

3. Reorganization or liquidation are hidden **procedures** lasting easily months if not years. The junior liability holders have to wait until first all senior debts have been settled. Thus **time value of money** impacts recovery rates, especially when claimants have to wait a long period for their recoveries.

4. The "**priority rules**", describing the settlement order of the different liabilities, make recovery rates more security dependent than from the defaulted company's credit quality. A consequence of these priority rules is that senior debts are settled before junior ones.

5. Recovery rates Z belong to $[0,1]$. There are some rare exceptions like when the assets value is zero and the liability holders still have to pay for lawyer's costs and thus may have negative recoveries and recovery rates $Z \leq 0$, or when the company after being reorganized, its liability are worth more than before the reorganization, i.e. $Z \geq 1$.

6. Equity holders and management have bargaining power in the negotiations of the liabilities, specially when liability claimants believe to recover more in case the company is reorganized than liquidated.

7. At the time bankers make loans they believe their securities will outperform the markets and they hardly imagine the their securities will default, thus it is even more difficult to have at the contracts signing time a precise idea about the recoveries in case of default.

8. **Credit exposure** and **usage given default** are two concepts, which are also not well defined, but used in the definition of recovery rates. These two notions are for bonds or mortgages easy concepts, but when we have a credit risk exposure due to call options on equities or swaps, the story becomes quite complicate, because the credit exposure fluctuates in function of the equity prices or term structure, etc. About these kinds of dynamics of credit exposure the reader finds more information in [Am].

9. Recovery rates also depend on the **security type of the payments**: assets of the defaulted company, risk free treasury bonds, renegotiated bonds, other corporate bonds, equities, cash, etc.

10. In principle, one would like to have information about the recovery payoffs subsequent to the declaration of default. However, this information is at this time rarely available and liability claimants often have to wait.

11. Credit portfolio fund traders have some freedom in announcing losses: for tax purposes they may say that they have lost 60% of their investments, to the fund investors that they have lost 55% of the initial value. The absolute lost amount is the same, but relatively it is not.

12. Human psychology plays an important impact on recovery rate's data, because people expose their gains and hide their losses.

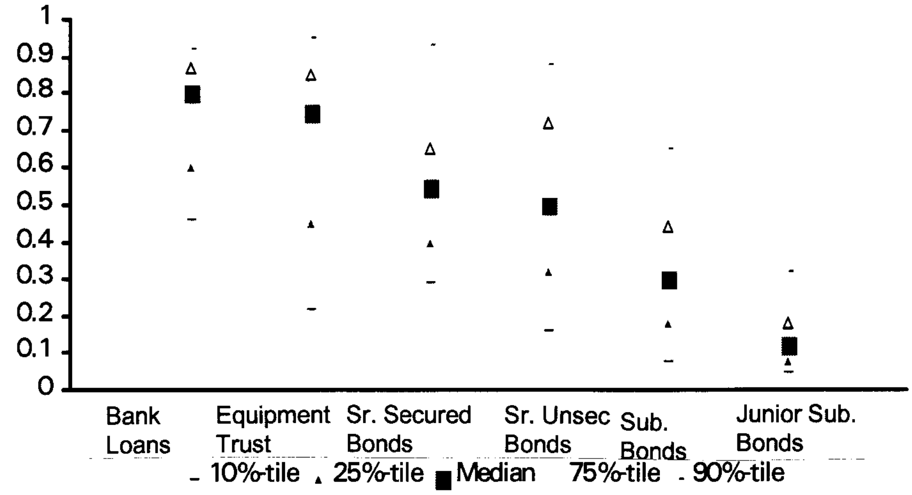
The above listed practical flexibility allows to use different recovery rate definitions in different environments and for different defaulted securities. For example a legal interpretation of recovery at default, assuming liquidation and absolute priority, would impose the same recovery for bonds and liabilities of equal seniority. Thus assuming liquidation, the recovery of face value is probably more realistic. On the other side if we are in a market-to-market evaluation book, then probably the recovery of market value makes more sense.

So far we have seen the advantages of a flexible definition for recovery rates Z but unfortunately the negative side of this flexibility makes the measuring and the modelling of recoveries (rates) difficult tasks.

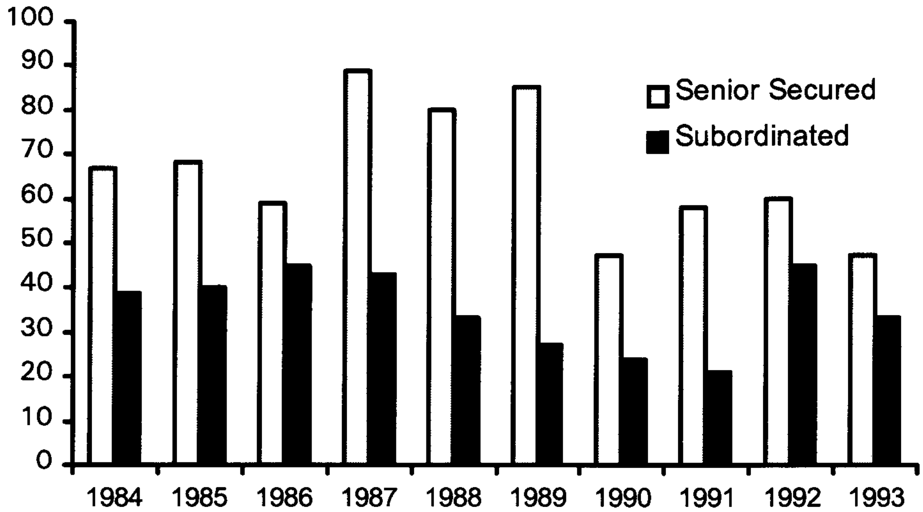
There are not many meaningful statistical data on recovery rates, probably also because losses are usually more hidden than published gains. Econometric analysis on recoveries (rates) show that these behave quite unsystematically. For this see also the following table from [JPM1, p. 78] and [CL], showing a recovery statistics by seniority class with face value 100.

	Carty & Lieberman		Altmann & Kishore	
Seniority Class	Average	Std. Dev.	Average	Std. Dev.
Senior Secured	53.80	26.86	57.89	22.99
Senior Unsecured	51.13	25.45	47.65	26.71
Senior Subordinated	38.52	23.81	34.38	25.08
Subordinated	32.74	20.18	31.34	22.42
Junior Subordinated	17.09	10.90	-	-

More precisely Moody's presents some recovery rate distributions deducted from different defaulted securities and average recoveries (face value 100) by seniority and year (see Figures).



Recovery Rate Distributions by Seniority



Recoveries by Seniority and Year

Because of this variety of different recovery rate definitions, in academics different recovery rate schemes have been studied: zero recovery, fractional recovery of face value or par (FRFV), fractional recovery of market value just before default (FRMV), recovery drawn from a beta distribution, etc. All these models basically assume that the defaulted company is liquidated and that the recovery rates are known at the time of default τ_i, i.e. recoveries and recovery rates are $\mathcal{F}_{\tau_i}$-measurable random variables. This is unfortunately not reality as we have seen from the listed recovery's properties! In the following we briefly present some ideas about our recovery valuation model.

1. From the recovery properties we know that recoveries (rates) are usually not known at the time of default τ_i, but normally just at some later time $T' > \tau_i$. **The recoveries** R and **recovery rates** Z **are therefore not** $\mathcal{F}_{\tau_i}$-measurable **but** $\mathcal{F}_{T'}$-**measurable random variables**, where T' is an $\mathbf{F}$-stopping time.

2. Recoveries R are usually paid within the bankruptcy settlement period $\tau_i \leq T_1 < ... < T_k \leq T'$, but being 85% of the defaulted companies reorganized, having thus a short settlement period (from about 3 to 6 months), we assume that all recoveries R are paid at the same time T', which is also assumed to be an $\mathcal{F}_{\tau_i}$-measurable random variable.

3. When at time τ_i company C_i defaults, the **default risk** on these securities **vanish**! After τ_i these securities are thus just affected by the common **recovery risk**. Since the different liability securities on the same defaulted company C_i are still on the markets, we expect because of **no-arbitrage** that some relation between the prices of defaulted securities must exist. This relation depends from the defaulted security payoffs, the company's asset portfolio, the market asset values, the asset liquidation strategy and from the priority rules. We thus expect a relation between risk neutral expected values of recoveries, corresponding to their risk neutral prices observed on the markets. Under some assumption recoveries R can be interpreted as T' **claims** and thus be priced.

The program of this section is therefore to model in a first step recoveries R as a function of the remaining asset value, priority rules, etc., then in a second step we develop a recovery valuation model, and finally comparing the recovery values against the investments I, we give some possible definitions of recovery rates Z.

7.2 Recoveries

We introduce again a complete filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ constructed in the same way and with the same properties as presented in Section 6.4 and satisfying the "usual hypothesis". For valuing recoveries we will be working in an arbitrage-free setting and thus we will consider from the beginning the behavior of the involved processes directly under an equivalent martingale measure $\mathbb{P}$. For studying recoveries R we have to study what occurs after the default τ_i of a company C_i.

The liabilities of company C_i can be divided into two classes: the liabilities $L_1, ..., L_l$ (L_i is the outstanding CHF exposure of the liability security i) financed by third parties and the equities E belonging to the company's owner. Classically we have the balance between asset values A and liability values $L \overset{def}{=} \sum_{k=1}^{l} L_k$

$$A = E + L. \tag{43}$$

Notation 57 *L^i denotes the set of all liabilities of the defaulted company C_i.*

In the recovery's properties list we have seen that at time τ_i when the default of company C_i is announced, investors usually do not know their recoveries R, but that they have **expectations** about the "recovery risks":

1. The post default procedure that regulators, liability holders, equity holders and management will apply within the bankruptcy settlement period;

2. The company's asset portfolio and its final sharing value;

3. The company's liability structure with corresponding settlement priorities;

4. The recoveries settlement end;

5. etc.

Investors have thus an **expectation about recoveries** R, but we again point out, they do not know the exact recoveries yet. These will be known, when this bankruptcy case is settled, at the **F**-stopping time $T' > \tau_i$. We therefore model recoveries R as $\mathcal{F}_{T'}$**-measurable random variables,** allowing an $\mathcal{F}_{\tau_i}$-expectation

$$\bar{R}_{\tau_i} \stackrel{def}{=} \mathbb{E}\left[R \mid \mathcal{F}_{\tau_i}\right]. \tag{44}$$

Remark 58 *We do not model recoveries as $\mathcal{E}_{T'}$-measurable random variables because the defaulted company C_i may have liability assets of other default prone companies $k \in \mathcal{S}^i$, which might default within the bankruptcy settlement period of C_i. The filtration $\mathcal{E}_t$ is therefore not sufficient for modelling recoveries R since it does not "see" defaults of other companies and its consequences on the defaulted asset's values.*

At this point we see two possibilities for modelling recoveries, first that we simply **assume** the recoveries R to be exogenously given as $\mathcal{F}_{T'}$-measurable random variables, or second that we endogenously model the recoveries R, considering the asset and liability structure of the company C_i, the local priority rules, etc.

We will see that for pricing purposes the first option is sufficient, but for giving more insights of practical post bankruptcy procedures, we model in what follows, recoveries R endogenously. The method we will apply is that at the bankruptcy's conclusion T', we know ex post everything about the settlement of this bankruptcy case. So starting from the end condition (recovery payoffs) at time T' and modelling backwards we will construct the expected recoveries at the default time τ_i. We make the conventions that

Assumption 59 *At the default time τ_i*

(L1) *Investors know all third party liabilities $L_1, ..., L_l$ of company C_i, i.e. $L_1, ..., L_l$ are $\mathcal{F}_{\tau_i}$-measurable;*

(L2) *All liability securities $L_1, ..., L_l, E$ on company C_i default simultaneously;*

(L3) *Investors know the settlement's end time T', i.e. T' is $\mathcal{F}_{\tau_i}$-measurable;*

(L4) *Investors know that all liabilities $L_1, ..., L_l, E$ are settled at time T' in cash.*

We believe that these are reasonable assumptions, because

(L1) All investors have the information that company C_i defaulted at time τ_i. They thus announce their lent exposure L_i towards this company, such that they recover as fast as possible their investments.

(L2) At time τ_i company C_i defaults and thus the default risk on all liability securities of C_i disappears.

(L3) [FM] observed that about **85%** of the defaulted companies are **reorganized** and about 15% liquidated. A reorganization with renegotiation of the company's liabilities mostly lasts few months, usually corresponding to the protection period of Chapter 11. So for 85% of defaulted companies we know at default when its liability securities have to be renegotiated. Also in case of liquidation regulators and liability claimants try to conclude as fast as possible.

(L4) Liability claimants might receive at the settlement time T' not only cash but also other assets like physical assets (real estates, cars, furniture, computers, ...), other corporate bonds, government bonds, equities or new issued debts in case the company is reorganized. All assets that liability holders receive have an intrinsic value, which cannot differ too much from a "reasonable" market value. If this is not the case other liability claimants will complain. In fact let's suppose that two liability holders have leaned the same amount with the same seniority. Let's further suppose that the two lenders recover different asset types. As a consequence of the priority rule both lenders should recover more or less the same value. So for example when the recovered assets of the first lender are worth more than the assets of the second claimant, then the second will probably run to the court and complain.

A key ingredient for modelling recovery payoffs is to know the asset portfolio of the company and its corresponding sharing value $A_{T'}$ at the settlement date T'. Unfortunately this value depends whether the company is reorganized or liquidated. In case the company is liquidated the owners might have to pay for social plans, sell assets in unfavorable market conditions, might have special and high cleaning and closing costs of polluted fabrics, etc. For example closing a chemical company probably costs much more than closing a book store or an internet software company. While when the company is reorganized there are some reorganization costs, like consultant's or lawyer's fees.

When the closing asset values are substantially lower than in the case of a reorganization, there will be a bargaining regret game between different players with different interests:

1. **Third liability holders** want to recover as much and as fast as possible their investments. They are minimizing their losses, since they have already lost and might lose more.

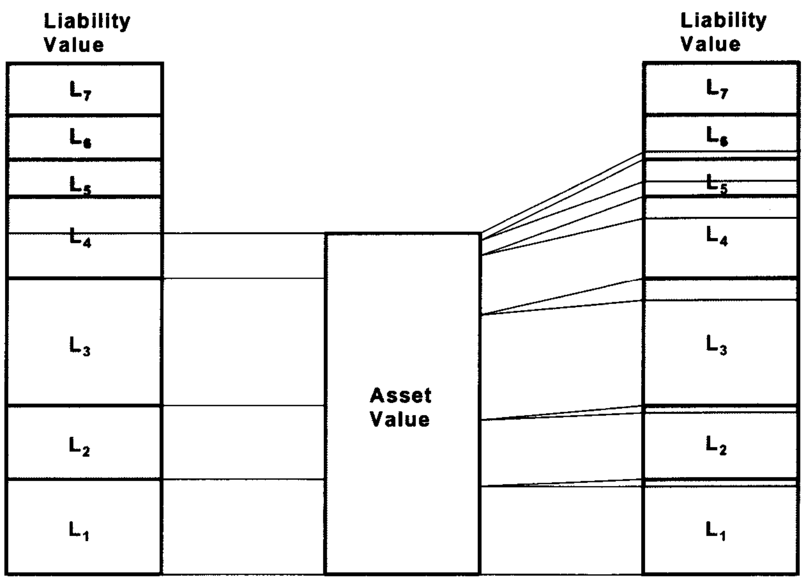
2. **Equity holders**, being the legal owner of the company, have lost everything since equities are the most junior liabilities. They have nothing to lose anymore, but immense gaining potentials. They might inject new capital in case they see good business opportunities in reorganizing the company. They are thus maximizing their gains.

3. **Management** has the business know-how. They might be able to reorganize the company to become profitable. Usually they have not lost much capital but they might lose their jobs.

Example 60 *The liabilities of a defaulted company were financed by outstanding third party's debt 90 millions CHF and equities. The asset portfolio value is expected to be worth in case the company is liquidated to 40 millions CHF and when it is reorganized* *to 80 millions CHF. This high difference in asset value gives to the management and to the equity holders a special bargaining power because they can propose for example that they will reorganize the company, inject further 40 millions CHF of new equities and that the liability claimants receive new liabilities valuing 60 millions CHF of the reorganized company.*

Remark 61 *Equity holders and management have similar interests, but not necessarily the same ones. Therefore in a renegotiation conflict generally one has equity holders and management against the third liability holders.*

The higher the difference between liquidation and reorganization value is, the more bargaining power the equity holders and the management have for proposing a reorganization with a discount D on their liability L payments, i.e. the remaining sharing asset's value corresponds to $L - D$. The liability claimants have to accept a write-down of $\frac{D}{L}$ on their total exposure L. From the below Figure we see that there are different methods (mappings) how to share the remaining asset value $L - D$ between the outstanding third party's liabilities $L_1, ..., L_l$.



Remaining Asset value Allocation

The write-down is not equal $(\frac{D}{L})$ for all liabilities, but the "priority rules" define how the remaining value $L - D$ is shared between the different liabilities $L_1, ..., L_l$. These "**priority rules**", specifying the **settling order** between the liabilities $L_1, ..., L_l$ with respective **settlement degrees** $s_1, ..., s_l$, have been installed for avoiding conflicts between liability holders because all liability claimants want recover as much as possible.

Example 62 *Senior debts are settled before junior ones; secured before unsecured.*

Example 63 *Salaries (including management or consultant fees) and social contribution are first settled, then 90% of outstanding taxes; then 85% of senior secured bonds, then 70% of senior unsecured bonds, 50% of junior debts, etc.*

The concept of order for liabilities belonging to the same recovery risks can be formally defined.

Definition 64 *The* ***ordering relation*** $\succcurlyeq$ *for two securities* $L_1, L_2 \in L^i$ *with corresponding settlement degrees* $0 \leq s_2 \leq s_1 \leq 1$ *is defined by*

$$L_1 \succcurlyeq L_2,$$

$$\Updownarrow$$

Liability L_2 *is payed out after liability* L_1 *has been settled for* $s_1 * L_1$.

Definition 65 *We call* $(L^i, \succcurlyeq)$ *an* ***absolute ordering system,*** *when for all* L_1, L_2 *and* $L_3 \in L^i$ *the following conditions are satisfied*

1. $\succcurlyeq$ *is reflexive:* $L_1 \succcurlyeq L_1$
2. $\succcurlyeq$ *is transitive:* $L_1 \succcurlyeq L_2$ *and* $L_2 \succcurlyeq L_3 \Rightarrow L_1 \succcurlyeq L_3$
3. $\succcurlyeq$ *is identical:* $L_1 \succcurlyeq L_2$ *and* $L_2 \succcurlyeq L_1 \Rightarrow L_1 = L_2$
4. $\succcurlyeq$ *is totally ordered: either* $L_1 \succcurlyeq L_2$ *or* $L_2 \succcurlyeq L_1$ *holds.*

Since the "priority rules" are in practice more or less applied, we make the following assumption.

Assumption 66 *At the default time* τ_i *we know the settlement order* $L_1 \succcurlyeq L_2 \succcurlyeq ... \succcurlyeq L_l$ *with corresponding settlement degrees* $s_1 \geq s_2 ... \geq s_l$.

Definition 67 *We say* ***absolute priority rule*** *when for the liabilities* $L_1 \succcurlyeq L_2 \succcurlyeq ... \succcurlyeq L_k \succcurlyeq ... \succcurlyeq L_l$ *we have* $s_1 = s_2 = ... = s_{k-1} = 1 \geq s_k \geq s_{k+1} = ... = s_l = 0$.

At time T' knowing all liabilities $L_i \in L^i$ with corresponding priority order and settlement degrees and knowing further the sharing asset value $L' \stackrel{def}{=} L - D$, we can recursively represent the recovery payoffs R_i for the liabilities L_i, $i = 1, ..., l$ by

$$\begin{aligned} R_i(L') &= \min\left\{L_i s_i,\ L' - \sum_{j<i} R_j(L')\right\} \\ &= L' - \sum_{j<i} R_j(L') - \left(L' - \sum_{j\leq i} L_j s_j\right)^+, \end{aligned} \tag{45}$$

which correspond in case of absolute priority to

$$\begin{aligned} R_i(L') &= \min\left\{L_i,\ \left(L' - \sum_{j<i} L_j\right)^+\right\} \\ &= L' - \sum_{j<i} R_j(L') - \left(L' - \sum_{j\leq i} L_j\right)^+. \end{aligned} \tag{46}$$

In the absolute priority representation (46) the defaulted security payoffs $R_i(L_{T'})$ can be synthetically built by one long asset L', short all liabilities of higher priority and short a call with strike all liabilities with higher and equal priority.

7.3 Recovery Valuation

The valuation of default prone securities with respective martingale pricing introduction and motivations will be presented in Section 8.3. Theoretically this recovery valuation subsection belongs to Section 8, but we anticipate it because it is the natural extension of the previous Subsection 7.2 and it will be the natural continuation for the next section, when defining recovery rates. In default prone securities markets also defaulted securities are present, since most defaulted securities maintain also a value after default. Default prone securities are affected by default and recovery risks, while defaulted securities are left with recovery risks. Some properties and phenomena about these defaulted securities markets are described by [Kj] and [Wh].

For pricing purposes instead of describing the recovery's dynamics as proposed in the previous Subsection 7.2 one can simply assume the recoveries R to be $\mathcal{F}_{T'}$-measurable random variables.

Since the liability securities of company C_i are after default $t \geq \tau_i$ still dealt on the markets, we assume that the no arbitrage assumption still yields, what is equivalent to the assumption of existence of a martingale probability measure $\mathbb{P}$. The following intuitive result must hold: the remaining expected sharing value $(L' = L - D)$ corresponds to the expected recovery values.

Proposition 68 *Under Assumption 59 and 66 and assuming no arbitrage, then for $t \in [\tau_i, T']$ we have*

$$\begin{aligned} \mathbb{E}\left[\delta^r_{t,T'} L' \mid \mathcal{F}_t\right] &= \mathbb{E}\left[\delta^r_{t,T'} \sum_{j \leq l} R_j\left(L'\right) \mid \mathcal{F}_t\right] \\ &= \sum_{j \leq l} \mathbb{E}\left[\delta^r_{t,T'} R_j\left(L'\right) \mid \mathcal{F}_t\right], \end{aligned} \tag{47}$$

where $\delta^r_{t,T'} \overset{def}{=} \exp\left(-\int_t^{T'} r_s \, ds\right)$ (see (63) in Section 8.3) and $R_j\left(L'\right)$ as in (45).

Remark 69 $\mathbb{E}\left[\delta^r_{t,T'} R_j\left(L'\right) \mid \mathcal{F}_t\right]$ *represents the recovery value at time t of the liability security L_j.*

Proof. The left side of the equality 47 represents the expected value at time t of the amount $L - D$ to return at time T' to the third party's liability. On the right side we have the expectation how the remaining sharing value $L - D$ is shared between the liabilities $L_k \in L^i$. Both amounts have to correspond since there is no arbitrage. ∎

Remark 70 *When under the martingale measure $\mathbb{P}$ the recovery payoffs $R_j\left(L'\right)$ are independent from the risk free short rate r then we have further*

$$\begin{aligned} \mathbb{E}\left[\delta^r_{t,T'} R_j\left(L'\right) \mid \mathcal{F}_t\right] &= \mathbb{E}\left[\delta^r_{t,T'} \mid \mathcal{F}_t\right] \, \mathbb{E}\left[R_j\left(L'\right) \mid \mathcal{F}_t\right] \\ &= P(t,T')\mathbb{E}\left[R_j\left(L'\right) \mid \mathcal{F}_t\right]. \end{aligned} \tag{48}$$

From equation (48) we clearly see that a defaulted security has principally two uncertainties influencing their values: the recovered amount and the time when the recoveries are paid.

The recovery procedure description of the last subsection shows that the remaining wealth sharing regret game is quite complex. The value of the recovery payoffs $\mathbb{E}\left[\delta^{r}_{t,T'} R_j\left(L'\right) \mid \mathcal{F}_t\right]$ can be simpler represented when further assuming

Assumption 71 *1. Constant risk free rate r.*

2. Asset prices A_t are driven by a diffusion process with constant parameters

$$dA_t = A_t\left(r\ dt + \sigma_A\ dW_t\right). \tag{49}$$

3. Recoveries are paid after a fixed period after default $T' \overset{def}{=} \tau_i + \delta$, where δ is an $\mathcal{F}_{\tau_i}$-measurable random variable.

4. Discount D on the sharing asset value corresponds to $D \overset{def}{=} cA_{T'}$, with c an $\mathcal{F}_{\tau_i}$-measurable random variable.

5. Absolute priority: $L_1 \succeq ... \succeq L_l$ (see Definition 67).

Remark 72 *The discount D is assumed to be proportional to the final asset value because it is thought as transactions, liquidations, lawyers or consultants costs. The final sharing asset value amounts therefore to $L'_{T'} \overset{def}{=} (1-c)\ A_{T'}$.*

We have the following result.

Theorem 73 *Under the Assumptions 59 and 71 and no-arbitrage, the recovery payoffs at time T' for liability L_i can be recursively calculated by*

$$\begin{aligned} R_i\left(L'_{T'}\right) &= \min\left\{L_i,\ \left(L'_{T'} - \sum_{j<i} L_j\right)^+\right\} \\ &= L'_{T'} - \sum_{j<i} R_j\left(A'_{T'}\right) - \left(L'_{T'} - \sum_{j\leq i} L_j\right)^+. \end{aligned} \tag{50}$$

Furthermore its recovery value at time $t \in [\tau_i, \tau_i + \delta]$ corresponds recursively to

$$\mathbb{E}\left[\delta^{r}_{t,T'} R_i\left(L'_{T'}\right) \mid \mathcal{F}_t\right] = BS_t\left[i-1\right] - BS_t\left[i\right] \tag{51}$$

where $BS_t[i]$ is standing for Black-Scholes formula (see [LL, p. 70]) with different strikes $K_i \overset{def}{=} \sum_{j\leq i} L_j$

$$BS_t\left[i\right] \overset{def}{=} BS_t\left[L'_t, \sigma_A, K_i, r, \tau_i + \delta - t\right] \tag{52}$$

and

$$BS_t\left[0\right] \overset{def}{=} L'_t.$$

Remark 74 *The recovery values of liability L_i corresponds according to (51) to the value of a long call on L'_t with strike $\sum_{j\leq i-1} L_j$ and short a call with strike $\sum_{j\leq i} L_j$. In an arbitrage-free market one can therefore syntectical construct all these liability securities.*

Remark 75 *Proposition 68 can be seen as a consequence of Theorem 73, because the sum of the recovery values is a telescopic sum corresponding to* $(1-c)\ A_{T'} - BS_t\,[l]$*, where* $BS_t\,[l]$ *is the equity's value at time* t.

Remark 76 *This simple recovery model is very similar to the Merton model ([Mr]), but we want to stress that although the results look very similar, the model is essentially different. In his model Merton values the credit risk of a corporate debt with a certain maturity* T*, when the debt is returned financed by the assets. This is a very strong assumption, that is hardly satisfied. In our recovery model the company is already defaulted and at time* T' *the liabilities are actually paid in cash or other physical assets.*

Remark 77 *We assumed that the liabilities are paid after a fixed time after default* $T' = \delta + \tau_i$*. Similarly as for default times one can relax this assumption and allow* T' *to have a distribution with density* $f(x)$*. The value of the liability* L_i *looks then like*

$$\mathbb{E}\left[\int_t^{\infty} \delta^r_{t,T'}\ f(T')\ R_i\left(L'_{T'}\right)\ dT' \mid \mathcal{F}_t\right] = \int_t^{\infty} f(T') \left\{ \begin{array}{c} BS_t\left[L'_t, \sigma_A, K_{i-1}, r, T'-t\right] \\ -BS_t\left[L'_t, \sigma_A, K_i, r, T'-t\right] \end{array} \right\} dT'. \tag{53}$$

Proof. For proving Theorem 73 we use the payoff recovery representations (46) and calculate their values at time $t \in [\tau_i, \tau_i + \delta]$ recursively by

$$\begin{aligned} \mathbb{E}\left[\delta^r_{t,T'} R_j\left(L'_{T'}\right) \mid \mathcal{F}_t\right] &= \mathbb{E}\left[\delta^r_{t,T'}\left\{L'_{T'} - \sum_{j<i} R_j\left(L'_{T'}\right) - \left(L'_{T'} - \sum_{j\le i} L_j\right)^+\right\} \mid \mathcal{F}_t\right] \\ &= L'_t - \sum_{j<i} \mathbb{E}\left[\delta^r_{t,T'} R_j\left(L'_{T'}\right) \mid \mathcal{F}_t\right] - \mathbb{E}\left[\delta^r_{t,T'}\left(L'_{T'} - \sum_{j\le i} L_j\right)^+ \mid \mathcal{F}_t\right], \end{aligned}$$

denoting the last term

$$\mathbb{E}\left[\delta^r_{t,T'}\left(L'_{T'} - \sum_{j\le i} L_j\right)^+ \mid \mathcal{F}_t\right] \overset{def}{=} BS_t\left[L'_{T'}, \sigma_A, \sum_{j\le i} L_j, r, +\delta - t\right],$$

where $BS_t\,[S, \sigma_S, K, r, T-t]$ is standing for Black and Scholes formula (see [LL, p. 70]). Recursively the most senior liability security L_1 has the value at time $t \in [\tau_i, T']$

$$\begin{aligned} \mathbb{E}\left[\delta^r_{t,T'} R_1\left(L'_{T'}\right) \mid \mathcal{F}_t\right] &= \mathbb{E}\left[\delta^r_{t,T'}\left\{L'_{T'} - \left(L'_{T'} - L_1\right)^+\right\} \mid \mathcal{F}_t\right] \\ &= L'_t - \mathbb{E}\left[\delta^r_{t,T'}\left(L'_{T'} - L_1\right)^+ \mid \mathcal{F}_t\right] \\ &= L'_t - BS_t\left[L'_{T'}, \sigma_A, L_1, r, \tau_i + \delta - t\right] \\ &= L'_t - BS_t\,[1]\,. \end{aligned}$$

Using the fact that we know the value of L_1 we can evaluate L_2 by

$$\begin{aligned} \mathbb{E}\left[\delta^r_{t,T'} R_2\left(L'_{T'}\right) \mid \mathcal{F}_t\right] &= \mathbb{E}\left[\delta^r_{t,T'}\left\{L'_{T'} - \left(L'_{T'} - \left(L'_{T'} - L_1\right)^+\right) - \left(L'_{T'} - L_1 - L_2\right)^+\right\} \mid \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\delta^r_{t,T'}\left\{\left(L'_{T'} - L_1\right)^+ - \left(L'_{T'} - L_1 - L_2\right)^+\right\} \mid \mathcal{F}_t\right] \\ &= BS_t\,[1] - BS_t\,[2]\,. \end{aligned}$$

The remaining liability recovery values can be recursively calculated in a similar way. ■

Remark 78 *In our simple recovery valuation model we have considered the company's asset value A_t to be driven by one diffusion process (49). The result can be extended allowing A_t to be driven like in an APT model by a linear combination of several diffusion processes of the type (49). This multi factor setup is then similar to basket options, which pricing theory (see [MR, pp. 222-225]) can be adapted for obtaining approximations of the recovery values.*

7.4 Recovery Rates

For calculating the recovery rates Z, according to (40), we need to know not only the recoveries R (values) but also we need to define according to what (I of equation (40)) we compare these recoveries. For this purpose we introduce several exposure definitions.

Definition 79 *The lent amount (in CHF) of a loan is called* ***face value*** *(FV).*

Example 80 *The face value of a zero-coupon bond corresponds to the size (in CHF) of the payment at maturity.*

Since all security's payoffs can be constructed as a linear combination of zero coupon bonds with different maturities, we can define for all credit sensitive securities a credit exposure.

Definition 81 *The* ***credit exposure*** *(CE) of a security corresponds to the amount of* ***loan equivalents*** *(LE), where 1 LE is equivalent to a loan credit exposure of face value 1 CHF.*

Example 82 *Coupon bonds make coupon payments of a given fraction of face value at equally spaced dates up to the maturity, when the face value is also returned. The credit exposure of a coupon bond corresponds to its face value and the outstanding interest payment of the current interest period.*

Example 83 *At maturity T of a call C with strike K on a share S, the call issuing bank defaults. The CE of this call C corresponds then to the call price $CE = C_T = (S_T - K)^+$. We observe that the credit exposure depends on the underlying share price S, so for $t \leq T$ we have a credit exposure dynamics.*

Remark 84 *By definition the face value FV is smaller or equal to the credit exposure CE:*

$$FV \leq CE. \tag{54}$$

With these two definitions of exposures (FV and CE) and recoveries (R) we finally define recovery rates (Z).

Definition 85 *The* ***fractional recovery rate of face value*** *is the $\mathcal{F}_{T'}$-measurable variable*

$$Z^{FV} \overset{def}{=} \frac{\delta^r_{\tau_i,T'} R}{FV_{\tau_i}}, \tag{55}$$

and the $\mathcal{F}_{T'}$-measurable ***fractional recovery rate of credit exposure*** *is*

$$Z^{CE} \overset{def}{=} \frac{\delta^r_{\tau_i,T'} R}{CE_{\tau_i}}. \tag{56}$$

We discount the recoveries R back to the default time τ_i for comparing recoveries and exposed amount (FV or CE) at the same time τ_i. This is equivalent, supposing that at default we would receive the amount $\delta^r_{\tau_i,T'}R$ of risk free assets, which are worth R at time T'. More important for pricing purposes are the risk neutral expectations, which are defined for $\tau_i \leq t$ by

Definition 86 *The* ***expected fractional recovery rate of face value*** *corresponds to*

$$\bar{Z}_t^{FV} \stackrel{def}{=} \mathbb{E}\left[Z^{FV} \mid \mathcal{F}_t\right] = \frac{\mathbb{E}\left[\delta^r_{\tau_i,T'}R \mid \mathcal{F}_t\right]}{FV_{\tau_i}}, \tag{57}$$

and the ***expected fractional recovery rate of credit exposure*** *is defined as*

$$\bar{Z}_t^{CE} \stackrel{def}{=} \mathbb{E}\left[Z^{CE} \mid \mathcal{F}_t\right] = \frac{\mathbb{E}\left[\delta^r_{\tau_i,T'}R \mid \mathcal{F}_t\right]}{CE_{\tau_i}}, \tag{58}$$

FV_{τ_i} and CE_{τ_i} are $\mathcal{F}_{\tau_i}$-measurable and thus we can take these quantities out from the risk neutral expectation's operator. Further satisfying $\delta^r_{\tau_i,T'} = \delta^r_{\tau_i,t}\delta^r_{t,T'}$ and being $\delta^r_{\tau_i,t}$ $\mathcal{F}_t$-measurable we have

$$\bar{Z}_t^{FV} = P(\tau_i,t)\ \frac{\mathbb{E}\left[\delta^r_{t,T'}R \mid \mathcal{F}_t\right]}{FV_{\tau_i}}. \tag{59}$$

When we have independence of R and r under the risk neutral martingale measure $\mathbb{P}$, then we have further

$$\bar{Z}_t^{FV} = \frac{P(\tau_i,T')}{FV_{\tau_i}}\mathbb{E}\left[R \mid \mathcal{F}_t\right],$$

where R is the only unknown, because the price of $P(\tau_i,T')$ comes from a market.

The expected face value recovery rate can be simply represented by using the recovery model described in Theorem 73.

Corollary 87 *Under the same assumptions of Theorem 73, for liability L_i the expected fractional recovery rate of face value corresponds to*

$$\bar{Z}_t^{FV}(i) = P(\tau_i,t)\frac{BS_t\left[L'_t,\sigma_A,K_{i-1},r,\tau_i+\delta-t\right] - BS_t\left[L'_t,\sigma_A,K_i,r,\tau_i+\delta-t\right]}{L_i}, \tag{60}$$

where $BS_t\,[.]$ is described in Theorem 73 and $K_i = \sum_{j\leq i} L_j$.

Proof. Combining the $\bar{Z}_t^{FV}$ representation (59), $FV_{\tau_i}(i) = L_i$ and (51) of Theorem 73 we obtain (60). ■

If we have a market-to-market evaluation book, it is more convenient to compare the (expected) security's recoveries R $(\overline{R})$ directly to the securities market values $MV_{\tau_{i-}}$ just before default τ_{i-}.

Definition 88 *The (****expected****)* ***fractional recovery rate of market value*** *is defined to be the $\mathcal{F}_{T'}$-measurable variable*

$$Z^{MV} \stackrel{def}{=} \frac{\delta^r_{\tau_i,T'}R}{MV_{\tau_{i-}}}, \tag{61}$$

respectively

$$\bar{Z}_t^{MV} \stackrel{def}{=} \frac{\mathbb{E}\left[\delta^r_{\tau_i,T'}R \mid \mathcal{F}_t\right]}{MV_{\tau_{i-}}}. \tag{62}$$

When one studies the valuation of default prone securities in this fractional recovery rate of market value setting, one will soon studying Forward Backward Stochastic Differential Equations (FBSDE), because the recovery rates Z, depending on the price of the security just before default, influence the price of the security itself. [DSS] showed how the BSDE pricing equation is composed and showed the existence of a solution.

7.5 Special Recoveries and Recovery Rates

When investors know the recoveries R (rates Z) at the contract's issuing date and in some other special cases stronger assumption on recovery (rates) can be imposed, like

Assumption 89 *The recoveries R (rates Z) are $\mathcal{E}_0$-measurable random variables.*

Or in case we do not expect any recovery $R = 0$ for a certain junior unsecured asset we can simply impose.

Assumption 90 *Complete loss in case of default: $Z = 0$.*

Assumption 90 is the easiest case since no uncertainty about the default payoff or the payoff time is left, since there is no payoff. Assumption 90 can be very reasonable for example for default digital puts, where 1 CHF is paid at maturity T if the counterpart defaults.

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8 Default & Recovery Valuation in Incomplete Markets

8.1 Introduction

In the credit risk characteristic's Section 4, we have seen that credit related securities are typically dealt in **incomplete markets**. For the valuation of default prone securities we identified in the introduction Section 3 different modelling key steps:

1. Development of a market consistent multi company default time model, describing the default date and the company that defaults.

2. Development of a recovery model in case of default.

3. Embed default and recovery models into risk neutral martingale pricing model, where prices are evaluated as expectations of discounted cash flows.

4. Calibrate a risk neutral martingale measure by comparing observed market and model prices.

Credit related securities may be split into three main asset classes: default-free, default prone and finally defaulted securities. Defaulted asset valuation has been analyzed in the recovery valuation Section 7.3. Further under an equivalent martingale measure $\mathbb{P}$ we have developed a multi company default and a recovery model. We need therefore to motivate that these default prone security's markets are free of arbitrage (Section 8.2) for embedding the multi default and the recovery model into a martingale pricing framework. For achieving this we first recall some basics about default-free martingale pricing (Section 8.3), which will then be extended to a default prone pricing model for giving a martingale representation of default prone securities (Section 8.4).

Following the valuation scheme presented in Figure 1 of Section 3, we give in Section 8.6 some ideas of how to "invert the default term structure" by comparing market prices with model prices. Using the no-arbitrage concept we filter out some information about the credit qualities, the default and recovery premiums and thus extract some information about the market used martingale measure $\mathbb{P}$. Finally in Section 8.7 we want to describe equivalent martingale measure transformations $\widetilde{\mathbb{P}} \rightarrow \mathbb{P}$.

8.2 Existence of a Martingale Measure

As presented in the multi company default model Section 6, living business men daily phenomena of credit related markets, split intuitively credit risks into two components of different natures: **systematic** and **unsystematic** risks. The credit risk dependencies between firms, caused by some common underlying systematic risk factors, can be observed in credit related markets:

1. Clustering of defaults: [Wt] and [HK] showed that in some periods there are more defaults than in others, depending on the states of the economic cycles.

2. Companies with similar rating which are belonging to similar country and/or industry sectors have similar credit spreads dynamics.

3. Defaulted liability security's prices satisfy some natural internal price relation.

These market properties and the studies of [Wt] and [HK] indicate the presence of a **substantial systematic credit risk component**.

With these properties (1.-3.) and considering how practitioners build their default models and that markets usually pay a premium only for the systematic risks, we conclude that we expect an internal relationship between the various credit sensitive security's market prices for avoiding arbitrage. The arbitrage-free assumption is heuristically equivalent to the existence of an equivalent martingale measure $\mathbb{P}$. For a precise statement about this equivalence the reader is referred to [DS].

8.3 Martingale Valuation of Default Free Securities

In this section we describe the default free securities available in the markets and their martingale valuation technology. Since in this section we do not want to address the transformation from the physical measure $\widetilde{\mathbb{P}}$ to an equivalent risk martingale measure $\mathbb{P}$ we simply **assume** that

Assumption 91 *$\mathbb{P}$ is a risk neutral martingale measure.*

Definition 92 *The **space of securities** is the vector space $\mathcal{S}$ of càdlàg, adapted, finite variation processes.*

Definition 93 *The space of **marketed securities** is a subspace $\mathcal{M}$ of the vector space $\mathcal{S}$ of all securities, with a linear mapping $\Pi : \mathcal{M} \subset \mathcal{S} \to \mathcal{P}$, the vector space of optional processes. For each marketed security $M \in \mathcal{M}$, $\Pi(M)$ is called its **price process**.*

Definition 94 *r denotes the **default-free spot, short, discount-free or instantaneous interest rate process**, which is assumed to be a positive, right-continuous, optional and integrable process on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$.*

Definition 95 *The **discount process** $\delta^r_{t,T}$ is the continuous process defined by*

$$\delta^r_{t,T} \overset{def}{=} \exp\left(- \int_{[t,T]} r_s \, ds \right). \tag{63}$$

Remark 96 *The **bank account** or the **capitalization rate** is specified by the process*

$$B_t \overset{def}{=} \frac{1}{\delta^r_{0,t}},$$

satisfying the stochastic differential equation $dB_t = B_t r_t \, dt$, with initial condition $B_0 = 1$.

Definition 97 *The price of a **default-free zero-coupon bond** $P \in \mathcal{M}$ of maturity T corresponds to*

$$P(t,T) \overset{def}{=} \mathbb{E}\left[\delta^r_{t,T} \mid \mathcal{F}_t \right], \tag{64}$$

where $\mathbb{E}$ is the expectation operator under an equivalent risk neutral equivalent martingale measure $\mathbb{P}$ and $\mathcal{F}_t$ is the filtration as described in the multi company default Section 6.4.

Remark 98 *When* r *is deterministic then (64) reduces to* $P(t,T) = \delta^r_{t,T}$.

Under the equivalent risk neutral martingale measure $\mathbb{P}$ for all marketed securities $M \in \mathcal{M}$ we have that

$$M + \Pi(M)$$

is an ($\mathbb{P}$,$\mathbf{F}$)-martingale, i.e. for all $0 \leq t \leq T$

$$\mathbb{E}\left[M_T + \Pi(M)_T \mid \mathcal{F}_t\right] = M_t + \Pi(M)_t.$$

The price process for a security $M \in \mathcal{M}$, stopping paying dividends at time T (thus $\Pi(M)_T = 0$), can then be represented by

$$\Pi(M)_t = \mathbb{E}\left[M_T - M_t \mid \mathcal{F}_t\right].$$

More specifically the price process $\Pi(M)_t$ of a security M with final payment M_T and cumulative dividend process D_s (satisfying technical conditions) can be represented for all $0 \leq t \leq T$ by

$$\Pi(M)_t = \mathbb{E}\left[\int\limits_{[t,T]} \delta^r_{t,s}\, dD_s + \delta^r_{t,T}\, M_T \mid \mathcal{F}_t\right]. \tag{65}$$

We will apply this pricing technique to default prone securities.

8.4 Valuation of Default Prone Securities

The valuation of default prone securities is the final goal. We have built in the previous sections a multi company default model, a recovery model and we have described the martingale pricing technique. In this section we will embed these model components into a martingale pricing model for obtaining a "consistent" default prone securities valuation model. The martingale pricing techniques have to be applied to default prone security's cash flows, which are conditional to the credit qualities of the underlying companies. We define

Definition 99 *A* ***marketed default prone security*** $S \in \mathcal{M}$ *is a triple* (D, X, R), *where*

D represents the ***set of companies****, whose defaults may impact the payoffs of* S*;*

X stands for the ***promised payoffs*** *in case of no defaults;*

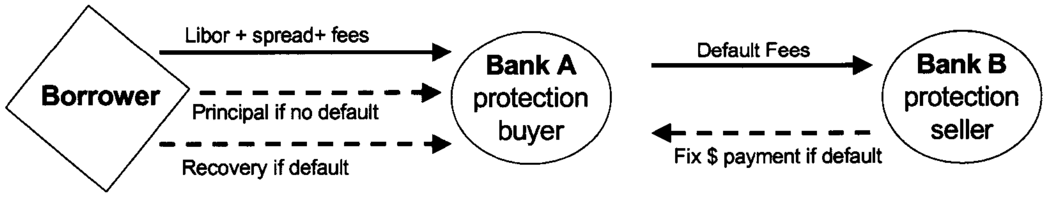
R represents the ***recovery payoffs*** *of* X *in case of any default.*

Notation 100 *We denote the set of marketed default prone securities by* $\mathcal{C} \subset \mathcal{M}$.

Before starting with the default adapted martingale pricing theory we present some credit sensitive contracts. Further definitions and descriptions of these contracts can be found in [JPM2], [RB], [RB2] or [Sp2].

Definition 101 *A* ***default prone zero-coupon bond*** *of company* XY $(\in D)$ *promises at maturity* T *a payoff of* $X = 1$ *CHF. In case company* XY *defaults, at maturity* T *probably not the full principal amount is returned but just* R *CHF* ≤ 1 *CHF.*

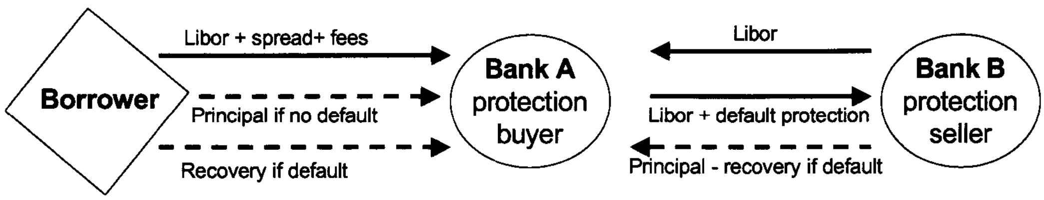
Definition 102 *A* ***default digital put option*** *(see Figure) on company XY ($\in D$) is a bilateral financial contract between the protection seller, that received an upfront premium and the protection buyer that receives one unit ($R = 1CHF$ and thus $X = 0$) in case XY defaults within a certain period T.*



Default Digital Put

Remark 103 *Default digital puts protect against default risks but not recovery risks, since a fix amount is received when XY defaults either at default τ or at maturity T.*

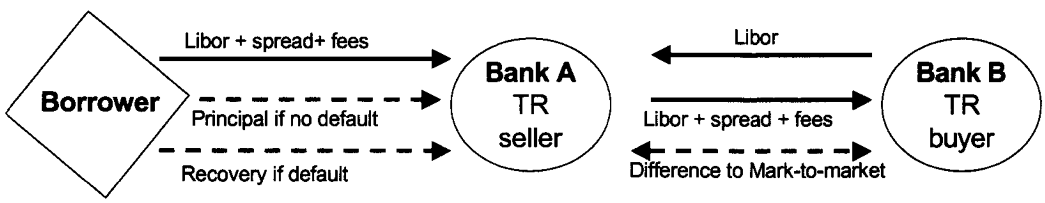
Definition 104 ***Credit default swap*** *(see Figure) is a bilateral financial contract in which one counterpart (protection buyer) pays a periodic fee, typically expressed in basis points per annum on the notional amount, in return for a contingent payment by the other counterpart (protection seller) after a credit event of the reference entity XY. The contingent payment (Principal - Recoveries) is designed to mirror the loss incurred by creditors of the reference entity XY in the event of its default. The settlement mechanism may be cash or physical.*



Credit Default Swap

Remark 105 *Default swaps hedge completely default and recovery risks of a specific borrower XY. However the protection buyer is not immune against other unfavorable market changes like interest rate movements.*

Definition 106 ***Total return swap*** *(see Figure) is a bilateral financial contract in which the total return of a specified asset is exchanged for another cash flow. The TR seller pays the total return (interest plus fees plus price appreciation less price depreciation) of a specified asset, the reference obligation and (usually) receives Libor plus a spread from the other counterpart (the TR buyer). Price appreciation or depreciation may be calculated and exchanged at maturity or on an interim basis.*



Total Return Swap

Remark 107 *The total return swaps hedge completely default, recovery and market risks of a specific security.*

Definition 108 ***Asset swap*** *is a package of credit securities and a corresponding swap that transforms the cash flows of the non-par securities (bond or loan), into a par (floating interest rate) structure. Asset swaps typically transforms fixed-rate bonds into par floaters, bearing a net coupon of Libor plus spread.*

Definition 109 ***First to default swap*** *is a credit default swap, where the protection seller takes on exposure the first entity* $C_1 \in \{1, ...S\} = D$ *suffering a credit event within a basket* $\{1, ...S\}$ *of* S *companies. The credit position in each name in the basket* D *is typically equal to the notional of the first to default swap. Losses are capped at the notional amount and the protection seller does not have exposures to subsequent credit events* C_i *for* $i > 1$.

Definition 110 *The **first-loss credit swap** is a protection for a portfolio of credit default swaps, whereby the protection seller commits to indemnify the protection buyer for a predefined amount of losses incurred following one* C_1 *or more credit events* $C_1, ..., C_i$ *in a specified reference portfolio* $\{1, ...S\}$.

Definition 111 ***Synthetic securitization*** *is a technique that combines securitization technology with credit derivatives to transfer risk on a portfolio while retaining the assets on the balance sheet. Advantages over traditional securitization include reduced costs, ease of execution and retention of on-balance sheet funding advantages.*

Definition 112 *The **substitution option** is a bilateral financial contract in which one party buys the right to substitute a specified asset or one of a specified group of assets for another at a point in time or contingent upon a credit event.*

Definition 113 ***Credit option is a put or call option*** *on the price of either (a) a floating rate note, bond, or (b) and asset swap package, consisting of a credit risk security with any payment characteristics and a corresponding derivative contract that exchanges the cash flows of that security for a floating rate cash flow stream. Typically three- or six-month Libor plus a spread.*

Definition 114 ***Credit spread option*** *is a bilateral financial contract in which the protection buyer pays a premium, usually upfront, and receives the present value of the difference between the spread prevailing the exercise date between the yield of the reference obligation and some benchmark yield (usually Treasuries or Libor) and the strike spread, if positive (a credit spread cap or call), or alternatively if negative (a credit spread floor or put).*

The risk neutral martingale pricing formula (65) can be adapted to payoffs conditional to default or surviving. For example assume that one default prone zero coupon bond P' on company $i \in \mathcal{S}^0$ paying 1 CHF at maturity T, becomes at default $\tilde{\tau}_i$ $R \leq 1$ (an $\mathcal{F}_{\tilde{\tau}_i}$-measurable random variable independent of r) equivalent default free zero coupon bonds P, i.e.

$$\begin{cases} 1 & if \quad T < \tilde{\tau}_i \\ R\, P(\tilde{\tau}_i, T) & if \quad T \geq \tilde{\tau}_i \end{cases}$$

According to (65) the price of P' at time $t=0$ corresponds to

$$\begin{aligned}
P'_R(0,T) &= \mathbb{E}\left[1_{\{T<\tilde{\tau}_i\}}\ \delta^r_{0,T}\ + 1_{\{T\geq\tilde{\tau}_i\}}\ R\ \delta^r_{0,\tilde{\tau}_i}\ P(\tilde{\tau}_i,T)\right] \\
&= \mathbb{E}\left[1_{\{T<\tilde{\tau}_i\}}\ \delta^r_{0,T}\ \right] + \mathbb{E}\left[R\right]\ \mathbb{E}\left[1_{\{T\geq\tilde{\tau}_i\}}\ \delta^r_{0,\tilde{\tau}_i}\ P(\tilde{\tau}_i,T)\right] \\
&= \mathbb{E}\left[1_{\{T<\tilde{\tau}_i\}}\ \delta^r_{0,T}\ \right] + \mathbb{E}\left[R\right]\ \mathbb{E}\left[1_{\{T\geq\tilde{\tau}_i\}}\ \delta^r_{0,T}\right] \\
&= \mathbb{E}\left[1_{\{T<\tilde{\tau}_i\}}\ \delta^r_{0,T}\ \right] + \mathbb{E}\left[R\right]\ \mathbb{E}\left[\left(1-1_{\{T<\tilde{\tau}_i\}}\right)\ \delta^r_{0,T}\right] \\
&= \left(1-\mathbb{E}\left[R\right]\right)\ \mathbb{E}\left[1_{\{T<\tilde{\tau}_i\}}\ \delta^r_{0,T}\ \right] + \mathbb{E}\left[R\right]\ \mathbb{E}\left[\delta^r_{0,T}\right] \\
&= \left(1-\mathbb{E}\left[R\right]\right)\ P'_0(0,T) + \mathbb{E}\left[R\right]\ P(0,T),
\end{aligned} \tag{66}$$

where we see that under the above specific assumptions the default prone zero coupon bond price $P'_R(0,T)$ with expected recovery $\mathbb{E}[R]$ decomposes into $\mathbb{E}[R]$ default free bonds $P(0,T)$ and $(1-\mathbb{E}[R])$ default prone bonds with zero recoveries $P'_0(0,T)$. The prices of default prone bonds $P'_R(0,T)$ can be calculated by (66) when we observe market prices of default-free bonds $P(0,T)$, zero-recovery default prone bonds $P'_0(0,T)$ and the expected recoveries $\mathbb{E}[R]$.

Notation 115 *We mark default prone security's prices with a dash $'$, like $P'_R(0,T)$.*

The simplest credit default contract is the European digital credit put option D' on company $i\in\mathcal{S}^0$, which pays 1 CHF at maturity T in case counterpart i defaults before T. With help of (65) its price D' at time $t=0$ looks like

$$\begin{aligned}
D'(0,T) &= \mathbb{E}\left[1_{\{\tilde{\tau}_i\leq T\}}\ \delta^r_{0,T}\right] \\
&= \mathbb{E}\left[\left(1-1_{\{\tilde{\tau}_i>T\}}\right)\ \delta^r_{0,T}\right] \\
&= \mathbb{E}\left[\delta^r_{0,T}\right] - \mathbb{E}\left[1_{\{\tilde{\tau}_i>T\}}\ \delta^r_{0,T}\ \right] \\
&= P(0,T) - P'_0(0,T).
\end{aligned} \tag{67}$$

When the payoff of 1 CHF takes place at the default time $\tilde{\tau}_i$ the contract is called American digital credit put option and its value corresponds to

$$\begin{aligned}
D'(0,T) &= \mathbb{E}\left[1_{\{\tilde{\tau}_i\leq T\}}\ \delta^r_{0,\tilde{\tau}_i}\right] \\
&= \mathbb{E}\left[\left(1-1_{\{\tilde{\tau}_i>T\}}\right)\ \delta^r_{0,\tilde{\tau}_i}\right] \\
&= \mathbb{E}\left[\delta^r_{0,\tilde{\tau}_i}\right] - \mathbb{E}\left[1_{\{\tilde{\tau}_i>T\}}\ \delta^r_{0,\tilde{\tau}_i}\ \right] \\
&= P(0,\tilde{\tau}_i) - P'_0(0,\tilde{\tau}_i).
\end{aligned} \tag{68}$$

The price relations between the various elementary contracts show that one can synthetically develop some securities with other securities and that these synthetic constructions might be useful for hedging credit risks. The difference between the American and the European digital put is that the default time $\tilde{\tau}_i$ matters, as can be seen from equations (67, 68).

The default prone security valuation examples (66, 67, 68) show that key difficulties for pricing default prone securities can be resumed in

1. Find analytical expressions for $\mathbb{E}\left[1_{\{T<\tilde{\tau}_i\}}\ \delta^r_{t,T}\ \mid\mathcal{F}_t\right]$;

2. Relax assumptions on recoveries R: i) R is not an $\mathcal{F}_{\tilde{\tau}_i}$-measurable random variable and ii) R is not necessarily independent from $1_{\{T\geq\tilde{\tau}_i\}}\ \delta^r_{0,\tilde{\tau}_i}$ under the risk neutral martingale measure $\mathbb{P}$.

For more general default prone securities $X \in \mathcal{M}$ we identify mainly three types of cash streams:

1. $1_{\{T<\tilde{\tau}_i\}} \int_{[t,T]} \delta^r_{t,s}\ dD_s$ regular payments before maturity T in case of no default;

2. $1_{\{T<\tilde{\tau}_i\}}\delta^r_{t,T}\ X$ regular payoff at maturity T in case of no default;

3. $1_{\{T\geq\tilde{\tau}_i\}}R$ (R being $\mathcal{F}_{T'}$-measurable) recovery payment in case of default at time $T' \geq \tilde{\tau}_i$.

The value of a default prone security X can thus be represented using the martingale formula (65)

$$\begin{aligned}
\Pi(X)_t &= \mathbb{E}\left[1_{\{T<\tilde{\tau}_i\}} \int_{[t,T]} \delta^r_{t,s}\ dD_s + 1_{\{T<\tilde{\tau}_i\}}\delta^r_{t,T}\ X + 1_{\{T\geq\tilde{\tau}_i\}}R\ \delta^r_{t,T'} \ \mid \mathcal{F}_t\right] \qquad (69)\\
&= \mathbb{E}\left[1_{\{T<\tilde{\tau}_i\}} \int_{[t,T]} \delta^r_{t,s}\ dD_s + 1_{\{T<\tilde{\tau}_i\}}\delta^r_{t,T}\ X \mid \mathcal{F}_t\right] + \mathbb{E}\left[1_{\{T\geq\tilde{\tau}_i\}}R\ \delta^r_{t,T'} \ \mid \mathcal{F}_t\right],
\end{aligned}$$

where the first term of (69) is the value of a default prone security with zero recovery $R=0$ and the second term represents the value of the recoveries R in case of default. The recovery term can be represented by using $1_{\{T\geq\tilde{\tau}_i\}} = 1 - 1_{\{T<\tilde{\tau}_i\}}$ like

$$\begin{aligned}
\mathbb{E}\left[1_{\{T\geq\tilde{\tau}_i\}}R\ \delta^r_{t,T'} \mid \mathcal{F}_t\right] &= \mathbb{E}\left[\left(1 - 1_{\{T<\tilde{\tau}_i\}}\right) R\ \delta^r_{t,T'} \mid \mathcal{F}_t\right] \qquad (70)\\
&= \mathbb{E}\left[R\ \delta^r_{t,T'} \mid \mathcal{F}_t\right] + \mathbb{E}\left[1_{\{T<\tilde{\tau}_i\}}R\ \delta^r_{t,T'} \mid \mathcal{F}_t\right].
\end{aligned}$$

The first term of (69) has been studied in the recovery valuation Section 7.3, while the second has to be further developed.

We will see that for pricing purposes it is not necessary to know the exact recoveries R at the default time τ_i, but we just have to know their risk neutral expectations for $t \geq \tau_i$

$$\bar{R}_t \stackrel{def}{=} \mathbb{E}\left[\delta^r_{\tau_i,T'}R \mid \mathcal{F}_t\right]. \qquad (71)$$

When we want to hedge recovery risks, we need more information about recoveries R than just their expectations $\bar{R}_t$.

Remark 116 *This non measurability of recoveries R at the default time τ_i makes the default prone securities markets incomplete, because for each possible recovery rate $Z \in [0,1]$, we need an asset affected by the specific recovery rate risk $Z \in [0,1]$. For perfect hedging we would need therefore an infinite number of securities, but corporate liability markets have just a finite number of securities, which shows that these market must be incomplete.*

From now on we will have to assume.

Assumption 117 (H) *Every $(\mathbb{P},\mathbf{E})$-square-integrable martingale is a $(\mathbb{P},\mathbf{F})$-square-integrable martingale.*

This hypothesis implies that any $(\mathbb{P},\mathbf{E})$-Brownian motion B remains an $(\mathbb{P},\mathbf{F})$-Brownian motion in the enlarged filtration. [BY] and [MS] showed that condition (H) has the following properties.

Proposition 118 *(H) is equivalent to*

(H*) *For any $t \geq 0$, the σ-field $\mathcal{E}_\infty$ and $\mathcal{F}_t$ are conditionally independent given $\mathcal{E}_t$.*

(H1) $\forall t \geq 0, \forall G_t \in \mathcal{F}_t, \quad \mathbb{E}[G_t \mid \mathcal{E}_\infty] = \mathbb{E}[G_t \mid \mathcal{E}_t]$.

(H2) $\forall F \in \mathcal{H}_\infty, \forall G_t \in \mathcal{F}_t, \quad \mathbb{E}[FG_t \mid \mathcal{E}_t] = \mathbb{E}[F \mid \mathcal{E}_t]\mathbb{E}[G_t \mid \mathcal{E}_t]$.

(H3) $\forall t \geq 0, \forall F \in \mathcal{E}_\infty, \quad \mathbb{E}[F \mid \mathcal{F}_t] = \mathbb{E}[F \mid \mathcal{E}_t]$.

The price of a default prone security X (=payoff at time T and X being $\mathbf{F}_T$-measurable) on company i satisfies the following Proposition where we assume zero recoveries $R = 0$ in case of default.

Proposition 119 *Assume that condition (H) is satisfied and let $\delta^r_{t,T}\ X$ be an $\mathcal{F}_T$-measurable integrable random variable. We further define the càdlàg process V for $t < T < \infty$ by*

$$V_t \stackrel{def}{=} \mathbb{E}\left[\delta^{r+\lambda^i}_{t,T}\ X \mid \mathcal{F}_t\right], \tag{72}$$

and for $t \geq T$ by $V_t = 0$ with the default intensity process λ^i_s as defined by equation (34) of the main Theorem 73.

Then the claim's price process

$$\Pi(X)_t = \mathbb{E}\left[1_{\{T<\tilde{\tau}_i\}}\ \delta^r_{t,T}\ X \mid \mathcal{F}_t\right] \tag{73}$$

can be represented by

$$\Pi(X)_t = 1_{\{t<\tilde{\tau}_i\}} V_t - 1_{\{t<\tilde{\tau}_i\}}\ \mathbb{E}\left[\Delta V_{\tilde{\tau}_i} \mid \mathcal{F}_t\right]. \tag{74}$$

Remark 120 *As we can see from formula (72), default acts on the price of default prone securities as a change in the discount rate. Equations (72) and (74) are well known in the traditional credit risk pricing literature, where the term $\mathbb{E}\left[\Delta V_{\tilde{\tau}_i} \mid \mathcal{F}_t\right]$ vanishes since in their model the process V_t, defined by (72), does not jump at the default time $\tilde{\tau}_i$. In our model we have jumps in (72) at τ_k, since investors receive at these specific times new important updates about the credit quality of the surviving companies.*

Proof. Being $\delta^r_{t,T}\ X\ \mathcal{F}_T$-measurable, we have that the default-free price $\mathbb{E}\left[\delta^r_{t,T}\ X \mid \mathcal{F}_t\right]$ of the sure payoff X at time T, follows a continuous martingale. Further being the default intensities λ^i continuous processes, except at the default events τ_k, where the default intensities λ^i might jump because of the default surprise, we have that the martingale

$V_t = \mathbb{E}\left[\delta_{t,T}^{r+\lambda^i} \; X \mid \mathcal{F}_t\right]$ is continuous, except at the default events τ_k, where it might jump by $\Delta V_{\tau_k} = V_{\tau_k} - V_{\tau_{k-}}$.

We define the process U_t by

$$U_t \stackrel{def}{=} 1_{\{t<\tilde{\tau}_i\}} V_t \tag{75}$$

and we check that

$$U_t = \mathbb{E}\left[1_{\{t<\tilde{\tau}_i\leq T\}} \; \Delta V_{\tilde{\tau}_i} + 1_{\{T<\tilde{\tau}_i\}} \; \delta_{t,T}^{r} \; X \mid \mathcal{F}_t\right]. \tag{76}$$

or equivalently

$$U_t = \mathbb{E}\left[\int_{[t,T]} \Delta V_s \; dN_s^i + 1_{\{T<\tilde{\tau}_i\}} \; \delta_{t,T}^{r} \; X \mid \mathcal{F}_t\right]. \tag{77}$$

We further define the $\mathbf{F}$-martingale m by

$$m_t \stackrel{def}{=} \mathbb{E}\left[\delta_{0,T}^{\lambda^i} \; \delta_{t,T}^{r} \; X \mid \mathcal{F}_t\right],$$

satisfying $V_t = \delta_{0,t}^{-\lambda^i} \; m_t = e^{\Lambda_t^i} \; m_t$ with $\Lambda_t^i \stackrel{def}{=} \int_0^t \lambda_s^i \; ds$.

Using now Itô's product rule for V_t, we obtain

$$dV_t = d\left(e^{\Lambda_t^i}\right) \; m_{t\_} + e^{\Lambda_t^i} \; dm_t = V_{t\_} \; e^{-\Lambda_t^i} \; de^{\Lambda_t^i} + e^{\Lambda_t^i} \; dm_t. \tag{78}$$

Applying Itô's formula on U_t, we get

$$dU_t = (1 - N_{t\_}^i) \; dV_t - V_{t\_} dN_t^i - \Delta V_t \Delta N_t^i.$$

and inserting equation (78) we obtain

$$dU_t = (1 - N_{t\_}^i) \; \left(V_{t\_} \; e^{-\Lambda_t^i} \; de^{\Lambda_t^i} + e^{\Lambda_t^i} \; dm_t\right) - V_{t\_} dN_t^i - \Delta V_t \Delta N_t^i.$$

After rearranging, we get

$$dU_t = dC_t - \Delta V_t \Delta N_t^i, \tag{79}$$

where C_t is the $\mathbf{F}$-martingale

$$\begin{aligned} dC_t &= (1 - N_{t\_}^i) \; \left(V_{t\_} \; e^{-\Lambda_t^i} \; de^{\Lambda_t^i} + e^{\Lambda_t^i} \; dm_t\right) - V_{t\_} dN_t^i \\ &= \left(1 - N_{t\_}^i\right) \; e^{\Lambda_t^i} \; dm_t + dD_t, \end{aligned}$$

since

$$\begin{aligned} dD_t &= -V_{t\_} \; \left\{dN_t^i - \left(1 - N_{t\_}^i\right) \; \left(\; e^{-\Lambda_t^i} \; de^{\Lambda_t^i}\right)\right\} \\ &= -V_{t\_} d\left(N_t^i \; - \Lambda_{t\wedge\tilde{\tau}_i}^i\right) \\ &= -V_{t\_} d\left(N_t^i \; - \int_0^{t\wedge\tilde{\tau}_i} \lambda_s^i \; ds\right) \\ &= -V_{t\_} \; dM_t^i, \end{aligned}$$

is an $\mathbf{F}$-martingale being $M_t^i = N_t^i - \int_0^{t\wedge\tilde{\tau}_i} \lambda_s^i \, ds$ an $\mathbf{F}$-martingale by Theorem 73. Now, because $U_T = 1_{\{T<\tilde{\tau}_i\}} X$, equation (79) implies equation (77) and hence representation (74) holds. ■

The price of a default prone security on company i, paying X at maturity T and with expected recoveries $\bar{R}_s$ in case the company defaults at time s, as defined in (71) satisfies the following Proposition.

Proposition 121 *Assuming (H) and define the càdlàg process V for $t < T < \infty$ by*

$$V_t \overset{def}{=} \mathbb{E}\left[\int_{[t,T]} \delta_{t,s}^{\lambda^i} \, \bar{R}_s \, \lambda_s^i \, ds + \delta_{t,T}^{r+\lambda^i} \, X \mid \mathcal{F}_t \right], \tag{80}$$

and for $t \geq T$ by $V_t = 0$.

Then the claim's price process

$$\Pi(X)_t = \mathbb{E}\left[1_{\{T\geq\tilde{\tau}_i\}} \, \delta_{t,\tilde{\tau}_i}^{r} \, \bar{R}_{\tilde{\tau}_i} + 1_{\{T<\tilde{\tau}_i\}} \, \delta_{t,T}^{r} \, X \mid \mathcal{F}_t \right]$$

can be represented by

$$\Pi(X)_t = 1_{\{t<\tilde{\tau}_i\}} V_t - 1_{\{t<\tilde{\tau}_i\}} \mathbb{E}\left[\Delta V_{\tau_k} \mid \mathcal{F}_t\right]. \tag{81}$$

Proof. Similar as for the previous Proposition 119. ■

Remark 122 *Consider two investors, an $\mathbf{F}$-investor and an $\mathbf{E}$-investor, where $\mathcal{F}_t = \mathcal{E}_t \vee \mathcal{G}_t$, with $\mathcal{E}_t \subsetneq \mathcal{F}_t$. By construction the $\mathbf{F}$-investor has more information than the $\mathbf{E}$-investor, thus the $\mathbf{F}$-investor can make some arbitrage with the $\mathbf{E}$-investor. One might be interested in the value of this* ***surplus information***

$$S(X) \overset{def}{=} \mathbb{E}\left[X \mid \mathcal{E}_t \vee \mathcal{G}_t\right] - \mathbb{E}\left[X \mid \mathcal{G}_t\right], \tag{82}$$

for default prone securities $X \in \mathcal{C}$.

8.5 Valuation with Risk Factors

In this section, following the idea presented in the multi companies default time Section 6, we chose a multifactor Cox Ingersoll Ross setup, which is in Section 9 extended to a simulation procedure. Furthermore this choice facilitates the calculations since negative interest rates or default intensities are easily avoided. The underlying risk factors and indicators $X = (X^1, ..., X^{K+S}) \in \mathbb{R}^{K+S}$ are **assumed** to be driven under the risk neutral martingale measure $\mathbb{P}$ by **independent** diffusion processes, i.e. for $i = 1, ..., K+S$

$$dX_t^i = \left(\alpha_i - \beta_i X_t^i\right) dt + \sigma_i \sqrt{X_t^i} \, dW_t^i, \tag{83}$$

where $\alpha_i, \beta_i, \sigma_i \in \mathbb{R}$ satisfying $\alpha_i > \frac{1}{2}\sigma_i^2$ to ensure strict positivity and $W = (W^1, ..., W^n) \in \mathbb{R}^k$ is a standard k-dimensional $(\mathbb{P},\mathbf{F})$-Brownian motion.

Remark 123 *A more general setup of these risk factors like (10) can be chosen, but one has to impose further constraints on the parameters such that interest rates or default intensities processes remain positive.*

Remark 124 *We assumed independent processes X_t^i, because in case they are dependent we simply transform the basis such that we obtain a basis of independent underlying risk factors.*

The first K factors $(X^1, ..., X^K) \in \mathbb{R}^K$ are reserved for describing the systematic risk, while for $i = K+1, ..., K+S$ the factor X^i represents the unsystematic risk component of company $i - K$.

The risk factors $X \in \mathbb{R}^{K+S}$ generate the Brownian filtration usually called economic information and denoted by $\mathcal{E}_t$. The short rate r is assumed to follow

$$r_t = \sum_{l=1}^{K} r^l \; X_t^l, \tag{84}$$

with r^l positive constants ensuring $r_t > 0$.

We recall that for $t \in [\tau_j, \tau_{j+1})$ given the information $\mathcal{F}_t$, the probability that company $i \in \mathcal{S}^j$ defaults at time τ_{j+1} after time t and within time Δ corresponds to

$$\mathbb{P}[\tau_{j+1} \leq t + \Delta, C_{j+1} = i \mid \mathcal{F}_t](w) = \int_0^{\Delta} d^{(j+1)}(w, x, t, i) \; dx, \tag{85}$$

and the probability of having a default event τ_{j+1} respectively

$$\mathbb{P}[\tau_{j+1} \leq t + \Delta \mid \mathcal{F}_t](w) = \int_0^{\Delta} d^{(j+1)}(w, x, t) \; dx. \tag{86}$$

We set the default time densities for $i, j = 0, ..., S$, $t \in [\tau_j, \tau_{j+1})$ and for $x \geq 0$ to

$$d^{(j+1)}(w, x, t, i) = 1_{\{x \geq 0\}} \lambda_{t+x}^i e^{-(t+x) \sum_{k \in \mathcal{S}^{j+1}} \lambda_{t+x}^k}, \tag{87}$$

where for the processes λ_t^i we assume also a multi factor model of the form

$$\lambda_t^i = 1_{[t \leq \tilde{\tau}_i)} \left\{ \underbrace{\sum_{l=1}^{K} \Lambda^{il} X_t^l}_{systematic} + \underbrace{\Lambda^{iK+i} X_t^{K+i}}_{unsystematic} + \underbrace{\sum_{k=1}^{S} 1_{[\tau_k, \tau_{k+1})} f^i \left[t, C_k, \Lambda, X_t^l \right]}_{default\ feedbacks} \right\}, \tag{88}$$

with constants $\Lambda^{il} \geq 0$ ($\Lambda^{il} = 0$ for the unsystematic factors $l = K, ..., K+i-1, K+i+1, ..., K+S$ and denoting $\Lambda = (\Lambda^{i1}, ..., \Lambda^{iK+S})$). The systematic term of (88) models the default behavior of company i in relation of the general economy; the unsystematic term of (88) represents the company specific default risk and the default feedback term models the feedbacks caused by defaults of other companies.

The feedback function $f^i \left[t, C_k, \Lambda, X_t^l \right]$ (third term of (88)), should depend on the factors t, C_k, Λ, X_t^l because

1. The more likely the company C_k is defaulted because of an unsystematic failure $(\Lambda^{kK+k}X_{\tau_k}^{K+k} \gg \sum_{l=1}^{K} \Lambda^{kl}X_{\tau_k}^{l})$, the less the surviving companies have similar troubles, the less their default intensity should react on this default, the more they can conquer new market positions. Concluding their default intensity should be stable of become better.

2. The more likely the company C_k is defaulted because of a systematic failure, i.e. $\Lambda^{C_kK+C_k}X_{\tau_k}^{K+C_k} \ll \sum_{l=1}^{K} \Lambda^{C_kl}X_{\tau_k}^{l}$, the higher the other companies are influenced by the same negative market conditions. Hence their default intensity should rise.

3. The more the surviving company is dependent from the same underlying systematic risk factors as the defaulted company $(\sum_{l=1}^{K} \Lambda^{C_kl}\Lambda^{il} \gg 0)$, the higher the feedback.

4. Investors mostly overreact when they receive the information of a new default (τ_k, C_k). There is thus a jump in the default intensities at the default time, which should decay (linearly or exponentially) with time t.

Example 125 *On can choose default feedbacks as a function of the number of defaulted companies*

$$f^i\left[t, C_k, \Lambda, X_t^l\right] = \frac{1}{k}e^{-\alpha t},$$

where the parameter α gives the information how fast the feedback decays. In this model the more companies default, the higher the default feedback, the more likely other companies will default.

Example 126 *An example of possible feedback-function for company i when company C_k defaults*

$$f^i\left[t, C_k, \Lambda, X_t^l\right] = \frac{\sum_{l=1}^{K} \Lambda^{C_kl}X_{\tau_k}^{l}}{\Lambda^{C_kK+C_k}X_{\tau_k}^{K+C_k}} \sum_{l=1}^{K} \Lambda^{C_kl}\Lambda^{il}X_t^l.$$

For modelling recoveries we similarly set up for company $i = 1, ..., S$ the market asset value processes A_t^i defined as

$$A_t^i = \underbrace{\sum_{l=1}^{K} A^{il}\ X_t^l}_{systematic} + \underbrace{A^{iK+i}\ X_t^{K+i}}_{unsystematic}, \tag{89}$$

where $A^{il} \in \mathbb{R}$ ($A^{il} = 0$ for $l = K, ..., K+i-1, K+i+1, ..., K+S$).

Remark 127 *This specification of asset, default intensities and risk free short rate allows only positive correlation among the default intensities and the short rate, which is empirically the more interesting case. However it is possible to slightly change the setup for obtaining also negative correlations.*

Under this specification (see [CIR]) the default-free bond prices $P(t,T)$, because the risk factors X_s^l are independent under $\mathbb{P}$, looks like

$$
\begin{aligned}
P(t,T) &= \mathbb{E}\left[\delta_{t,T}^{r} \mid \mathcal{F}_t\right] = \mathbb{E}\left[e^{-\int_t^T r_s\, ds} \mid \mathcal{F}_t\right] \qquad (90)\\
&= \mathbb{E}\left[e^{-\int_t^T \sum_{l=1}^K r^l\, X_s^l\, ds} \mid \mathcal{F}_t\right] = \mathbb{E}\left[e^{-\sum_{l=1}^K r^l \int_t^T\, X_s^l\, ds} \mid \mathcal{F}_t\right]\\
&= \mathbb{E}\left[\prod_{l=1}^K e^{-r^l \int_t^T\, X_s^l\, ds} \mid \mathcal{F}_t\right] = \prod_{l=1}^K \mathbb{E}\left[e^{-r^l \int_t^T\, X_s^l\, ds} \mid \mathcal{F}_t\right]\\
&= \prod_{l=1}^K A^l\left[T-t, r^l\right] e^{-B^l\left[T-t,r^l\right]},
\end{aligned}
$$

with the functions

$$
\begin{aligned}
A^l\left[T-t, r^l\right] &= \left[\frac{2\gamma e^{\frac{1}{2}(\gamma+\beta_l)(T-t)}}{(\gamma+\beta_l)\left(e^{\gamma(T-t)}-1\right)+2\gamma}\right]^{\frac{2\alpha_l}{\sigma_l^2}} \qquad (91)\\
B^l\left[T-t, r^l\right] &= \frac{2\left(e^{\gamma(T-t)}-1\right)}{(\gamma+\beta_l)\left(e^{\gamma(T-t)}-1\right)+2\gamma}\\
\gamma &= \sqrt{\beta_l^2 + 2r^l\sigma_l^2}
\end{aligned}
$$

Similarly for continuous default intensities λ we can represent the default prone bond's prices by

$$
\begin{aligned}
P'(t,T) &= \mathbb{E}\left[\delta_{t,T}^{r+q\lambda} \mid \mathcal{F}_t\right] = \mathbb{E}\left[e^{-\int_t^T (r_s+q\lambda_s)\, ds} \mid \mathcal{F}_t\right] \qquad (92)\\
&= \prod_{l=1}^K A^l\left[T-t, r^l+q^l\lambda^l\right] e^{-B^l\left[T-t,r^l+q^l\lambda^l\right]},
\end{aligned}
$$

where q represents the quantity loss given default. As we can see in equation (67), default is reflected in the price with a higher discount rate $r_s + q\lambda_s$, where $q\lambda_s$ corresponds to the instantaneous risk neutral expected loss.

8.6 Calibrating a Martingale Measure

For the practical evaluation of prices we need to know the market used equivalent martingale measure $\mathbb{P}$ for calculating the various expectations. This measure $\mathbb{P}$ is reflected in market prices of credit sensitive securities. We need therefore to estimate $\mathbb{P}$ from "benchmark" market prices. Since these markets are incomplete there exist infinite many equivalent martingale measures. We are not able to identify which would be the best martingale measure $\mathbb{P}$ for pricing credit default sensitive securities.

Let $n = S + K$ be the number of randomness sources from the underlying risk factors X_t, then if we **assume** that there is a **liquid market for all default prone securities** and in order to use the ideas presented in [Bt2] we fix $S + K$ credit securities $\zeta^1, ..., \zeta^n$ whose payoffs are of the form

$$\zeta^i = \Phi^i(t, X_{T_i}^1, ..., X_{T_i}^n), \quad i = 1, ..., n, \qquad (93)$$

and such that their prices, presented in the valuation Section 8.5, behave like

$$\Pi_t^i = F^i(t, X_t^1, ..., X_t^n), \quad i = 1, ..., n, \tag{94}$$

We use these "**benchmarks**" claims for valuing other default prone contingent claim securities $\zeta = \Phi(R_T^1, ..., R_T^l)$, which price processes are also assumed to follow

$$\Pi_t = F(t, X_t^1, ..., X_t^n). \tag{95}$$

The **benchmarks** securities determine the "correct" actual market risk premium $\pi_t^i, i = 1, ..., n$, being a solution of the system of equations

$$\alpha - r\mathbf{1}_n = \sigma\ \pi, \tag{96}$$

where α and σ are the return rates and the volatility matrix of the n benchmark securities and $\mathbf{1}_n$ denotes the n-dimensional unit 1 vector.

For this setting [Bt2] proposes the following procedure for the "inversion of the yield curve"

1. Assume that the market price of risks π, satisfying (96), is of the form

   $$\pi = \pi(t, X_t^1, ..., X_t^n, \beta), \tag{97}$$

   where $\beta \in \mathbb{R}^p$ is the parameter vector for calibrating.

2. Compute for all benchmark claims $i = 1, ..., n$ the theoretical pricing functional $F^i(t, X_t^1, ..., X_t^n, \beta)$ by the credit valuation model as a function of the parameter vector β.

3. Observing today's value of the underlying processes $X_0 = (X_0^1, ..., X_0^n)$, compute today's theoretical prices of the contracts as

   $$\Pi^i(0, \beta) = F^i(0, X_0, \beta), \quad i = 1, ..., n. \tag{98}$$

4. Observe concrete market prices for the $i = 1, ..., n$ benchmark securities $\Pi^{i*}(0)$.

5. Choose the "implied" parameter vector $\beta^* \in \mathbb{R}$ such that the theoretical prices $\Pi^i(0, \beta^*)$ are "as closed as possible" to the observed concrete market prices $\Pi^{i*}(0)$,

   $$\Pi^{i*}(0) \approx \Pi^i(0, \beta^*) \quad i = 1, ..., n. \tag{99}$$

   For example by solving the least squares minimization problem

   $$\min_{\beta \in \mathsf{R}^p} \left[\sum_{i=1}^{n} \left\{ \Pi^{i*}(0) - \Pi^i(0, \beta) \right\}^2 \right], \tag{100}$$

   we obtain one possible solution of (99).

6. Other claims, which pricing equation satisfies equation (95), can then be evaluated by

   $$\pi = \pi(t, X_t^1, ..., X_t^n, \beta^*) \tag{101}$$

   and its price at time $t = 0$ corresponds to $\pi(t, X_0^1, ..., X_0^n, \beta^*)$.

Remark 128 *With this "inversion of the yield curve" procedure we calibrate one possible equivalent martingale measure $\mathbb{P}$, but there are other methods for estimating other martingale measures $\widetilde{\mathbb{P}}$, since the markets are incomplete and hence there are infinite equivalent martingale measures $\widetilde{\mathbb{P}}$!*

8.7 Equivalent Measure Changes

Artzner and Delbaen showed in the paper [AD2] that the default intensity for a company also exists under equivalent risk neutral martingale measure. Here in this section, we recall some results of [Ks, p. 73] and [Bp, p. 64] that showed an extension of the [AD2] result, valid for our multi company default model frame. They basically showed that if the default intensities exist under the physical measure $\widetilde{\mathbb{P}}$, where the intensities have the natural interpretation of instantaneous default probabilities, then the default intensities also exist under equivalent risk neutral martingale measures $\mathbb{P}$. Moreover when we know the default intensities under both equivalent martingale measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$, then we also know the measure transformation. In fact [Bp, p. 64] showed

Theorem 129 *Let $N_t = (N_t^1, ..., N_t^S)$ be an S-variate point process defined on $(\Omega, \mathcal{F})$ with two probability measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ such that $\mathbb{P}_t << \widetilde{\mathbb{P}}_t$ for all $t \geq 0$, where $\mathbb{P}_t$ and $\widetilde{\mathbb{P}}_t$ are the restrictions on $\mathcal{F}_t^N$. Suppose that N_t admits $\mathbf{F}$-intensity $\lambda_t = (\lambda_t^1, ..., \lambda_t^S)$.*

Then there exist $\mathcal{F}_t^N$-predictable processes $(\mu_t^1, ..., \mu_t^S)$ with non-negative components such that $(\lambda_t^1 \mu_t^1, ..., \lambda_t^S \mu_t^S)$ is then the $(\widetilde{\mathbb{P}}, \mathcal{F}_t)$-intensity of N_t. Moreover the Radon-Nykodin derivative corresponds to

$$\frac{d\mathbb{P}_t}{d\widetilde{\mathbb{P}}_t} = L_t = \prod_{i=1}^{S} L_t^i, \tag{102}$$

where

$$L_t^i = e^{\int_0^t (1-\mu_s^i)\lambda_s^i \, ds} \left(1_{\{\tilde{\tau}_i > t\}} + 1_{\{\tilde{\tau}_i \leq t\}} \mu_{\tilde{\tau}_i}^i \right), \tag{103}$$

with $\tilde{\tau}_i$ denoting as usual the default time of company i.

Proof. See [Bp, Theorem 8 p. 64]. ■

Moreover for a fixed maturity $T < \infty$

Theorem 130 *Using the same notation as Theorem 129, and assuming moreover $\mathbb{E}[L_T] = 1$ for a fixed maturity $T < \infty$, we define a new probability measure $\widetilde{\mathbb{P}}$ by*

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = L_T.$$

Then $(\lambda_t^1 \mu_t^1, ..., \lambda_t^S \mu_t^S)$ is the $(\widetilde{\mathbb{P}}, \mathcal{F}_t)$-intensity of $(N_t^1, ..., N_t^S)$ over $[0, T]$.

Proof. See [Bp, Theorem 8 p. 64]. ■

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9 Simulation of Security's Prices

9.1 Introduction

Measuring default and recovery risks of an entire credit book of 1000 or more contracts with different borrowers, becomes very soon a quite difficult task for analytical calculations. For this practitioners often develop simulation procedures (see [JPM1] or [KMV1]) for reproducing complex behaviors of financial markets. Portfolio managers are interested in knowing possible values of their credit book during the exposure period $T > 0$ at several business relevant times

$$t_i \stackrel{def}{=} \frac{T}{m} i, \quad i = 1, .., m-1,$$

for managing their different exposure such that some liquidity criteria, regulatory requirements or returns over risks criteria are satisfied or optimized. More information about portfolio optimization, risk capital concept, capital allocation and risk management objectives can be read in [PBR].

Unfortunately being credit events extremely rare, one needs many simulation paths for obtaining an accurate approximation for the portfolio profit and loss distribution. For reproducing accurately credit events one therefore has to build efficient simulation engine.

In the first Section 9.2 we describe how to simulate the underlying risk factors, which are used then in Section 9.3, where we describe how we simulate our multi company default model of Section 6. (see also Section 11.5, where we described its discrete version). In Section 9.4 we present the recovery simulation engine for valuing the recoveries in case of a default. Finally in the last section 9.5 we merge these two simulation models for obtaining a default prone securities valuation engine.

9.2 Simulating Underlying Risk Factors

The underlying risk factors X in $\mathbb{R}^{S+K}$ with initial condition x_0, are driven by the dynamics as presented in (83), i.e. for $i = 1, ..., K+S$

$$dX_t^i = \left(\alpha_i - \beta_i X_t^i\right) dt + \sigma_i \sqrt{X_t^i}\, dW_t^i, \tag{104}$$

where $\alpha_i, \beta_i, \sigma_i \in \mathbb{R}$ satisfy $\alpha_i > \frac{1}{2}\sigma_i^2$ to ensure strict positivity and where $W = (W^1, ..., W^n)$ in $\mathbb{R}^n$ is a standard n-dimensional $(\mathbb{P},\mathbf{F})$-Brownian motion.

Unfortunately by making the dynamics (104) discrete for obtaining a discrete path $(X_{t_i})_{i=0}^m$, it may happen that some X^i might become negative, causing a problem for the square root. For avoiding this negative points we propose a slightly adapted simulation procedure. We denote $\Delta t = t_{k+1} - t_k$ and suppose that we have drawn a positive path $(X_{t_i})_{i=0}^k$ until $t_k < T$ and hence for obtaining the $k+1$ point we must

1. Draw the increment for the $S+K$ dimensional Brownian motion $\Delta W_{t_{k+1}} \stackrel{def}{=} W_{t_{k+1}} - W_{t_k}$ from an $S+K$ dimensional multivariate independent normal distribution with mean 0 and variance Δt $N\left(0, \sqrt{\Delta t}\right)$.

2. Calculate X at time t_{k+1}, i.e. for all $i = 1, ..., S+K$ with

$$X_{t_{k+1}}^i = \left[\left(\alpha_i - \beta_i X_{t_k}^i\right) \Delta t + \sigma_i \sqrt{X_{t_k}^i}\, \Delta W_{t_{k+1}}^i\right]^+. \tag{105}$$

3. Replace $k+1$ by $k+2$ and turn to Step 1. until $t_{k+2} > T$.

Remark 131 *In case for some $i = 1, ..., S+K$ the expression*

$$\left(\alpha_i - \beta_i X^i_{t_k}\right) \Delta t + \sigma_i \sqrt{X^i_{t_k}}\, \Delta W^i_{t_{k+1}}$$

becomes negative, we simply set $X^i_{t_{k+1}} = 0$ and at the next draw the $X^i_{t_{k+2}} = \alpha_i \Delta t$ will be again positive since we have a positive drift $\alpha_i > \frac{1}{2}\sigma_i^2 > 0$.

Remark 132 *The simulation of correlated risk factors X can be obtained by simply drawing correlation Brownian motion increments. More precisely*

Algorithm 133 *1. Perform a Cholesky decomposition of the correlation matrix C (See [JPM1]) to produce a matrix A (the "square root" of C) such that*

$$A^T * A = C. \tag{106}$$

2. For each period $t_k \geq 0$ draw a vector $\Delta V_t = (\Delta V_t^1, ..., \Delta V_t^N)$ of length N, where each element ΔV^i of ΔV is a draw from an independent $N(0, \sqrt{\Delta t})$ distribution.

3. Produce for each period a new vector ΔW_t,

$$\Delta W_t = A^T * \Delta V_t. \tag{107}$$

This simulation procedure delivers a discrete economic path $(X_{t_i})_{i=0}^m$, which generates the "economic" information $\mathcal{E}_t$.

9.3 Simulating Multi Default with Feedbacks

In this subsection we present the procedure for simulating the default sequence for S companies, i.e. at what time which company defaults. We follow the idea presented in Sections 6 and 11.5 for proposing a simulation procedure. Our goal of this section is to obtain several (let's say p) consistent default sequences $\{(\tau_k, C_k),\ k = 0, ..., S\}$. For the simulation one needs therefore to first draw p economic paths $(X_{t_i})_{i=0}^m$, where for each economic path $(X_{t_i})_{i=0}^m$ we must finally obtain one default sequence $\{(\tau_k, C_k),\ k = 0, ..., S\}$. For this we suppose that one economic path is given and that we are at time $t = \tau_p$ for $p = 0, ..., S-1$, knowing $\mathcal{F}_{\tau_p} = \mathcal{E}_{\tau_p} \vee \mathcal{G}_{\tau_p}$ (i.e. X_t for $0 \leq t \leq \tau_p$ and $\{(\tau_v, C_v),\ v = 0, ..., p\}$). We also recall the notation used in the Appendix 11.5, for denoting the different time grid points: $t_p^k = \tau_p + k\ \Delta$. At this point we are hence interested at what time (τ_{p+1}) the next company (C_{p+1}) defaults. For answering this question we propose the following procedure:

1. Knowing the economy $\mathcal{E}_{t_p^k}$ for all $t_p^k \geq 0$ and the default sequence up to time τ_p $\{(\tau_v, C_v),\ v = 0, ..., p\}$, calculate with representations (87, 88) the cumulative default event probability

$$F^{(p+1)}(t_p^k) \quad = \quad \mathbb{P}\left[\delta_{p+1} \in \left[0, t_p^k - \tau_p\right] \mid \mathcal{E}_{t_p^k} \vee \mathcal{G}_{\tau_p}\right](w)$$

$$
\begin{aligned}
&= \int\limits_{[0,t_p^k-\tau_p]} d^{(p+1)}(w,x,t_p^k)\,dx \\
&= \int\limits_{[0,t_p^k-\tau_p]} \sum_{i\in\mathcal{S}^p} d^{(p+1)}(w,x,t_p^k,i)\,dx \\
&= \sum_{i\in\mathcal{S}^p}\sum_{l=1}^{K+S} \Lambda^{il} \int\limits_{[0,t_p^k-\tau_p]} X_s^l\, e^{-s\sum_{v\in\mathcal{S}^p}\sum_{l=1}^{K+S}\Lambda^{iv}X_s^v}\,ds \\
&= \sum_{i\in\mathcal{S}^p}\sum_{l=1}^{K+S} \Lambda^{il} \sum_{s=0}^{k} X_{t_p^s}^l\, e^{-\sum_{v\in\mathcal{S}^p}\sum_{l=1}^{K+S}\Lambda^{iv}X_{t_p^s}^v}\Delta t.
\end{aligned}
\tag{108}
$$

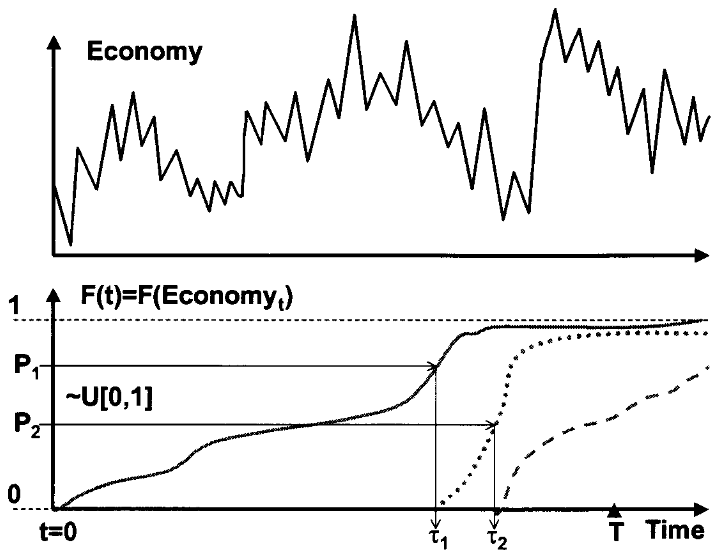
2. Simulate the $p+1$ default event time τ_{p+1} :

   (i) Draw a random variable $D_{p+1} \backsim U[0,1]$.

   (ii) Invert the cumulative default event probability distribution function $F^{(p+1)}(t)$ (see Figure), for obtaining the $p+1$ default event time τ_{p+1}, i.e.

   $$\tau_{p+1} = \left[F^{(p+1)}\right]^{-1}(D_{p+1}).$$

   (iii) When τ_{p+1} is after the interested business horizon: $\tau_{p+1} > T$, then STOP algorithm for this economic path.



Default Event Simulation

3. Draw the company C_{p+1} out of $\mathcal{S}^p$ that defaults at τ_{p+1}:

   (i) For company $i \in \mathcal{S}^p$ the instantaneous default probability at the default time τ_{p+1} corresponds to $\lambda^i_{\tau_{p+1}}$.

(ii) Define for $i \in \mathcal{S}^p$ the intervals $I^i = \left[\frac{\lambda^{i-1}_{\tau_{p+1}}}{\sum_{v\in\mathcal{S}^p}\lambda^v_{\tau_{p+1}}}, \frac{\lambda^{i}_{\tau_{p+1}}}{\sum_{v\in\mathcal{S}^p}\lambda^v_{\tau_{p+1}}}\right)$ with $\lambda^{v-1}_{\tau_{p+1}} = 0$ for $v = \min\mathcal{S}^p$ and $\bigcup_{v\in\mathcal{S}^p} I^v = [0,1)$.

(iii) Draw another random variable $K_{p+1} \backsim U[0,1]$.

(iv) Company $i \in \mathcal{S}^p$ defaults ($C_{p+1} = i$) iff $K_{p+1} \in I^i$.

4. Adapt all default intensities with new information $\mathcal{F}_{\tau_{p+1}} = \mathcal{E}_{\tau_{p+1}} \vee \mathcal{G}_{\tau_{p+1}}$:

   (i) $\lambda^{C_{p+1}}_t = 0$ for $t \geq \tau_{p+1}$.

   (ii) Adapt λ^v_t for $v \in \mathcal{S}^{p+1}$ for default feedbacks according to (88).

5. Replace index p with $p+1$, and go back to Step 1., calculating the default event τ_{p+2} cumulative distribution function $F^{(p+2)}(t)$, starting from τ_{p+1} and having further the knowledge that C_{p+1} defaulted at τ_{p+1}. Repeat this algorithm until you drop out at Step 2. (iii) when for the $p+1$ default event τ_{p+1} we have $\tau_p \leq T < \tau_{p+1}$.

Remark 134 *Depending on the result's accuracy, the number of default prone counterparts S, their average default probability, the type of risk measures (expectations, percentiles laying in the tail), etc. one needs to draw more or less default sequences.*

Remark 135 *The economic path $(X_{t_i})_{i=0}^m$ is not updated at each default time because we just consider default probability feedbacks on the companies and not on the entire economic system.*

Remark 136 *By construction the default event times τ_p are linked to the economy and are also surprises in the economic filtration $\mathcal{E}_t$. At each moment investors know what might be the possible feedbacks in case one company defaults, but since they cannot predict at what time which company defaults, the default feedbacks are also surprises.*

Remark 137 *We believe that the above simulation procedure produces more realistic default sequence than the procedure, where the companies default, which asset values at horizon lies lower than their default thresholds because*

1. *It allows default feedbacks;*
2. *It allows defaults within the interesting time horizon and not just at the time horizon;*
3. *It is easier to link the total default probability to economic cycles than to link all individual default probabilities to economic cycles.*

9.4 Simulating Recoveries

In this section we want to develop a recovery simulation engine following our recovery model presented in Section 7. After the time τ_p, when company C_p defaulted, we have to model the losses or respectively the recoveries of the liabilities $L_1^{C_p}, ..., L_l^{C_p}$ of company $C_p = i$.

With one economic path $(X_{t_i})_{i=0}^m$ we have developed in the previous section one default sequence $\{(\tau_k, C_k),\ k = 1, ..., S\}$. With the same economic path $(X_{t_i})_{i=0}^m$ we will develop the recoveries (rates) on the defaulted securities, i.e. whose underlying companies defaulted before the interesting business time horizon T, let's say the companies C_k for $k = 1, ..., D \leq S$. For simplification we present the procedure for valuing the recoveries on the liabilities $L_1^i, ..., L_l^i$ of just one company $C_k = i$ that defaulted at time τ_k. For $t < \tau_k$ the liabilities $L_1^i, ..., L_l^i$ have to be valued as default prone assets, while from τ_k on as defaulted assets. Before presenting the simulation procedure we have to recall few definitions and assumptions:

1. The company's market asset value processes A_t^i, as assumed in (89), follows an APT approach like
$$A_t^i = \sum_{l=1}^{K+S} A^{il}\ X_t^l, \tag{109}$$
where $A = (A^{i1}, ..., A^{iK+S}) \in \mathbb{R}^{K+S}$ are constants. The constants A^{il} for $l = 1, ..., K$ describe the company's asset value dependence with the systematic risk factors movement, while $A^{iK+i}\ X_t^{K+i}$ represents the unsystematic asset dynamics, independent of all other risk factors.

2. $L_1^i, ..., L_l^i$ corresponds to the outstanding lent exposure to company i at time τ_k (as defined in Definition 81).

3. The discount D_i on the asset value $A_{T'}^i$ represents the costs due to reorganization or liquidation. Since these costs D_i are industry dependent (reorganizing or liquidating a computer company causes probably other costs than reorganizing respectively liquidating a car manufacturer), we assume that we know the reorganization (liquidation) costs for each industry, respectively that we know these cost for each risk factor X^l for $l = 1, ..., K$ as presented in Table 2.

Risk Factor	**Reorganization Costs**	**Liquidation Costs**
$\mathbf{X}^1$	d_1^R	d_1^L
$\vdots$	$\vdots$	$\vdots$
$\mathbf{X}^K$	d_K^R	d_K^L

Depending on reorganization or liquidation P, we define the final discount D_i on the company's i assets A_t^i by
$$D_i^P = (A^{i1}, ..., A^{iK}) \cdot (d_1^P, ..., d_K^P)^T. \tag{110}$$

4. For the settlement of these liabilities $L_1^i, ..., L_l^i$ we assume absolute priority (see Definition 67): $L_1^i \succ ... \succ L_l^i$.

After having recalled some definitions and assumptions, we propose the following steps for simulating recoveries on the outstanding liability securities $L_1^i, ..., L_l^i$:

1. [FM] estimated that 85% of defaulted companies are reorganized, hence draw whether company i is reorganized or liquidated: $P_i \sim$ binomial(85%).

2. Draw, depending on reorganization or liquidation P_i, the payoff settlement time $T_i' = \tau_k + \delta_i$, where $\delta_i \sim \mathcal{E}(\Gamma_i^{P_i})$ with

$$\Gamma_i^{P_i} = (A^{i1}, ..., A^{iK}) \cdot (\gamma_1^{P_i}, ..., \gamma_1^{P_i}), \tag{111}$$

where the parameters γ_j^i are as assumed in Table 3.

Risk Factor	Reorganization $\mathbb{E}$[Payoff Time]	Liquidation $\mathbb{E}$[Payoff Time]
$\mathbf{X}^1$	γ_1^R	γ_1^L
$\vdots$	$\vdots$	$\vdots$
$\mathbf{X}^K$	γ_K^R	γ_K^L

This simulation step is similar to Step 2. of the default sequence construction, where we draw the next default event (see Figure). Because of discreteness identify T_i' with the closest time grid point t_s and in case $T_i' > T$ we have to extend the drawn economic path $(X_{t_i})_{i=0}^m$ until T_i'.

3. Calculate, depending on reorganization or liquidation P_i, the asset sharing discount $D_i^{P_i}$ according to (110) and Table 2.

4. Calculate the asset sharing value $A_{T_i'}^{i'}$ at time T_i' by

$$A_{T_i'}^{i'} = \left(1 - D_i^{P_i}\right) \sum_{l=1}^{K+S} A^{il} \; X_{T_i'}^l. \tag{112}$$

5. Calculate, according to (46), the payoff for liability L_k^i at time T_i' recursively by

$$\begin{aligned} R_k\left(A_{T_i'}^{i'}\right) &= \min\left\{L_k, \left(A_{T_i'}^{i'} - \sum_{j<k} L_j\right)^+\right\} \\ &= A_{T_i'}^{i'} - \sum_{j<k} R_j\left(A_{T_i'}^{i'}\right) - \left(A_{T_i'}^{i'} - \sum_{j\leq k} L_j\right)^+. \end{aligned} \tag{113}$$

6. The recovery value at time $t_s \geq \tau_k$ corresponds finally to

$$R_k\left(A_{T_i'}^{i'}\right) \begin{cases} P(T_i', t_s)^{-1} & if \ T_i' \leq t_s \\ P(t_s, T_i') & if \ T_i' > t_s \end{cases}, \tag{114}$$

where $P(T_i', t_s)$ and $P(t_s, T_i')$ can be obtained by a numerical integration, using representation (90)

$$P(t,T) = \delta_{t,T}^r = e^{-\int_t^T r_s \, ds} = e^{-\int_t^T \sum_{l=1}^K r^l \; X_s^l \, ds} = \prod_{l=1}^K e^{-r^l \int_t^T \; X_s^l \, ds}. \tag{115}$$

9.5 Simulating Default Prone Securities Portfolio Values

In this section we merge the default sequence and the recovery simulation procedures described in the previous Sections 9.3 and 9.4 for obtaining a default securities valuation simulation procedure. When the management is interested in a buy and hold until maturity strategy of loans or corporate bonds then the "interim" losses due to market movements or credit quality shifts are not of relevant interest, but just the surviving of their portfolio counterparts until maturity is of interest. For this buy and hold strategy the portfolio management is hence just interested in modelling the possible losses due to default before maturity.

Before presenting the simulation procedure we have to characterize the portfolio exposures of the different market liability securities $L_1^i, ..., L_{l_i}^i \in L^i$ on several underlying default prone companies $i \in \mathcal{S}^0$. An investor usually does not hold the entire liability assets L_k^i but only a certain exposure E_k^i (see credit exposure Definitions 79 and 79) respectively a fraction w_k^i of L_k^i

$$w_k^i = \frac{E_k^i}{L_l^i}. \tag{116}$$

The investor's total portfolio exposure E is hence described by the vector

$$E = (E_1^1, ...E_{l_1}^1, ..., E_1^S, ..., E_{l_S}^S).$$

For calculating its total credit exposure $|E|$, since he runs only a credit risk in case the exposure is positive $\left(E_k^i\right)^+ > 0$, we need to consider also the netting agreements with the different dealing counterparts. When the investor has a netting agreement with counterpart i, positive and negative exposures are netted and the exposure at risk amounts to the sum of the exposures $\left(\sum_{k \in L^i} E_k^i\right)^+$, while without netting agreements, the exposure at risk amounts to $\sum_{k \in L^i} \left(E_k^i\right)^+$. We denote $N \subset \mathcal{S}^0$ the subset of companies with whom the investor has a netting agreement. The total exposure at risk corresponds therefore to

$$|E| = \sum_{i \in \mathcal{S}^0} \left\{ \begin{array}{ll} \sum_{k \in L^i} \left(E_k^i\right)^+ & i \notin N \\ \left(\sum_{k \in L^i} E_k^i\right)^+ & i \in N \end{array} \right. \tag{117}$$

Example 138 *A bank lent 10 millions CHF to company XY and are holding assets of XY worth 3 millions CHF. With netting agreement the bank has a net exposure at risk of 7 millions CHF, while without agreement they have the full 10 millions CHF exposure at risk.*

After having presented the investor's portfolio credit exposure, for calculating the incurred portfolio losses for a specific economic path $(X_{t_i})_{i=0}^m$ at the different times $t_i \overset{def}{=} \frac{T}{m} i, \quad i = 1, .., m,$we propose the following simulation procedures:

1. Construct default sequence as described in Sections 9.3: $\{(\tau_k, C_k),\ k = 1, ..., S\}$.

2. For all companies C_k such that $\tau_k \leq t_i$ calculate the recoveries $R_l^{C_k}$ for the liabilities $L_l^{C_k}$ payed off at time T'_{C_k} as proposed in Sections 9.4.

3. The loss $V_l^{C_k}$ on the liability security $L_l^{C_k}$ at time t_i then corresponds to

$$V_l^{C_k} = L_l^{C_k} P(\tau_k, t_i) - R_l^{C_k} \begin{cases} P(T'_{C_k}, t_i)^{-1} & if\ T'_{C_k} \leq t_i \\ P(t_i, T'_{C_k}) & if\ T'_{C_k} > t_i \end{cases} \tag{118}$$

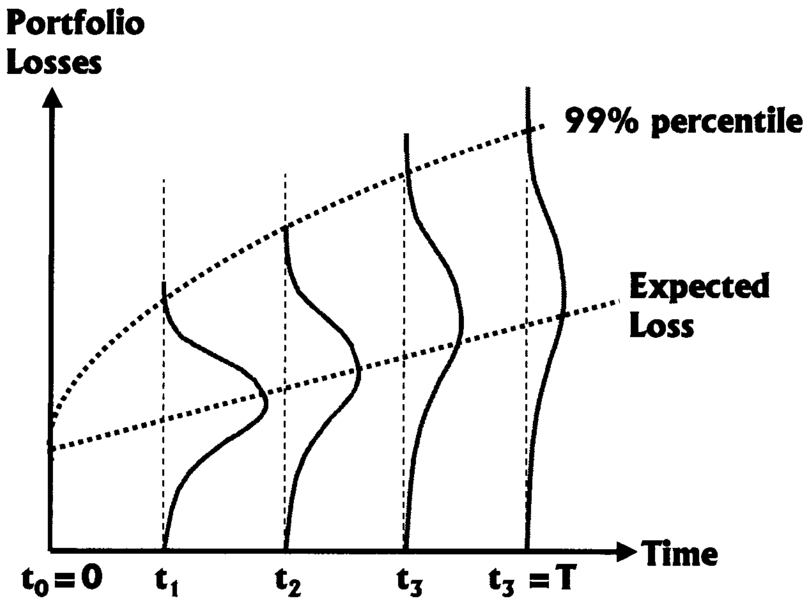
4. The investor holding only a fraction $w_l^{C_k} = E_l^{C_k} / L_l^{C_k}$of the liability security $L_l^{C_k}$ incurs a respective loss at time t_i on this position corresponding to

$$w_l^{C_k} V_l^{C_k} \tag{119}$$

5. Considering further the netting agreements, the total loss V_{t_i} the investor incurred because of defaults before time t_i correspond thus to

$$V_{t_i} = \sum_{k \in \mathcal{D}_{t_i}} \left\{ 1_{\{C_k \notin N\}} \sum_{l \in L^{C_k}} \left(w_l^{C_k} V_l^{C_k} \right)^+ + 1_{\{C_k \in N\}} \left(\sum_{l \in L^{C_k}} w_l^{C_k} V_l^{C_k} \right)^+ \right\}. \tag{120}$$

This simulation procedure delivers for each economic path X the respective portfolio losses at the different time grids $t_i \geq 0$. Hence drawing many economic paths X, we obtain portfolio loss distributions at each time grids $t_i \geq 0$ as showed in the next Figure.



Portfolio Default Loss Distributions

We denote the time t_i loss distribution's density by $f_{t_i}(x)$ for losses $x \geq 0$.

Unfortunately when one has to analyze the losses on a book of traded credit securities, the estimation of the possible losses due to default is not sufficient because the trader may incur losses due either by unfavorable market movements or by some credit quality changes. The simple calculation of the default loss distribution is therefore not sufficient and one needs to calculate the profit and loss (P&L) distribution for obtaining a loss distribution, which reflects also market and credit quality changes. For this one has to revalue at each time grid $t_i \geq 0$ the credit book value, i.e. revalue all position.

For this we have to calibrate our pricing model (Section 8.5) with actual market prices as presented in the Calibrating a Martingale Measure Section 8.6. So lets suppose that for the risk factor's initial condition x_0, we have estimated from market prices the calibration parameters β^*.

In our pricing model all prices of credit claims L_k^i have a representation like

$$\Pi_t^{L_k^i} = F^{L_k^i}(t, X_t^1, ..., X_t^n, \beta^*). \tag{121}$$

The portfolio has thus at time $t = 0$ a value of

$$F^{port}(0, X_0^1, ..., X_0^n, \beta^*) = \sum_{i \in \mathcal{S}^0} \sum_{L_k^i \in L^i} w_k^i F^{L_k^i}(0, X_0^1, ..., X_0^n, \beta^*), \tag{122}$$

where w_k^i is the investor's share on the kth liability security of company i, i.e. L_k^i

$$w_k^i = \frac{E_k^i}{L_k^i}. \tag{123}$$

Under the assumption that the w_k^i remain constant until $t \geq 0$ the portfolio value at the time $t \geq 0$ corresponds then to

$$F^{port}(t, X_t^1, ..., X_t^n, \beta^*) = \sum_{i \in \mathcal{S}^0} \sum_{L_k^i \in L^i} w_k^i F^{L_k^i}(t, X_t^1, ..., X_t^n, \beta^*). \tag{124}$$

Drawing again many states of the economy X and calculating

$$P\&L(t, X_t) = F^{port}(t, X_t^1, ..., X_t^n, \beta^*) - F^{port}(0, X_0^1, ..., X_0^n, \beta^*) \tag{125}$$

we will obtain a portfolio $P\&L$ distributions at the time points $t \geq 0$ similar as shown with the last Figure.

Remark 139 *With equation (125) we evaluate the possible portfolio value changes due to market or credit changes or finally due to default.*

Remark 140 *It would be of interest to know also what happens when we allow the share w_k^i on liability L_k^i to change, i.e. when we allow not just buy and hold strategies.*

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10 Conclusions

In the Credit Risk Characteristic's section we have seen that credit risk related securities typically show some market imperfections (illiquidity, high transaction costs, not transparent premiums, institutional arbitrage, etc.), such that the assumption of a perfect complete market is far too strong.

On the other side, in the same section, we have seen that credit related security prices show also some relevant systematic structure: security prices of similar credit risks behave similarly (example, companies belonging to the same industrial sector). Even securities of a defaulted company show some internal price relations, as we have shown in the Recovery section.

Because of these systematic market price relations, we are convinced that the weaker assumptions of no-arbitrage is adequate for modelling prices of defaulted and default prone securities. The no-arbitrage condition, as we have seen in the pricing section, is heuristically equivalent to the existence of infinitely many equivalent martingale measures, since markets are incomplete.

The no-arbitrage condition gives further the advantage that all security prices (default-free, default prone and defaulted) can be simply represented as risk neutral expectations of discounted cash flows, conditional to default or not.

Finally we observe that it is not enough studying only default prone security prices, but we also need to study their relations to default-free and securities of already defaulted companies. These auxiliary securities establish benchmarks for default prone securities, hence it is convenient to have one pricing representation method.

For modelling the conditional to default condition, we have built under an equivalent martingale measure a multi company default model, which allows default feedbacks. We are convinced, that these default feedbacks are important to be considered, because we believe that at the default of a company, investors receive new important informations. These informations might change investors' credit quality opinion of the surviving companies and thus revalue their risk adjusted prices. We further observed that the default dependencies, responsible for the default clustering and for the strong economic cycle dependence, are principally caused by three different sources: i) Companies' finance may be driven by similar underlying systematic risk factors; ii) Instantaneous disclosure of hidden systematic weaknesses, under which other companies may also suffer; iii) Default of one company may cause direct financial troubles to others.

The quantification of recoveries and their valuation in case a company defaults, is a further fundamental ingredient for modelling default prone security's prices. For achieving realistic recoveries, we have first observed market prices of defaulted securities, noting three important effects: i) All liability securities of a company simultaneously lose their default risk at the default time; remaining therefore just affected by the same recovery risk. ii) Reorganization (Chapter 11) mostly applies: 85% of the companies are reorganized within either three or six months, constraining the reorganization procedure period by law. iii) The priority rules play an important rule in the distribution of the remaining asset value.

Based on these three properties of defaulted securities, we have built a model for deducting from defaulted instrument's prices the amount that investors expect to recover. Comparing further these recoveries with the respective initial investments, we get a model for recovery rates, that are used in the default prone security's pricing model.

In the valuation section we have seen that the multi default model and the recovery model can be merged into a martingale pricing model, obtaining the pricing equations that default prone security prices have to follow.

Compared to the traditional default prone securities pricing equation, we obtain a new additional term, which reflects the surprise arrival of new information at default events.

The difference between the pricing theory with and without default feedback is therefore specially seen, when we study security's prices depending on more than one default prone company. This is the case when we analyze credit risk transfer portfolio's or structured credit derivative's prices.

For gauging the model with market prices, we proposed an "inversion of the default term structure", obtaining the market parameters of an equivalent martingale measure.

Finally for the business use, since analytical prices with default feedback cannot be easily calculated, we proposed three simulation procedures for obtaining default sequences with default feedbacks, recoveries and default prone security's prices.

Summarizing we have developed a credit portfolio valuation model, that can be applied in the business for credit risk controlling, managing and pricing.

Concluding, we believe that our credit model with default feedback has some advantages compared to other business multi default models, because:

1. It allows for default feedbacks, which one observe in credit markets;

2. Defaults happen not just at the horizon as for example modeled in the Merton type models;

3. It allows efficient and simple simulation procedures, since we can just simulate the next default event and then draw the company that defaults;

4. We can easier link the default event probability to economic variables.

5. It is practically feasible: replacing some basic credit modules of the Risk Aggregation Engine (RAE), we obtain a default feedback model. (RAE is a model we have developed and implemented at Swiss Re for aggregating consistently market and credit risks. For details see [Rm1] or [Rm2].)

11 Appendix

11.1 Specifying Default Intensities

In the multi default time section 6 we defined for all companies the default time densities $d^{(j+1)}(w,x,t,i)$ and we were looking for the default time intensities λ_t^i satisfying Theorem 52. In this section we want to study the inverse problem: for given **F**-predictable default time intensities λ_t^i we are looking for default time density solutions $d^{(j+1)}(w,x,t,i)$, satisfying the conditions of Theorem 52. For studying the behavior of the default time densities $d^{(j+1)}(w,x,t,i)$, we will give different patterns of **F**-predictable default time intensities λ_t^i and solve for the densities $d^{(j+1)}(w,x,t,i)$.

In the first model we give the intensities as constants $\lambda^i \in \mathcal{F}_0$, vanishing at the default of the respective company.

In a further model we allow to adapt the intensities λ^i at each default time τ_j, but within each no-defaults period (i.e. for $t \in [\tau_j, \tau_{j+1})$) the intensities remain constants $\lambda^{ij} \in \mathcal{F}_{\tau_j}$ and vanishes at the default of the respective company. The economic system includes again S default prone companies. Here in this section we take over the filtration $\mathcal{F}_t$ as defined in Section 6, however we **assume** that investor's information is just updated at each default event τ_j, i.e. $\mathcal{F}_t = \mathcal{E}_{\tau_j} \vee \mathcal{G}_t$.

Because the default time densities $d^{(j+1)}(.)$ are at most updated with new information $\mathcal{F}_t$ at each default time τ_j, their notation can be simplified

$$d^{(j+1)}(w,x,i) = d^{(j+1)}(w,x,\tau_j,i),$$

where we recall that the x is the time length since the last default τ_j.

We further denote the waiting time from the last default to the next default by $\delta_{j+1} \stackrel{def}{=} \tau_{j+1} - \tau_j$

More precisely in this section we **suppose** that these default time distributions admit **densities** $d^{(j+1)}(.)$ for all **F**-stopping times τ_j and for all $\Delta > 0$, such that for each company $i \in \mathcal{S}^j$:

$$\mathbb{P}[\delta_{j+1} \leq \Delta, C_{j+1} = i \mid \mathcal{F}_{\tau_j}](w) = \int_0^{\Delta} d^{(j+1)}(w,x,i)\, dx. \tag{126}$$

Similarly as in Proposition 44 we have that the **default event density** corresponds to

$$d^{(j+1)}(w,x) \stackrel{def}{=} \sum_{i \in S^j} d^{(j+1)}(w,x,i), \tag{127}$$

where we recall that S^j denotes the set of survived companies in the jth no default time period. Since the intensities are defined through the filtration $\mathcal{F}_t$ we have to adapt our main Theorem 52 of Section 6 as follows.

Theorem 141 *For all $i = 1, ..., S$ we define the processes A_t^i by (33), i.e. the integral of the processes λ_t^i , which are defined by*

$$\lambda_t^i \stackrel{def}{=} \sum_{j=0}^{S-1} \frac{d^{(j+1)}(w, t-\tau_j, i)}{1 - \int_0^{t-\tau_j} d^{(j+1)}(w,u)\, du} 1_{\{\tau_j < t \leq \tau_{j+1}\}}, \tag{128}$$

Then $A_{t\wedge\tau_n}$ is the **F***-default compensator of the default indicator process $N_{t\wedge\tau_n}$ for all $1 \leq n \leq S$, i.e. for all $i = 1, ..., S$ and for all $n = 1, ..., S$, $N^i_{t\wedge\tau_n} - A^i_{t\wedge\tau_n}$ is an* **F***-martingale.*

For proving the Theorem we need the following Lemma that can be found in [Pp] as an exercise.

Lemma 142 *Let X_t be a real valued* **F***-adapted process such that for all bounded* **F***-stopping times T, X_T is integrable and $\mathbb{E}[X_T] = \mathbb{E}[X_0]$. Then X_t is an* **F***-martingale.*

Proof. With Lemma 142, it is sufficient to show that

$$\mathbb{E}\left[N^i_{\sigma\wedge\tau_n} - A^i_{\sigma\wedge\tau_n}\right] = 0$$

for all $1 \leq n \leq S$ and all finite **F**-stopping times σ. We decompose the finite **F**-stopping times $\sigma \geq \tau_n$ as

$$\sigma \wedge \tau_{n+1} = (\tau_n + R_n) \wedge \tau_{n+1}, \quad on\ \{\sigma \geq \tau_n\}, \tag{129}$$

where R_n is an **F**-measurable nonnegative random variable. With the following decomposition

$$\lambda^i_s = \sum_{j=0}^{n-1} \lambda^{ij}_s 1_{\{\tau_j < s \leq \tau_{j+1}\}},$$

we first calculate

$$\begin{aligned}
\mathbb{E}\left[A^i_{\sigma\wedge\tau_n}\right] &= \mathbb{E}\left[\int_0^{\sigma\wedge\tau_n} \lambda^i_s \, ds\right] \\
&= \mathbb{E}\left[\int_0^{\sigma\wedge\tau_n} \sum_{j=0}^{n-1} \lambda^{ij}_s 1_{\{\tau_j < s \leq \tau_{j+1}\}} \, ds\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{n-1} \int_0^{R_j \wedge \delta_{j+1}} \lambda^{ij}_{\tau_j+s} 1_{\{\tau_j \leq \sigma\}} \, ds\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{n-1} \mathbb{E}\left[\int_0^{R_j \wedge \delta_{j+1}} \lambda^{ij}_{\tau_j+s} 1_{\{\tau_j \leq \sigma\}} \, ds \mid \mathcal{F}_{\tau_j}\right]\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{n-1} \mathbb{E}\left[\int_0^{R_j \wedge \delta_{j+1}} \lambda^{ij}_{\tau_j+s} \, ds \mid \mathcal{F}_{\tau_j}\right] 1_{\{\tau_j \leq \sigma\}}\right],
\end{aligned}$$

where we have used the fact that $\{\sigma \geq \tau_j\}$ is $\mathcal{F}_{\tau_j}$ measurable.

Second, using a telescopic sum development, the decomposition of the stopping time σ and the fact that $\{\sigma \geq \tau_j\}$ belongs to $\mathcal{F}_{\tau_j}$, we obtain

$$\mathbb{E}\left[N^i_{\sigma\wedge\tau_n}\right] = \mathbb{E}\left[\sum_{j=0}^{n-1} \left(N^i_{\sigma\wedge\tau_{j+1}} - N^i_{\sigma\wedge\tau_j}\right)\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\sum_{j=0}^{n-1} 1_{\{R_j \geq \delta_{j+1}\}} 1_{\{C_{j+1}=i\}} 1_{\{\sigma \geq \tau_j\}}\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{n-1} \mathbb{E}\left[1_{\{R_j \geq \delta_{j+1}\}} 1_{\{C_{j+1}=i\}} 1_{\{\sigma \geq \tau_j\}} \mid \mathcal{F}_{\tau_j}\right]\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{n-1} \mathbb{E}\left[1_{\{R_j \geq \delta_{j+1}\}} 1_{\{C_{j+1}=i\}} \mid \mathcal{F}_{\tau_j}\right] 1_{\{\sigma \geq \tau_j\}}\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{n-1} \mathbb{P}\left[\delta_{j+1} \leq R_j, C_{j+1} = i \mid \mathcal{F}_{\tau_j}\right] 1_{\{\sigma \geq \tau_j\}}\right].
\end{aligned}
$$

From the above two calculations, we see that $\mathbb{E}\left[N^i_{\sigma\wedge\tau_n}\right] = \mathbb{E}\left[A^i_{\sigma\wedge\tau_n}\right]$, when for all $\sigma \geq \tau_j$ the following equality holds

$$
\mathbb{E}\left[\int_0^{R_j} \lambda^{ij}_{\tau_j+s}\, ds \mid \mathcal{F}_{\tau_j}\right] 1_{\{\tau_j \leq \sigma\}} = \mathbb{P}\left[\delta_{j+1} \leq R_j, C_{j+1} = i \mid \mathcal{F}_{\tau_j}\right] 1_{\{\sigma \geq \tau_j\}},
$$

which can be written with help of the density function $d^{(j+1)}(.)$ definitions (see Definition equation (21)) and secondly using Fubini's Theorem for exchanging integrals

$$
\begin{aligned}
\int_0^{R_j} d^{(j+1)}(w,s,i)\, ds &= \int_0^{\infty} d^{(j+1)}(w,u) \int_0^{R_j\wedge u} \lambda^{ij}_{\tau_j+s}\, ds\, du \\
&= \int_0^{R_j} \lambda^{ij}_{\tau_j+s} \int_s^{\infty} d^{(j+1)}(w,u)\, du\, ds.
\end{aligned}
$$

Differentiating both sides we get

$$
d^{(j+1)}(w,R_j,i) = \lambda^{ij}_{\sigma} \int_{R_j}^{\infty} d^{(j+1)}(w,u)\, du
$$

and hence we see that the above equality is satisfied for

$$
\begin{aligned}
\lambda^{ij}_{\sigma} &= \frac{d^{(j+1)}(w,R_j,i)}{\int_{R_j}^{\infty} d^{(j+1)}(w,u)\, du} \\
&= \frac{d^{(j+1)}(w,R_j,i)}{1-\int_0^{R_j} d^{(j+1)}(w,u)\, du} \\
&= \frac{d^{(j+1)}(w,\sigma-\tau_j,i)}{1-\int_0^{\sigma-\tau_j} d^{(j+1)}(w,u)\, du}.
\end{aligned}
$$

This concludes the proof. ■

So we see that if we want to find default time densities $d(.)$ for given default time intensities λ, the default time densities $d(.)$ have to satisfy the relations (128).

11.2 Poisson Model

Reconsider the simplest non trivial extension of the one company default time model, as presented in Example 41, where you chose a $\lambda_i > 0$ for every default prone company $i = 1, ..., S$, such that

$$\mathbb{P}[\tau_i < T] = 1 - e^{-\lambda_i T}.$$

We have seen that in this model the default time density distribution for company i looks like $d(w,t,i) = \lambda_i e^{-\lambda_i t}$. Further we know that for each default intensity process λ_i there exist a default indicator process N_t^i, such that $N_t^i - \int_{[0,t]} \lambda_s^i \, ds$ is a $\mathbf{F}$-martingale, and default of company i is defined as the first jump of the point process N_t^i.

The first result of this subsection shows that this model extension obviously does not consider, except for very special non interesting cases, any feedbacks from the economy or from the information of previous defaulted companies.

Proposition 143 *Assume that the default of a company is modeled specifying the default intensities via corresponding default probabilities $\mathbb{P}[\tau_i < T] = 1 - e^{-\lambda_i T}$. Assume further that we want to have a dependence structure between the default prone companies given by the system of equations (128). Then for all no-defaults period $0 \leq p \leq S$ and for all companies $i, k \in \mathcal{S}^p$the following two equalities hold for almost all $t \geq 0$*

$$\frac{\lambda_t^i}{d(w,t,i)} = \frac{\lambda_t^k}{d(w,t,k)}; \tag{130}$$

$$\lambda_t^i = \lambda_t^k. \tag{131}$$

Proof. We show the proof in the case of two companies, which can be easily extended to S. From the equality of dependencies (128), we know that for $t \leq \tau_1$ the densities $d(.)$ have to satisfy

$$\lambda_t^i = \frac{d(w,t,i)}{1 - \int\limits_{[0,t]} d(w,x) \, dx}, \qquad i = 1, 2,$$

where $\int_0^t d(w,t) \, dx = \int_0^t (d(w,t,1) + d(w,t,2)) \, dx$ corresponds to the probability of a default event before for t. Since for $t \leq \tau_1$ the denominator is the same for both equations $i = 1, 2$, the equality (130) easily follows. Equation (131) follows now from the specific assumption on the default time distributions: $d(w,t,i) = \lambda_i e^{-\lambda_i t}, i = 1, 2$. ■

Remark 144 *We will later show that, if we model default dependence trough equation (128), then a generalization of equality (130) holds also in a more general model.*

11.3 Multi Companies Constant Intensity Model

In previous subsection we obtained a negative result, because we were looking for solutions of Poisson type default time densities: $d(w,t,i) = \lambda_i e^{-\lambda_i t}$. In this subsection we will not restrict to Poisson type solutions, but for more general ones. For each company $i = 1, ..., S$ we **assume** a constant $\mathbf{F}$-default intensity until default

$$\lambda_t^i = \lambda^i \mathbf{1}_{\{i \in \mathcal{S}_t\}},$$

which is $\mathbf{F}$-measurable. More precisely λ^i is $\mathcal{E}_0$-measurable and $\mathbf{1}_{\{i \in \mathcal{S}_t\}}$ is $\mathcal{G}_t$-measurable. We further **assume** that the dependence structure between the S default prone companies is given by the system of equations (128).

The questions we are interested in are as follows

1. Does there exist a unique solution?
2. How do the solutions $d^{(n+1)}(w,x,i), i = 1, ..., S$ and $d^{(n+1)}(w,x)$ look like?

An answer is given in the following Proposition

Proposition 145 *Under the assumptions of the present section, (128) has a unique solution within all no-defaults periods $0 \le p \le S-1$, for all $x \ge 0$,*

$$d^{(p+1)}(w,x,i) = \lambda^i e^{-x\sum_{j\in\mathcal{S}^p}\lambda^j}, \quad i \in \mathcal{S}^p \tag{132}$$

$$d^{(p+1)}(w,x,i) = 0, \quad i \in \mathcal{D}^p$$

and the $p+1$-event distribution

$$d^{(p+1)}(w,x) = \left(\sum_{j\in\mathcal{S}^p}\lambda^j\right) e^{-x\sum_{j\in\mathcal{S}^p}\lambda^j}, \tag{133}$$

Moreover for all $i,j \in \mathcal{S}^p$, for all $t \in [\tau_p, \tau_{p+1})$, must hold

$$\frac{d^{(p+1)}(w,t-\tau_p,i)}{\lambda^i} = \frac{d^{(p+1)}(w,t-\tau_p,j)}{\lambda^j}. \tag{134}$$

Proof. Proposition 145 can be quickly checked by replacing the proposed solutions (133, 132) into the system (128). Finally (134) holds since all equations of (128) have the same denominator: the no event probability. ∎

Within each no-default period $\tau_p \le t \le \tau_{p+1}$ we can calculate the probability of having a next default event τ_{p+1} before time t and which company is most likely bankrupt. In fact

Proposition 146 *Within all no-defaults periods $0 \le p \le S$, $\tau_p \le s \le t \le \tau_{p+1}$, the $(p+1)$th event probability corresponds to*

$$\mathbb{P}[\tau_{p+1} < t \mid \mathcal{F}_s] = 1 - e^{-(t-s)\sum_{j\in\mathcal{S}^p}\lambda^j} \tag{135}$$

and for all surviving companies $i \in \mathcal{S}^p$

$$\mathbb{P}[\tau_{p+1} < t, C_{p+1} = i \mid \mathcal{F}_s] = \mathbb{P}[C_{p+1} = i \mid \mathcal{F}_s]\ \mathbb{P}[\tau_{p+1} < t \mid \mathcal{F}_s], \tag{136}$$

where $\mathbb{P}[C_{p+1} = i \mid \mathcal{F}_s] = \frac{\lambda^i}{\sum_{j\in\mathcal{S}^p}\lambda^j}$.

Proof. The probability formulas (135, 136) can be obtained by the integration of the different densities (133, 132), presented in Proposition 145. In fact for all no-default periods $0 \leq p \leq S-1$ and for $\tau_p \leq s \leq t \leq \tau_{p+1}$ we have

$$\begin{aligned}
\mathbb{P}[\tau_{p+1} < t \mid \mathcal{F}_s] &= \mathbb{P}[\tau_{p+1} < t \mid \mathcal{F}_{\tau_p}, s < \tau_{p+1}] \\
&= \frac{\mathbb{P}[s < \tau_{p+1} < t \mid \mathcal{F}_{\tau_p}]}{\mathbb{P}[\ s < \tau_{p+1} \mid \mathcal{F}_{\tau_p}]} \\
&= \frac{\mathbb{P}[s < \tau_{p+1} < t \mid \mathcal{F}_{\tau_p}]}{1 - \mathbb{P}[\ s \geq \tau_{p+1} \mid \mathcal{F}_{\tau_p}]} \\
&= \frac{\left(\sum_{j \in \mathcal{S}^p} \lambda^j\right) \int_s^t e^{-v \sum_{j \in \mathcal{S}^p} \lambda^j} dv}{1 - \left(\sum_{j \in \mathcal{S}^p} \lambda^j\right) \int_0^s e^{-v \sum_{j \in \mathcal{S}^p} \lambda^j} dv} \\
&= \frac{e^{-s \sum_{j \in \mathcal{S}^p} \lambda^j} - e^{-t \sum_{j \in \mathcal{S}^p} \lambda^j}}{e^{-s \sum_{j \in \mathcal{S}^p} \lambda^j}} \\
&= 1 - e^{-(t-s) \sum_{j \in \mathcal{S}^p} \lambda^j}.
\end{aligned}$$

Similarly for all companies $i \in \mathcal{S}^p$ we have

$$\begin{aligned}
\mathbb{P}[\tau_{p+1} < t, C_{p+1} = i \mid \mathcal{F}_s] &= \mathbb{P}[\tau_{p+1} < t, C_{p+1} = i \mid \mathcal{F}_{\tau_p}, s < \tau_{p+1}] \\
&= \frac{\mathbb{P}[s < \tau_{p+1} < t, C_{p+1} = i \mid \mathcal{F}_{\tau_p}]}{\mathbb{P}[\ s < \tau_{p+1} \mid \mathcal{F}_{\tau_p}]} \\
&= \frac{\lambda^i \int_s^t e^{-v \sum_{j \in \mathcal{S}^p} \lambda^j} dv}{1 - \left(\sum_{j \in \mathcal{S}^p} \lambda^j\right) \int_0^s e^{-v \sum_{j \in \mathcal{S}^p} \lambda^j} dv} \\
&= \frac{\lambda^i \int_s^t e^{-v \sum_{j \in \mathcal{S}^p} \lambda^j} dv}{\sum_{j \in \mathcal{S}^p} \lambda^j e^{-s \sum_{j \in \mathcal{S}^p} \lambda^j}} \\
&= \frac{\lambda^i}{\sum_{j \in \mathcal{S}^p} \lambda^j} \frac{e^{-s \sum_{j \in \mathcal{S}^p} \lambda^j} - e^{-t \sum_{j \in \mathcal{S}^p} \lambda^j}}{e^{-s \sum_{j \in \mathcal{S}^p} \lambda^j}} \\
&= \frac{\lambda^i}{\sum_{j \in \mathcal{S}^p} \lambda^j} \left[1 - e^{-(t-s) \sum_{j \in \mathcal{S}^p} \lambda^j}\right] \\
&= \mathbb{P}[C_{p+1} = i \mid \mathcal{F}_s]\ \mathbb{P}[\tau_{p+1} < t \mid \mathcal{F}_s].
\end{aligned}$$

■

Remark 147 *From equality (136) we see that the events* $\{C_{p+1} = i\}$ *and* $\{\tau_{p+1} < t\}$ *are formally independent, given the information* $\mathcal{F}_s$.

Remark 148 *Furthermore from the formulas (135,136), we observe that for all no-defaults periods* $0 \leq p \leq S-1$ *and for all* $s \leq t \in [\tau_p, \tau_{p+1})$ *we have*

$$\mathbb{P}[\tau_{p+1} < t \mid \mathcal{F}_s] = \sum_{j \in \mathcal{S}^p} \mathbb{P}[C_{p+1} = j \mid \mathcal{F}_s]\ \mathbb{P}[\tau_{p+1} < t \mid \mathcal{F}_s] \tag{137}$$

and thus

$$\textstyle\sum_{j\in S^p}\mathbb{P}[C_{p+1}=j\mid\mathcal{F}_s]=\ 1,$$

which confirms that ***if there is a default event one and only one company defaults.***

Remark 149 *When the pth default happens, i.e. when one out of the $S-p-1$ companies defaults, then the default probability density belonging to the defaulted company vanishes and the dependency equation system (128) is also reduced by a further dimension.*

Remark 150 *Formula (136) suggests the following procedure for simulating multi company defaults:*

1. *Draw the* ***default time*** *τ_{p+1} from the default event distribution $d^{(p+1)}(w,x)$;*
2. *Dice* ***which*** *company defaults: company $i\in S^p$ defaults with probability*

$$\mathbb{P}[C_{p+1}=i\mid\mathcal{F}_s]=\frac{\lambda^i}{\sum_{j\in S^p}\lambda^j}.$$

At the default events τ_p one company defaults and thus its default density and intensity vanish. What happens to the default probabilities and default densities of the surviving companies $i\in S^{p+1}$ at the default times τ_p?

The default event probability density jumps at time τ_p by

$$\begin{aligned}
\Delta_{\tau_p}^{event} &= d^{(p+1)}(w,0)-d^{(p)}(w,\tau_p)\\
&= \left(\textstyle\sum_{j\in S^{p+1}}\lambda^j\right)e^{-0\sum_{j\in S^{p+1}}\lambda^j}-\left(\textstyle\sum_{j\in S^p}\lambda^j\right)e^{-\tau_p\sum_{j\in S^p}\lambda^j}\\
&= \left(\textstyle\sum_{j\in S^{p+1}}\lambda^j\right)-\left(\textstyle\sum_{j\in S^p}\lambda^j\right)e^{-\tau_p\sum_{j\in S^p}\lambda^j}\\
&= \left(\textstyle\sum_{j\in S^{p+1}}\lambda^j\right)\left(1-\frac{\sum_{j\in S^p}\lambda^j}{\sum_{j\in S^{p+1}}\lambda^j}e^{-\tau_p\sum_{j\in S^p}\lambda^j}\right)
\end{aligned}$$

and thus its relative jump corresponds to

$$\begin{aligned}
\tilde{\Delta}_{\tau_p}^{event} &= \frac{d^{(p+1)}(w,0)-d^{(p)}(w,\tau_p)}{d^{(p+1)}(w,0)}\\
&= 1-\frac{\sum_{j\in S^p}\lambda^j}{\sum_{j\in S^{p+1}}\lambda^j}e^{-\tau_p\sum_{j\in S^p}\lambda^j}.
\end{aligned}$$

At the p-th default event τ_p, the survivor's ($s\in S^{p+1}$) intensity jumps by

$$\begin{aligned}
\Delta_{\tau_p}^{s} &= d^{(p+1)}(w,0,s)-d^{(p)}(w,\tau_p,s) \qquad (138)\\
&= \lambda^s e^{-0\sum_{j\in S^{p+1}}\lambda^j}-\lambda^s e^{-\tau_p\sum_{j\in S^p}\lambda^j}\\
&= \lambda^s\left(1-e^{-\tau_p\sum_{j\in S^p}\lambda^j}\right),
\end{aligned}$$

where $\left(1-e^{-\tau_p \sum_{j\in \mathcal{S}^p} \lambda^j}\right)$ corresponds to the default event probability before time τ_{p+1} having the information $\mathcal{F}_{\tau_p}$. Therefore the relative jump to the new density for the survivors $s \in \mathcal{S}^{p+1}$is equal to

$$\begin{aligned}\tilde{\Delta}^s_{\tau_p} &= \frac{d^{(p+1)}(w,0,s) - d^{(p)}(w,\tau_p,s)}{d^{(p+1)}(w,0,s)} \\ &= \left(1-e^{-\tau_p \sum_{j\in \mathcal{S}^p} \lambda^j}\right).\end{aligned}$$

Remark 151 *The jumps $\tilde{\Delta}^s_{\tau_p}$are positive, therefore all survived companies $s \in \mathcal{S}^{p+1}$ become more likely to be knocked out at the next default event τ_{p+1}.*

11.4 Multi Company Jump Intensity Model

In this section we generalize the S dimensional constant intensity model, described in the previous section. The economic system contains S companies, whose intensities λ^i_t may jump at default event of another company τ_p. We represent the default intensities of company i for $t \geq 0$ by

$$\lambda^i_t \stackrel{def}{=} \sum_{p=0}^{S-1} \lambda^{ip} \mathbf{1}_{\{\tau_p \leq t < \tau_{p+1}\}} \mathbf{1}_{\{i\in \mathcal{S}^p\}}, \tag{139}$$

where the $\mathcal{F}_{\tau_p}$-measurable intensity λ^{ip} is constant over a no-defaults period p and vanishes at default of the company. The λ^i_t might jump at each default τ_p, because at this time new information $\mathcal{F}_{\tau_p} = \mathcal{E}_{\tau_p} \vee \mathcal{G}_{\tau_p}$ arrives.

Notation 152 *We denote the **intensity jumps of company** i at the event default times τ_p, $p = 1, ..., S$ by $\delta^{ip} \stackrel{def}{=} \lambda^{ip} - \lambda^{ip-1}$ and we further denote the **total intensity jump** from beginning until period p by $J^{ip} \stackrel{def}{=} \lambda^{ip} - \lambda^{i0}$.*

Remark 153 *These jumps δ^{ip} are by construction $\mathcal{F}_{\tau_p}$-measurable.*

Remark 154 *The jumps δ^{ip} are not just due to default and its causes, but also because of the changes about the present expectation of the economic evolution.*

Corollary 155 *With the above definitions and notations, we can show that for all no-defaults period $0 \leq p \leq S-1$ and $i \in \mathcal{S}^p$ the following equality holds*

$$\lambda^{ip}\mathbf{1}_{\{i\in\mathcal{S}^p\}} = \left\{\lambda^{i0} + \sum_{k=1}^{p} \delta^{ik}\right\} \mathbf{1}_{\{i\in\mathcal{S}^p\}}. \tag{140}$$

Proof. The indicator $\mathbf{1}_{\{i\in\mathcal{S}^p\}}$ guarantees that at default τ_k of company i ($C_k = i$) both sides of the equality vanish. Further because of the telescopic sum: $\sum_{k=1}^p \delta^{ik} = \sum_{k=1}^p \lambda^{ip} - \lambda^{ip-1} = \lambda^{ip} - \lambda^{i0}$, the Corollary becomes trivial. ■

Before the first default τ_1, the solutions of the densities were the same as in the constant model. Moreover we can say that before the first jump $\delta^{ip} \neq 0$ at τ_p, the densities were the same as in the multi company constant intensity model. From the first jump on $\delta^{ip} \neq 0$

at τ_p the solutions will differ. The dependencies between the companies are assumed as usual to be modeled via the equations (128) of Theorem 141, i.e. within each no-default period $0 \leq p \leq S-1$ for all $t \in [\tau_p, \tau_{p+1})$ the following system of equalities must hold

$$\lambda^{ip} = \frac{d^{(p+1)}(w, t-\tau_p, i)}{1 - \int_0^{t-\tau_p} d^{(p+1)}(w, u)\, du}, \qquad i \in \mathcal{S}^p. \tag{141}$$

Under these model conditions we are looking for the solutions: the default densities $d^{(p+1)}(w, x, i)$, which are described in the next Proposition.

Proposition 156 *Under the assumptions of the present section, the problem has a unique solution within all no-defaults periods $0 \leq p \leq S-1$, for all $x \in [0, \tau_{p+1} - \tau_p)$, given by*

$$d^{(p+1)}(w, x, i) = \lambda^{ip} e^{-x \sum_{j \in \mathcal{S}^p} \lambda^{jp}}, \quad i \in \mathcal{S}^p \tag{142}$$

for the defaulted

$$d^{(p+1)}(w, x, i) = 0, \quad i \in \mathcal{D}^p$$

and the $p+1$-event distribution

$$d^{(p+1)}(w, x) = \left(\sum_{j \in \mathcal{S}^p} \lambda^{jp}\right) e^{-x \sum_{j \in \mathcal{S}^p} \lambda^{jp}}, \tag{143}$$

Moreover for all $i, j \in \mathcal{S}^p$ and for all $t \in [\tau_p, \tau_{p+1})$ must hold

$$\frac{d^{(p+1)}(w, t-\tau_p, i)}{\lambda^{ip}} = \frac{d^{(p+1)}(w, t-\tau_p, j)}{\lambda^{jp}}. \tag{144}$$

Proof. For all no-default periods $0 \leq p \leq S-1$, i.e. for all $t \in [\tau_p, \tau_{p+1})$, every company $i \in \mathcal{S}^p$has a default intensity $\lambda^{ip} > 0$ and the different default intensities are linked by the system of equations (141). This means that we have to find the solutions $d^{(p+1)}(w, x, i)$, which are valid within the $p+1$-th default period. By replacing (142, 143) into (141) one can quickly verify the Proposition. Finally, the relation (144) immediately follows from the system of equations (141), since all denominators are the same. ■

Within each no-default period we can calculated the probability of having a further default event in a certain time interval and which company most likely defaults. In fact

Proposition 157 *Within all no-default periods $0 \leq p \leq S-1$ and for all $s \leq t$ in $[\tau_p, \tau_{p+1})$,*

$$\mathbb{P}[\tau_{p+1} < t \mid \mathcal{F}_s] = 1 - e^{-(t-s) \sum_{j \in \mathcal{S}^p} \lambda^{jp}} \tag{145}$$

and for all surviving companies $i \in \mathcal{S}^p$:

$$\mathbb{P}[\tau_{p+1} < t, C_{p+1} = i \mid \mathcal{F}_s] = \mathbb{P}[C_{p+1} = i \mid \mathcal{F}_s]\ \mathbb{P}[\tau_{p+1} < t \mid \mathcal{F}_s], \tag{146}$$

where we defined $\mathbb{P}[C_{p+1} = i \mid \mathcal{F}_s] \stackrel{def}{=} \frac{\lambda^{ip}}{\sum_{j \in \mathcal{S}^p} \lambda^{jp}}$.

Proof. Similarly as for Proposition 146, the probabilities (145, 146) can be obtained by integrating the densities (143, 142), described in the Proposition 156. ■

Remark 158 *From equality (146) we see that the events* $\{C_{p+1} = i\}$ *and* $\{\tau_{p+1} < t\}$ *are formally independent, given the information* $\mathcal{F}_s$.

Remark 159 *Moreover we observe that for all no-default periods* $0 \leq p \leq S$, *for all* $t \geq s \geq \tau_p$ *we have*

$$\mathbb{P}[\tau_{p+1} < t \mid \mathcal{F}_s] = \sum_{j \in \mathcal{S}^p} \mathbb{P}[C_{p+1} = j \mid \mathcal{F}_s] \; \mathbb{P}[\tau_{p+1} < t \mid \mathcal{F}_s], \tag{147}$$

and thus

$$\sum_{j \in \mathcal{S}^p} \mathbb{P}[C_{p+1} = j \mid \mathcal{F}_s] = \; 1,$$

which confirms that if there is a default event one and only one company defaults.

Similarly as in the constant default intensity model we can calculated the default time densities jumps $\Delta^s_{\tau_p}$ at time τ_p for the survived companies $s \in \mathcal{S}^p$

$$\begin{aligned}
\Delta^s_{\tau_p} &= d^{(p+1)}(w, 0, s) - d^{(p)}(w, \tau_p, s) \\
&= \lambda^{sp} e^{-0 \sum_{j \in \mathcal{S}^p} \lambda^{jp}} - \lambda^{sp-1} e^{-\tau_p \sum_{j \in \mathcal{S}^{p-1}} \lambda^{jp-1}} \\
&= \left(\lambda^{sp-1} + \delta^{sp}\right) - \lambda^{sp-1} e^{-\tau_p \sum_{j \in \mathcal{S}^{p-1}} \lambda^{jp-1}} \\
&= \lambda^{sp-1} \left(1 + \frac{\delta^{sp}}{\lambda^{sp-1}} - e^{-\tau_p \sum_{j \in \mathcal{S}^{p-1}} \lambda^{jp-1}}\right),
\end{aligned} \tag{148}$$

and its relative jump at time τ_p corresponds to

$$\begin{aligned}
\Delta^s_{\tau_p} &= \frac{d^{(p+1)}(w, 0, s) - d^{(p)}(w, \tau_p, s)}{d^{(p+1)}(w, 0, s)} \\
&= \left(1 - \frac{\delta^{sp}}{\lambda^{sp}}\right) \left(1 + \frac{\delta^{sp}}{\lambda^{sp-1}} - e^{-\tau_p \sum_{j \in \mathcal{S}^{p-1}} \lambda^{jp-1}}\right),
\end{aligned} \tag{149}$$

In absolute value the distribution density jump $\Delta^s_{\tau_p}$ (formula (148)) has an extra term $(\delta^{sp}/\lambda^{sp-1})$ than the one calculated in the constant default intensity model (formula (138)). This additional term is due to the additional default feedback δ^{sp}, which we allow in this model. Hence when all $\delta^{sp} = 0$, formula (148) reduces to formula (138). Further we observe that in this model the extra jump in the default densities $(\delta^{sp}/\lambda^{sp-1})$ is dependent, as expected, from the relative default intensity jump at time τ_p

$$\frac{\delta^{sp}}{\lambda^{sp-1}} = \frac{\lambda^{sp} - \lambda^{sp-1}}{\lambda^{sp-1}}.$$

In case investors expect at time τ_p that the default of company C_p has a negative (resp. positive or neutral) impact on the future business of the surviving company s, then $\delta^{sp} > 0$ (resp. $\delta^{sp} \leq 0$) and the survival's company future default probability will increase (resp. decrease).

For simulating default sequences one has to

1. Define the $S(S-1) + ... + p(p-1) + ... + 2$ possible default intensity shocks δ^{si} that could happen to the $p-1$ surviving companies after the pth default event τ_p of company C_p;

2. Calculate the default time intensities for the surviving companies: λ^{sp};
3. Calculate the default event density distribution after time $\tau_p : d^{(p+1)}(w,x)$;
4. Draw the next **default time** τ_{p+1} from the default event distribution $d^{(p+1)}(w,x)$;
5. Dice **which** company defaults: company $i \in S^p$ defaults with probability

$$\mathbb{P}[C_{p+1} = i \mid \mathcal{F}_s] = \frac{\lambda^{ip}}{\sum_{j \in S^p} \lambda^{jp}}.$$

6. Restart from Step 2. until all companies are defaulted.

Remark 160 *Instead of defining $S(S-1) + ... + p(p-1) + ... + 2$ possible default intensity shocks δ^{si}, one can just define $S(S-1)$ default intensity shocks δ^{si}, describing the default impacts of company i on company s.*

11.5 Multi Company Discrete Jump Intensity Model

In this section we generalize the S jump intensity model, described in the previous section. The economic system contains S companies, whose intensities λ_t^i may jump at default event of another company τ_p, but they may also jump at some time points $s \geq \tau_p$, when investors receive new information about the economic situation $\mathcal{E}_s$. More precisely we assume that the investors receive new economic information $\mathcal{E}_s$ Δ-periodically after each default event τ_p, i.e. at time $t_k^p = \tau_p + k\Delta$, $k = 0, ..., K^p$ with K^p such that $t_{K^p}^p \leq \tau_{p+1}$ and $t_{K^p+1}^p > \tau_{p+1}$. We represent the default intensities of company i for $t \geq 0$ by

$$\lambda_t^i \stackrel{def}{=} \sum_{p=0}^{S-1} \sum_{k=0}^{K^p} \lambda^{ipk} \mathbf{1}_{\{\tau_p \leq t < \tau_{p+1}\}} \mathbf{1}_{\{t_k^p \leq t < t_{k+1}^p\}} \mathbf{1}_{\{i \in S^p\}}, \tag{150}$$

where λ^{ipk} is the $\mathcal{F}_{t_k^p}$-measurable intensity, constant over the period $[t_k^p, t_{k+1}^p)$ and vanishes at default of the company ($i \notin S^p$).

Notation 161 *We denote the **intensity jumps of company** i at the time t_k^p, $p = 1, ..., S$ and $k = 1, ..., K^p$ by $\delta^{ipk} \stackrel{def}{=} \lambda^{ipk} - \lambda^{ipk-1}$*

Corollary 162 *With the above definitions and notations, we can show that for all no-defaults period $0 \leq p \leq S-1$ and $i \in S^p$ the following equality holds*

$$\lambda^{ipk} \mathbf{1}_{\{i \in S^p\}} = \left\{ \lambda^{ip0} + \sum_{j=1}^{K^p} \delta^{ipj} \right\} \mathbf{1}_{\{i \in S^p\}}. \tag{151}$$

Proof. Similar as Corollary 155. ■

The dependencies between the companies are assumed as usual to be modeled via the equations (128) of Theorem 141, i.e. within each no-default period $0 \leq p \leq S-1$ for all $t \in [\tau_p, \tau_{p+1})$ the following system of equalities must hold

$$\lambda^{ip} = \frac{d^{(p+1)}(w, t-\tau_p, i)}{1 - \int_0^{t-\tau_p} d^{(p+1)}(w,u)\, du}, \qquad i \in S^p. \tag{152}$$

More precisely since we have economic information $\mathcal{E}_{t_k^p}$ until t_k^p we have to introduce again our information variable t_k^p and thus for $t \in [t_k^p, t_{k+1}^p)$ the following system of equalities must be satisfied

$$\lambda^{ipk} = \frac{d^{(p+1)}(w, t - t_k^p, t_k^p, i)}{1 - \int_0^{t - t_k^p} d^{(p+1)}(w, u, t_k^p)\, du}, \qquad i \in \mathcal{S}^p. \tag{153}$$

In the next Proposition we will describe the solutions of (153): the default densities $d^{(p+1)}(w, x, t_k^p, i)$ at each t_k^p.

Proposition 163 *Under the assumptions of the present section, the problem has a unique solution within all no-defaults periods $0 \leq p \leq S - 1$, for all $t \in [t_k^p, t_{k+1}^p)$ $(x = t - t_k^p)$, given by*

$$d^{(p+1)}(w, x, t_k^p, i) = \lambda^{ipk} e^{-x \sum_{j \in \mathcal{S}^p} \lambda^{jpk}}, \quad i \in \mathcal{S}^p \tag{154}$$

for the already defaulted companies

$$d^{(p+1)}(w, x, i) = 0, \quad i \in \mathcal{D}^p$$

and the $p + 1$-event distribution

$$d^{(p+1)}(w, x, t_k^p) = \left(\sum_{j \in \mathcal{S}^p} \lambda^{jpk}\right) e^{-x \sum_{j \in \mathcal{S}^p} \lambda^{jpk}}, \tag{155}$$

Moreover for all $i, j \in \mathcal{S}^p$ and for all $t \in [\tau_p, \tau_{p+1})$ must hold

$$\frac{d^{(p+1)}(w, t - t_k^p, t_k^p, i)}{\lambda^{ipk}} = \frac{d^{(p+1)}(w, t - t_k^p, t_k^p, j)}{\lambda^{jpk}}. \tag{156}$$

Proof. Same as for Proposition 156. ∎

Within each no-default period we can calculated the probability of having a further default event in a certain time interval and which company most likely defaults. In fact

Proposition 164 *Within all no-default periods $0 \leq p \leq S - 1$, for all $k = 1, ..., K^p$ and for all $s \leq t$ in $[t_k^p, t_{k+1}^p)$,*

$$\mathbb{P}[\tau_{p+1} < t \mid \mathcal{F}_s] = 1 - e^{-(t-s) \sum_{j \in \mathcal{S}^p} \lambda^{jpk}} \tag{157}$$

and for all surviving companies $i \in \mathcal{S}^p$:

$$\mathbb{P}[\tau_{p+1} < t, C_{p+1} = i \mid \mathcal{F}_s] = \mathbb{P}[C_{p+1} = i \mid \mathcal{F}_s]\ \mathbb{P}[\tau_{p+1} < t \mid \mathcal{F}_s], \tag{158}$$

where we defined $\mathbb{P}[C_{p+1} = i \mid \mathcal{F}_s] \stackrel{def}{=} \frac{\lambda^{ipk}}{\sum_{j \in \mathcal{S}^p} \lambda^{jpk}}$.

Proof. Similarly as for Proposition 146. ∎

Remark 165 *This last discrete model suggest the default sequence simulation procedure described in Section 9.3.*

12 Notations

$\mathbb{E}$ Expectation operator

$\mathbb{P}$ Probability operator

$\mathbb{R}$ Real numbers

$\mathcal{E}$ Filtration of economy

$\mathcal{E}$ Filtration of economy & default history

$\mathcal{G}$ Filtration of default history

$\mathcal{D}_t$ Defaulted companies until time t

$\mathcal{D}^i$ Defaulted companies after the i default

$\mathcal{S}_t$ Survived companies after time t

$\mathcal{S}^i$ Survived companies before $i+1$ default

$\mathcal{M}$ Set of marketed securities

$\mathcal{C}$ Set of marketed default prone securities

C_t Company that defaults at time t

N_t Default indicator process

1_A Set indicator

W_t Brownian motion

X_t Underlying risk factor processes

U_t Underlying firm specific risk process

I Investment

E Equity process

R Recoveries

L Losses

L_i Liabilities

A Asset process

P Bond Price

P' Default bond price

Z Recovery rates

Q Loss rates

S Surplus information value process

d Default time densities

FV Face value

CE Credit exposure

MV Market value

$BS(.)$ Black-Scholes call option pricing formula

Ω Probability set

$\Pi(M)$ Price of M

Δ time length

τ_i Default times

δ_{i+1} Time to the next default

$\delta^r_{s,t}$ Discount rate

λ Default intensities

μ Instantaneous drift

σ Volatility dependence matrix

$\sigma(A)$ σ-algebra generated by A

$\preceq$ preferred

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Curriculum Vitae

Personal data

Name	Marcel Beat Rüegg
Date of birth	26.11.1970
Citizen of	Lumino (Ticino, Switzerland) and St.Gallenkappel St.Gallen, Switzerland)
Address	Gaissenrain 1, 8802 Kilchberg (Zurich)
Marital status	Married since 1992
Children	1 daughter, born 29.1.2000
Military service	Specialist in meteorology until 1997 Avalanche specialist since 1998

Professional experience

1998 - to date	**Risk Analyst in Corporate Integrated Risk Management with Swiss Re**, working on various projects: development of Euler capital allocation principle and credit insurance measurement tool, tested several credit portfolio management tools (KMV portfolio manager, CreditMetrics), developed with Tom Wilson ([Wt]) market and credit aggregation engine taking into account their underlying economic interdependencies.
1996 - 1998	**Quantitative consultant in Financial Risk Management with UBS**, having worked on various projects: Risk capital measurement, UBS overall capital aggregation, capital allocation to business divisions, development of performance measures like RARORC, RAROREC, developed and took part in implementing a system for measuring the credit risk of derivative contracts, developed and calculated the risk capital needs for the operational risks
1996 -1998	and premises.
1996 -1998	**ASVZ sports coach**
1987 - 1996	**Private mathematics tuition**
July - August 1991	**Zürcher Kantonalbank, inheritance department**
July - August 1990	**Zürcher Kantonalbank, cash department**
July - August, 1986 - 1989	**Sanitation engineer with Preisig** in Oerlikon
July 1985	**Voluntary community work** in Molare

Education

1996 - 2000	**Financial mathematics dissertation at ETH Zurich:** *Default and Recovery Risks Valuation in Incomplete Markets*
1991 - 1996	**Studies in Mathematics at ETH Zurich** with optional subjects:Theoretical physics, Functional analysis, Differential geometry and Financial mathematics **Master thesis in financial mathematics** on *Optimal Consumption and Portfolio Choice under Borrowing Constraints*; awarded the Walter Saxer-Versicherungs-Hochschulpreis in 1996
1987 - 1991	**Bellinzona high school** (Ticino), Latin/English options
1983 - 1987	**Castione middle school** (Ticino)
1981 - 1983	**Lumino primary school** (Ticino)
1978 - 1981	**Regensdorf primary school** (Zurich)

Publications & papers

June 1999	*The Use of Risk Adjusted Capital to Support Business Decision-Making*,Casualty Actuarial Society and Casualty Actuaries in Reinsurance, with co-authors Gary Patrick, Stefan Bernegger
January 1999	*Risk Adjusted Capital Allocation*, Swiss Re publication
September 1996	*Optimal Consumption and Portfolio Choice under Borrowing Constraints*, Master thesis

Interests

Hobbies	Sport (rowing, mountaineering), chess and literature.